On a divisibility property of Fibonacci polynomials

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January 6, 2006

1 Introduction

The Fibonacci sequence $(u_1, u_2, u_3, \ldots) = (1, 1, 2, 3, 5, \ldots)$, given by the recursion $u_{n+2} = u_{n+1} + u_n$, has the closed expression $u_n = \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}}$ (Binet's formula), where $\theta = \frac{\sqrt{5}+1}{2}, \bar{\theta} = \frac{1-\sqrt{5}}{2}$, see e.g. [K], [HW]. From this follows the factorization

$$u_n = \prod_{1 \neq d | n} \Phi_d(\theta, \bar{\theta}),$$

where $\Phi_d(a, b)$ is the bivariate homogeneous version of the cyclotomic polynomial, see [C],[BMS]. The numbers $\Phi_d(\theta, \bar{\theta})$ are integers, since $\Phi_d(a, b)$ is symmetric in a, b hence $\Phi_d(\theta, \bar{\theta})$ is a polynomial function over $\mathbb{Z}$ of the elementary symmetric functions of θ and $\bar{\theta}$, which are : $\theta + \bar{\theta} = 1,\ \theta\bar{\theta} = -1$.

In a completely analoguous way, one may factorize the Fibonacci polynomials $F_n(x)$, given by the recursion $F_{n+2}(x) = x\, F_{n+1}(x) + F_n(x)$, $F_0(x) = 1$, $F_1(x) = x$, and the closed formula $F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where α, β are the roots of the companion polynomial $t^2 - xt - 1$ of the recursion. Denoting the factors by $\theta_n(x)$, one has therefore $F_n(x) = \prod_{d|n} \theta_d(x)$ and $\theta_d(x) = \Phi_d(\alpha, \beta)$.

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These factors are called *fibotomic polynomials* in [L], where is proven their irreducibility, with arguments based on their roots.

We propose here to recover this result, to show the positivity of these irreducible polynomials, by generalizing these results to the bivariate Fibonacci polynomials and by using algebraic arguments. The key lemma uses the substitution $a \mapsto x+1, b \to -x$ which allows to pass from the recursion $u_{n+2} = au_{n+1} + bu_n$ to the recursion $u_{n+2} = (x+1)u_{n+1} - xu_n$ defining the polynomials $x^{n-1} + x^{n-2} + \cdots + 1 = \frac{x^n-1}{x-1}$.

2 Bivariate Fibonacci polynomials

The bivariate Fibonacci polynomials are defined by $F_n(a,b) = aF_{n-1}(a,b) + bF_{n-2}(a,b)$ for $n \geq 2$, and $F_0(a,b) = 0, F_1(a,b) = 1$.

Theorem 2.1 *There are irreducible polynomials $\psi_n(a,b)$ in $\mathbb{Z}[a,b]$ such that for any $n \geq 1$*

$$F_n(a,b) = \prod_{d|n} \psi_d(a,b).$$

Note that this defines uniquely the ψ_d. The theorem may be deduced from the results in [L], but we shall give a quick proof by reducing it to cyclotomic polynomials. The key lemma is the following.

Lemma 2.1 *Let $\mathbb{Z}[a,b]$ be graded by $\deg(a) = 1, \deg(b) = 2$. The polynomials $F_n(a,b)$ are homogeneous. The multiplicative monoid of homogeneous elements in $\mathbb{Z}[a,b]$ is isomorphic to its image under σ, where σ is the substitution $\mathbb{Z}[a,b] \mapsto \mathbb{Z}[x], \sigma(a) = x+1, \sigma(b) = -x$. The image of σ is the semigroup of palindromic polynomials in $\mathbb{Z}[x]$.*

We call *palindromic* a polynomial which is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_r x^r$, with $(a_n, a_{n-1}, \ldots, a_r) = (a_r, \ldots, a_{n-1}, a_n)$. For use below, such a polynomial will get the degree $n+r$, if $a_n \neq 0$; note that this is well-defined, and that palindromic polynomials form a submodule of $\mathbb{Z}[x]$.

Proof The $F_n(a,b)$ are clearly homogenous. We show that σ induces a $\mathbb{Z}$-module isomorphism from the submodule of homogeneous polynomials of degree n of $\mathbb{Z}[a,b]$ onto the submodule of palindromic polynomials of degree n of $\mathbb{Z}[x]$ (as defined above).

It is enough to give bases of both submodules, which correspond under σ. For $\mathbb{Z}[a,b]$, it is the set of monomials $a^n, a^{n-2}b, a^{n-4}b^2, \ldots$

For $\mathbb{Z}[x]$, it is the set of their images under σ, that is, $(x+1)^n$, $-(x+1)^{n-2}x, (x+1)^{n-4}x^2, \ldots$ These polynomials are clearly palindromic of degree n, and form a basis as required. The lemma follows. ■

Proof of theorem It is immediate to verify by induction that $\sigma(F_n(a,b)) = x^{n-1}+x^{n-2}+\cdots+1$. Denoting by $\Phi_n(x)$ the cyclotomic polynomial, it is well-known that $\Phi_n(x)$ is palindromic and that $x^{n-1}+x^{n-2}+\cdots+1 = \prod_{d|n} \Phi_d(x)$ (we put $\Phi_1(x) = 1$ for our purposes). The lemma thus implies the theorem, with the ψ_n defined by $\sigma(\psi_n(a,b)) = \Phi_n(x), \psi_n$ homogeneous. Moreover, the ψ_n must be irreducible. ■

Corollary 2.1 *The $\psi_n(a,b)$ have coefficients in $\mathbb{N}$.*

Proof It is well-known that $\Phi_n(x) = \prod(x-\theta)$, where the product is over all primitive $n-$th roots of unity. Note that θ and $\bar{\theta}$ are simultaneously primitive roots of unity, and are distinct (except if $n = 2$, a case which is self-evident). Hence $\Phi_n(x)$ is a product of $x^2 - (\theta+\bar{\theta}) + 1$.

Extending the lemma from $\mathbb{Z}$ to $\mathbb{R}$, we see that $x^2 - (\theta+\bar{\theta}) + 1$ is palindromic, equal to $(x+1)^2 + (2-\theta-\bar{\theta})(-x)$; hence it is the image under σ of $a^2 + (2-\theta-\bar{\theta})b$ which has nonnegative coefficients, since for $|\theta| = 1, 2-\theta-\bar{\theta} \in [0,4]$.

Thus the lemma implies that $\Phi_n(x)$ is a product of polynomials over $\mathbb{R}_+$, which proves the corollary. ■

This results suggests the following open problem: *find a combinatorial interpretation for the polynomials $\psi_n(a,b)$*. By the specialization of the lemma, it would lead to a combinatorial interpretation for the cyclotomic polynomials.

Corollary 2.2 *The $\psi_n(a,b)$ and $F_n(a,b)$ have the following factorization into irreducible polynomials over $\mathbb{C}$: $\psi_2(a,b) = F_2(a,b) = a$ and for $n \geq 3, \psi_n(a,b) = \prod_{\substack{(k,n)=1 \\ 1\leq k<\frac{n}{2}}} \left(a^2 + 4b\sin^2\frac{\pi k}{n}\right), F_n(a,b) = s_n \prod_{1\leq k<\frac{n}{2}} \left(a^2 + 4b\sin^2\frac{\pi k}{n}\right)$ with $s_n = a$ if n is even, $s_n = 1$ if n is odd.*

Proof We have simply to continue the previous proof, with again $n \geq 3$. The primitive n-th roots of unity are the $e^{2ik\pi/n}$, with $1 \leq k \leq n$ and $(k,n) = 1$.

Hence, we obtain all of them up to conjugation if we take $1 \leq k < \frac{n}{2}$, since $n \geq 3$, hence -1 is not primitive $n-$th root of unity. Now, if $\theta = e^{2i\pi k/n}$,

then $2-\theta-\theta = 2-2\cos\frac{2k\pi}{n} = 4\sin^2\frac{k\pi}{n}$, and therefore $\psi_n(a,b)$ is as required, and similarly for $F_n(a,b)$, since $\psi_1(a,b) = 1, \psi_2(a,b) = a$. ∎

The classical Fibonacci polynomials $F_n(x)$ are defined by $F_0(x) = 0$, $F_1(x) = 1, F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for $n \geq 2$.

Corollary 2.3 *There are irreducible polynomial $\theta_n(x) \in \mathbb{Z}[x]$ such that for any $n \geq 1, F_n(x) = \prod_{d|n} \theta_d(x)$. The roots of the $\theta_n(x)$ are the imaginary numbers $\pm 2i\sin\frac{\pi k}{n}$ and $(k,n) = 1$. The coefficients of $\theta_n(x)$ are nonnegative and $\deg(\theta_n) = \varphi(n)$.*

Proof We have simply to put $a = x, b = 1$ in the previous results: $F_n(x) = F_n(x,1)$. The irreducibility follows from the fact that the $F_n(a,b)$ are homogeneous for the degree defined in the Lemma and the formula $F_n(a,b) = b^{n-1/2}F_n\left(\frac{a}{\sqrt{b}}\right)$. ∎

We recover the irreducibility result of [L]; he shows that the $\theta_n(x)$ (called there fibotomic polynomials) are irreducible, by using an argument based on their roots.

Corollary 2.4 *Let (u_n) be a sequence of elements of a commutative ring such that $u_1 = 1, u_2 = a$ and $u_{n+2} = au_{n+1} + bu_n$ for $n \geq 0$. Then there is a sequence (α_n) of elements of the ring such that for any $n \geq 1, u_n = \prod_{d|n} \alpha_d$.*

Proof We have just to put $\alpha_n = \psi_n(a,b)$ in the theorem. ∎

Taking in the corollary $\mathbb{Z}$ as the ring and $a = b = 1$, we obtain the Fibonacci sequence $(u_1, u_2, u_3, \ldots) = (1, 1, 2, 3, 5, \ldots)$.

Corollary 2.5 *There is a sequence of natural numbers $(\alpha_n)_{n\geq 1}$ such that the Fibonacci numbers (u_n) satisfy $u_n = \prod_{d|n} \alpha_d$.*

Note that, by [HW] Th. 179 the Fibonacci numbers (u_n) satisfy $\gcd(u_n, u_m) = u_{\gcd(n,m)}$ (in particular, $u_n \mid u_m$ if $n \mid m$). Corollary 2.5 may be deduced from this, by standard arithmetical arguments, which we give in the Appendix.

Appendix: a divisibility property

Let $(u_n)_{n\geq 1}$ be a sequence of elements of a factorial ring R such that $\gcd(u_n, u_m) = u_{\gcd(n,m)}$. We show that there exists a sequence $(\alpha_n)_{n\geq 1}$ in R such that for $n \geq 1$, $u_n = \prod_{d|n} \alpha_d$.

As for cyclotomic polynomials, one obtains by Möbius inversion $\alpha_n = \prod_{d|n} u_d^{\mu\left(\frac{n}{d}\right)}$, where the quotients have to be taken in the field of fractions K of R. So, all we have to show is that α_n, defined by this equality, is in R.

Taking a fixed irreductible element p of R, by projection onto the powers of p, and considering the exponents of p, the problem is reduced to the following additive problem: let $(v_n)_{n\geq 1}$ be a sequence of natural integers such that $\min(v_n, v_m) = v_{gcd(n,m)}$; show that β_n defined by $\beta_n = \sum_{d|n} v_d \mu(n/d)$ is ≥ 0.

For this, let $p_1, \ldots, p_k$ be the distinct primes dividing n. For $A \subseteq \{1, \ldots, n\}$, let $p_A = \prod_{i\in A} p_i$ and $v_A = u_{n/p_A}$. Then $\beta_n = \sum_A v_A(-1)^{|A|}$ by the formula above. Moreover, the formula for $\min(v_n, v_m)$ translates into $v_{A\cup B} = \min(v_A, v_B)$, since $\gcd(n_A, n_B) = n_{A\cup B}$. We may assume that $v_{\{1\}} \leq v_{\{2\}} \leq \ldots \leq v_{\{k\}}$. Thus we obtain that for $A \neq \emptyset, v_A = v_{\{\min A\}}$. Hence $\beta_n = u_n + \sum_{i=1}^{k} v_{\{i\}} \sum_{\min(A)=i} (-1)^{|A|}$.

Now, if $i \leq k-1$, one has, by letting $B = A\backslash\{i\}$, $\sum_{\min(A)=i} (-1)^{|A|} = -\sum_{B\subseteq\{i+1,\ldots,k\}} (-1)^{|B|} = 0$, as is well-known. Thus $\beta_n = u_n - v_{\{k\}} = u_n - u_{n/p_k} \geq 0$, since $\min(u_n, u_{n/p_k}) = u_{\gcd(n,n/p_k)} = u_{n/p_k}$.

Tables

The bivariate Fibonacci polynomials $F_n(a, b), n = 1, \ldots, 12$:

1, a, a^2+b, a^3+2ab, $a^4+3a^2b+b^2$, $a^5+4a^3b+3ab^2$, $a^6+5a^4b+6a^2b^2+b^3$, $a^7+6a^5b+10a^3b^2+4ab^3$, $a^8+7a^6b+15a^4b^2+10a^2b^3+b^4$, $a^9+8a^7b+21a^5b^2+20a^3b^3+5ab^4$, $a^{10}+9a^8b+28a^6b^2+35a^4b^3+15a^2b^4+b^5$, $a^{11}+10a^9b+36a^7b^2+56a^5b^3+35a^3b^4+6ab^5$.

The irreducible polynomials $\psi_n(a,b), n = 1, \ldots, 12$:

1, a, a^2+b, a^2+2b, $a^4+3a^2b+b^2$, a^2+3b, $a^6+5a^4b+6a^2b^2+b^3$, $a^4+4a^2b+2b^2$, $a^6+6a^4b+9a^2b^2+b^3$, $a^4+5a^2b+5b^2$, $a^{10}+9a^8b+28a^6b^2+35a^4b^3+15a^2b^4+b^5$, $a^4 + 4a^2b + b^2$.

The classical Fibonacci polynomials $F_n(x), n = 1, \ldots, 12$:

$1,\ x,\ x^2 + 1,\ x^3 + 2x,\ x^4 + 3x^2 + 1,\ x^5 + 4x^3 + 3x,\ x^6 + 5x^4 + 6x^2 + 1,$ $x^7+6x^5+10x^3+4x$, $x^8+7x^6+15x^4+10x^2+1$, $x^9+8x^7+21x^5+20x^3+5x$, $x^{10} + 9x^8 + 28x^6 + 35x^4 + 15x^2 + 1, x^{11} + 10x^9 + 36x^7 + 56x^5 + 35x^3 + 6x$.

The irreducible polynomials $\theta_n(x), n = 1, \ldots, 12$:

1, x, $x^2 + 1$, $x^2 + 2$, $x^4 + 3x^2 + 1$, $x^2 + 3$, $x^6 + 5x^4 + 6x^2 + 1$, $x^4 + 4x^2 + 2$, $x^6 + 6x^4 + 9x^2 + 1$, $x^4 + 5x^2 + 5$, $x^{10} + x^8 + x^6 + x^4 + x^2 + 1, x^4 + 4x^2 + 1$.

The sequence (α_n) of divisors of the Fibonacci numbers, $n = 1, \ldots, 12$: $1, 1, 2, 3, 5, 4, 13, 7, 17, 11, 89, 6$. This sequence is listed as number A061446 in [SP].

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