

Representation of Numbers in a Rational Base and the Mahler's Problem

Shigeki Akiyama, Christiane Frougny & Jacques Sakarovitch

Univ. Niigata, Paris VIII and LIAFA, CNRS / ENST

Part I

A short view on Mahler's problem

The fractional part of the powers of rational numbers

Notation

$\theta \in \mathbb{R}$ $\{\theta\}$ fractional part of θ

Problem

$\theta \in \mathbb{R}, \theta > 1$ Distribution of $S(\theta) = \{\theta^n\}_{n \in \mathbb{N}}$?

Theorem

For almost all θ , $S(\theta)$ is uniformly distributed.

The fractional part of the powers of rational numbers

Very few results are known for specific values of θ .

Proposition

θ Pisot $\implies 0$ is the only limit point of $S(\theta)$ (in \mathbb{R}/\mathbb{Z}).

Experimental results show that $S(\theta)$ looks :

- *uniformly distributed* for transcendental θ ,
- *very chaotic* for rational θ .

Theorem (Pisot ?? — Vijayaraghavan 40)

θ rational $\implies S(\theta)$ has infinitely many limit points.

Parameterisation of the problem

Fix the rational $\frac{p}{q}$, $p > q \geq 2$ coprime integers.

New problem

$\xi \in \mathbb{R}$ Distribution of $M_{\frac{p}{q}}(\xi) = \left\{ \xi \left(\frac{p}{q} \right)^n \right\}_{n \in \mathbb{N}}$?

Theorem

For almost all ξ , $M_{\frac{p}{q}}(\xi)$ is uniformly distributed.

The (generalized) Mahler approach

Notation

$I \subsetneq [0, 1[$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{ \xi \in \mathbb{R} \mid M_{\frac{p}{q}}(\xi) \text{ is eventually contained in } I \} .$$

Two directions of research:

Look for I as **large** as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is **empty**.

Look for I as **small** as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is **non empty**.

Theorem (Mahler 68)

$\mathbf{Z}_{\frac{3}{2}}([0, \frac{1}{2}[)$ *is at most countable.*

Open problem

Is $\mathbf{Z}_{\frac{3}{2}}([0, \frac{1}{2}[)$ non empty?

The search for big l with empty $Z_{\frac{p}{q}}(l)$

Theorem (Flatto, Lagarias, Pollington 95)

The set of reals s

such that $Z_{\frac{p}{q}}\left([s, s + \frac{1}{p}[\right)$ is empty

is dense in $[0, 1 - \frac{1}{p}]$.

Theorem (Bugeaud 04)

The same set is of Lebesgue measure $1 - \frac{1}{p}$.

The search for small l with non empty $Z_{\frac{p}{q}}(l)$

Theorem (Pollington 81)

$Z_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right]\right)$ is non empty.

The search for small l with non empty $Z_{\frac{p}{q}}(l)$

Theorem (Pollington 81)

$Z_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right]\right)$ is non empty.

Theorem (A.-F.-S. 05)

Let $p \geq 2q - 1$. There exists $Y_{\frac{p}{q}} \subset [0, 1[$ of measure $\frac{q}{p}$
such that $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is (countable) infinite.

The search for small I with non empty $Z_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$Z_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right]\right)$ is non empty.

Theorem (A.-F.-S. 05)

Let $p \geq 2q - 1$. There exists $Y_{\frac{p}{q}} \subset [0, 1[$ of measure $\frac{q}{p}$
such that $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is (countable) infinite.

Indeed $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions}\}$.

The search for small I with non empty $Z_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$Z_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right]\right)$ is non empty.

Theorem (A.-F.-S. 05)

Let $p \geq 2q - 1$. There exists $Y_{\frac{p}{q}} \subset [0, 1[$ of measure $\frac{q}{p}$
such that $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is (countable) infinite.

Indeed $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions}\}$.

What this means is what this talk is about.

Part II

Number systems and finite automata

Number systems (integer base p)

$$N \in \mathbb{N}$$

Representation of N in base p : word in $\{0, 1, \dots, p - 1\}^$*

Number systems (integer base p)

$$N \in \mathbb{N}$$

Representation of N in base p : word in $\{0, 1, \dots, p-1\}^*$

$$\langle N \rangle_p = a_k a_{k-1} \dots a_1 a_0$$

$$N = \sum_0^k a_i p^i$$

Number systems (integer base p)

$$N \in \mathbb{N}$$

Representation of N in base p : word in $\{0, 1, \dots, p-1\}^*$

$$\langle N \rangle_p = a_k a_{k-1} \dots a_1 a_0 \qquad N = \sum_0^k a_i p^i$$

$$L_p = \{ \langle N \rangle_p \mid N \in \mathbb{N} \} = A^* \setminus 0A^*$$

The base 3 number system

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$$

together with

$$A = \{0, 1, 2\}$$

	0	111	13
1	1	112	14
2	2	120	15
10	3	121	16
11	4	122	17
12	5	200	18
20	6	201	19
21	7	202	20
22	8	210	21
100	9	211	22
101	10	212	23
102	11	220	24
110	12	221	25

The base 3 number system

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

$$L_3 = \{\langle N \rangle_3 \mid N \in \mathbb{N}\} = A^* \setminus 0A^*$$

The base 3 number system

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$$

together with

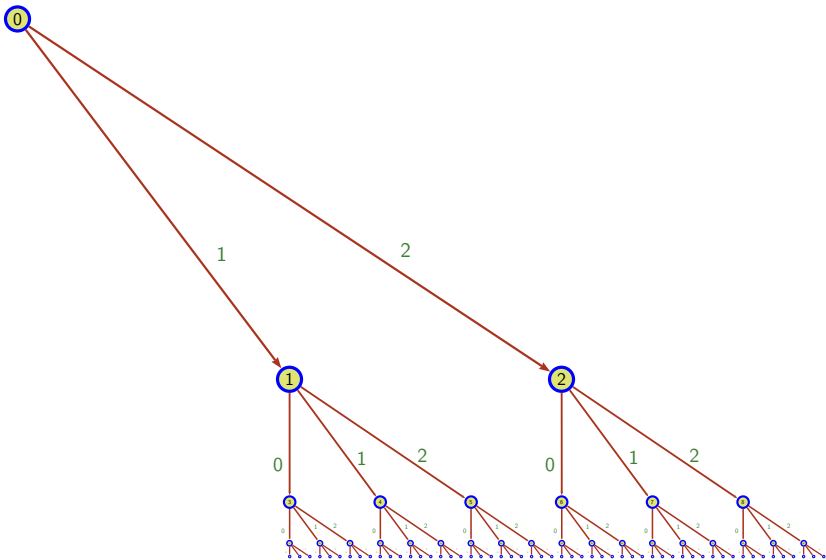
$$A = \{0, 1, 2\}$$

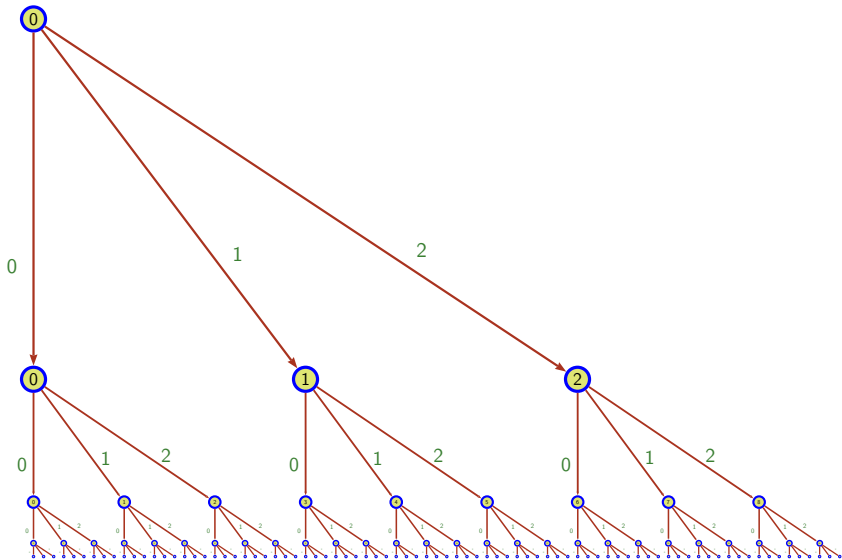
0000	0	0111	13
0001	1	0112	14
0002	2	0120	15
0010	3	0121	16
0011	4	0122	17
0012	5	0200	18
0020	6	0201	19
0021	7	0202	20
0022	8	0210	21
0100	9	0211	22
0101	10	0212	23
0102	11	0220	24
0110	12	0221	25

The base 3 number system

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

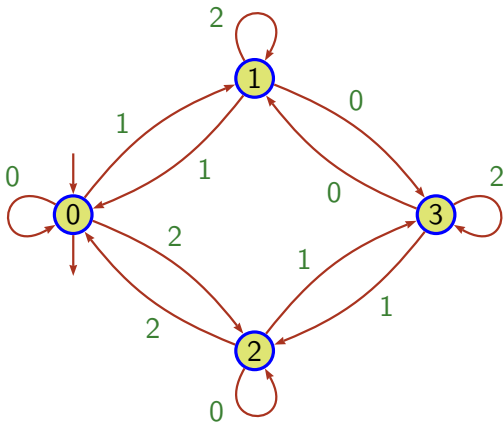
$$L'_3 = \{\langle N \rangle_3 \mid N \in \mathbb{N}\} = A^*$$





Number systems and finite automata (1)

The p -representations of the integers divisible by k
is recognised by a finite automaton.



Number systems and finite automata (1)

$$X \subseteq \mathbb{N}$$

Number systems and finite automata (1)

$$X \subseteq \mathbb{N}$$

Definition

X *recognisable* $\Leftrightarrow X =$ finite union of arithmetic prog.

X *p -recognisable* $\Leftrightarrow \langle X \rangle_p$ recognised by a finite automaton

Number systems and finite automata (1)

$$X \subseteq \mathbb{N}$$

Definition

X *recognisable* $\Leftrightarrow X =$ finite union of arithmetic prog.

X *p -recognisable* $\Leftrightarrow \langle X \rangle_p$ recognised by a finite automaton

Proposition

X *recognisable* $\implies X$ *p -recognisable* — in every base p

Number systems and finite automata (1)

$$X \subseteq \mathbb{N}$$

Definition

X *recognisable* $\Leftrightarrow X =$ finite union of arithmetic prog.

X *p -recognisable* $\Leftrightarrow \langle X \rangle_p$ recognised by a finite automaton

Proposition

X *recognisable* $\implies X$ *p -recognisable* — in every base p

Fact

X *p -recognisable* $\not\Rightarrow X$ *recognisable*

Number systems and finite automata (1)

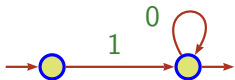
$$X \subseteq \mathbb{N}$$

Proposition

X recognisable $\implies X$ p -recognisable — in every base p

Fact

X p -recognisable $\not\implies X$ recognisable



$$L(\mathcal{A}) = 10^* \quad \pi(L(\mathcal{A})) = \{3^n \mid n \in \mathbb{N}\}$$

Number systems and finite automata (1)

$$X \subseteq \mathbb{N}$$

Proposition

X recognisable $\implies X$ p -recognisable — in every base p

Fact

X p -recognisable $\not\Rightarrow X$ recognisable

Theorem (Cobham 69)

p, q two multiplicatively independent integers.

X p -recognisable and q -recognisable $\implies X$ recognisable.

Number systems and finite automata (2)

Addition is a normalisation

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 1\ 1\ 2\ 0\ 0\ 1 \end{array}$$

Number systems and finite automata (2)

Addition is a normalisation

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 1\ 1\ 2\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 4\ 1\ 2\ 3\ 1 \end{array}$$

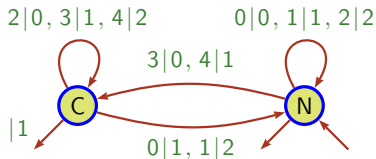
Number systems and finite automata (2)

Addition is a normalisation

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 1\ 1\ 2\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 4\ 1\ 2\ 3\ 1 \end{array}$$

$$\nu: \{0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$



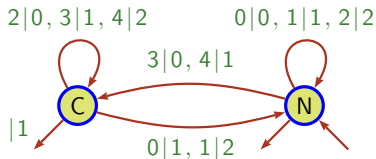
Number systems and finite automata (2)

Addition is a normalisation

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 1\ 1\ 2\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} 2\ 1\ 1\ 1\ 0 \\ 2\ 0\ 1\ 2\ 1 \\ \hline 4\ 1\ 2\ 3\ 1 \end{array}$$

$\nu: \{0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$



$$\leftarrow \frac{1}{1} C \leftarrow \frac{4}{1} N \leftarrow \frac{1}{2} C \leftarrow \frac{2}{0} C \leftarrow \frac{3}{0} N \leftarrow \frac{1}{1} N \leftarrow$$

Number systems and finite automata (2)

Proposition

*In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter right sequential transducer.*

Number systems and finite automata (2)

Proposition

*In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter right sequential transducer.*

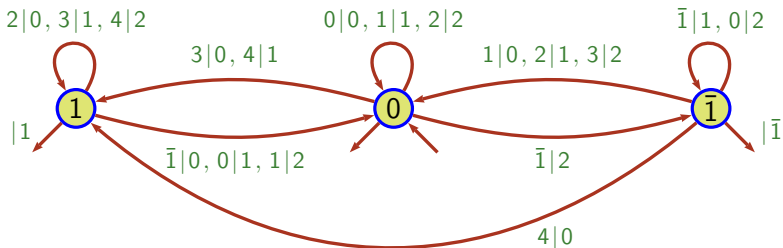
$$\nu: \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$

Number systems and finite automata (2)

Proposition

*In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter right sequential transducer.*

$$\nu: \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$

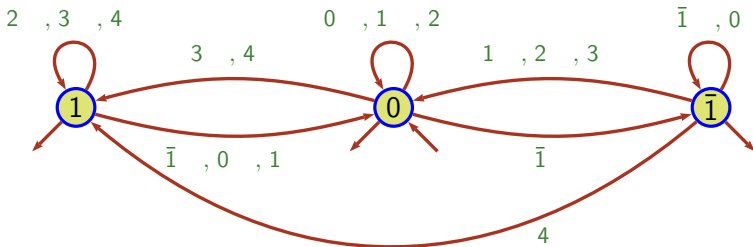


Number systems and finite automata (2)

Proposition

In any integer base,
normalisation from any alphabet of digits is realised
by a letter-to-letter *right sequential* transducer.

$$\nu: \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$

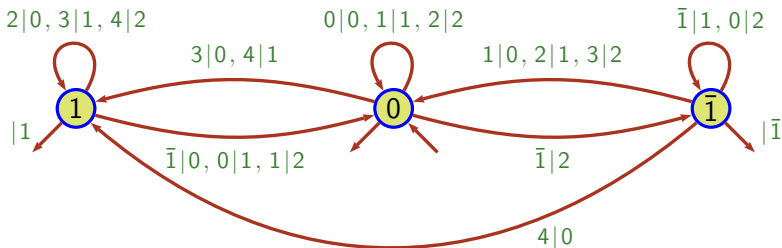


Number systems and finite automata (2)

Proposition

In any integer base,
normalisation from any alphabet of digits is realised
by a *letter-to-letter* right sequential transducer.

$$\nu: \{\bar{1}, 0, 1, 2, 3, 4\}^* \longrightarrow \{0, 1, 2\}^*$$



How are the representations in base 3 computed ?

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

How are the representations in base 3 computed ?

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Greedy algorithm $N \in \mathbb{N} \quad \exists k \quad 3^{k+1} > N \geq 3^k$

$$N_k = N$$

$$N_{k-1} = N_k - a_k 3^k \quad a_k \in A, \quad 3^k > N_{k-1}$$

$$N_{k-2} = N_{k-1} - a_{k-1} 3^{k-1} \quad a_{k-1} \in A, \quad 3^{k-1} > N_{k-2}$$

...

...

$$N = \sum_0^k a_i 3^i$$

$$\langle N \rangle_3 = a_k a_{k-1} \dots a_1 a_0$$

How are the representations in base 3 computed ?

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

$$\text{Greedy algorithm} \quad 17 \in \mathbb{N} \quad 3^{2+1} > 17 \geq 3^2$$

$$N_2 = 17$$

$$k = 2$$

$$N_1 = 17 - 1 \cdot 3^2 = 8$$

$$a_2 = 1 \in A, \quad 3^2 > 8$$

$$N_0 = 8 - 2 \cdot 3^1 = 2 = a_0$$

$$a_1 = 2 \in A, \quad 3^1 > 2$$

$$17 = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$$

$$\langle 17 \rangle_3 = 122$$

How are the representations in base 3 computed ?

$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, 2\}$

How are the representations in base 3 computed ?

$$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Greedy algorithm $x \in \mathbb{R} \quad \exists k \quad 3^{k+1} > x \geq 3^k$

$$x_k = x$$

$$x_{k-1} = x_k - a_k 3^k \quad a_k \in A, \quad 3^k > x_{k-1}$$

$$x_{k-2} = x_{k-1} - a_{k-1} 3^{k-1} \quad a_{k-1} \in A, \quad 3^{k-1} > x_{k-2}$$

...

...

$$x = \sum_{-\infty}^k a_i 3^i$$

$$\langle x \rangle_3 = a_k a_{k-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots$$

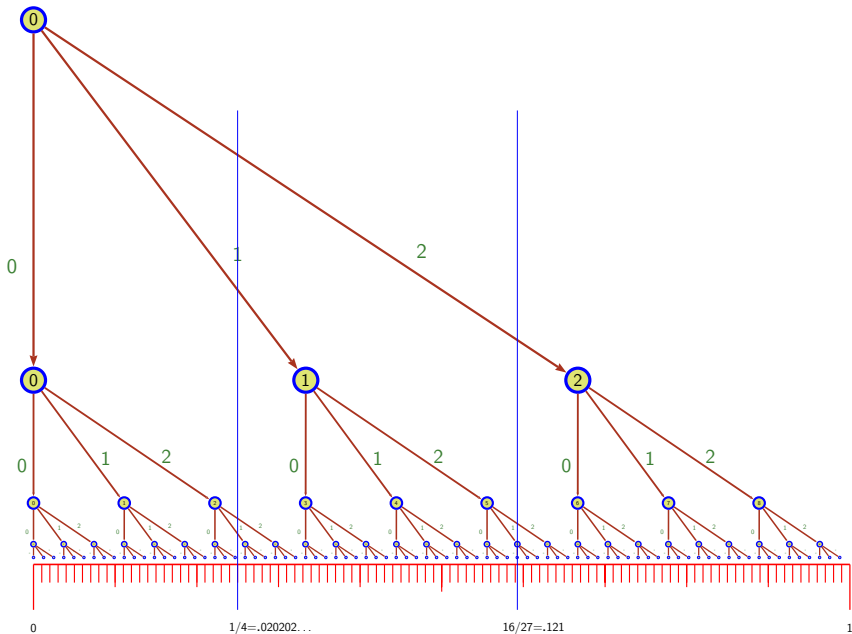
How are the representations in base 3 computed ?

$$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Greedy algorithm $x \in [0, 1[$

$$x_1 = x \quad a_i = \lfloor 3x_i \rfloor \quad x_{i+1} = \{3x_i\}$$

$$x = \sum_{i=1}^{\infty} a_i 3^{-i} \quad \langle x \rangle_3 = .a_1 a_2 a_3 \dots$$



The base β number system

$$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

The base β number system

$\beta > 1$ any real number

$V = \{v_i = (\beta)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$

The base β number system

$\beta > 1$ any real number

$V = \{v_i = (\beta)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$

Greedy algorithm $x \in [0, 1[$

$$x_1 = x \qquad a_i = \lfloor \beta x_i \rfloor \qquad x_{i+1} = \{\beta x_i\}$$

$$x = \sum_{i=1}^{\infty} a_i \beta^{-i} \qquad \langle x \rangle_{\beta} = .a_1 a_2 a_3 \dots$$

The base β number system

$\beta > 1$ any real number

$V = \{v_i = (\beta)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$

β Pisot \implies Parry Theorem

$$L_\beta = \{\langle x \rangle_\beta \mid x \in \mathbb{R}\} \in \text{Rat } A^\mathbb{N}$$

The base β number system

$\beta > 1$ any real number

$V = \{v_i = (\beta)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, \dots, \lfloor \beta \rfloor\}$

β Pisot \implies Parry Theorem

$$L_\beta = \{\langle x \rangle_\beta \mid x \in \mathbb{R}\} \in \text{Rat } A^\mathbb{N}$$

Theorem (Frougny, Berend-Frougny)

β is Pisot *iff* in base β ,

normalisation from any alphabet of digits

is realised by a letter-to-letter (finite) transducer.

Part III

Representation of integers in a rational base

The base $\frac{3}{2}$ number system – first approach

$$W = \{w_i = \left(\frac{3}{2}\right)^i \mid i \in \mathbb{Z}\} \quad A = \{0, \dots, \lfloor \frac{3}{2} \rfloor\} = \{0, 1\}$$

Greedy algorithm $x \in \mathbb{R} \quad \exists k \quad \left(\frac{3}{2}\right)^{k+1} > x \geq \left(\frac{3}{2}\right)^k$

$$x_k = x$$

$$x_{k-1} = x_k - a_k \left(\frac{3}{2}\right)^k \quad a_k \in A, \quad \left(\frac{3}{2}\right)^k > x_{k-1}$$

$$x = \sum_{-\infty}^k a_i \left(\frac{3}{2}\right)^i \quad \langle x \rangle_W = a_k a_{k-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots$$

$$\langle 2 \rangle_W = 10.010000010 \dots$$

The base 3 number system – another look

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

“Euclide” algorithm $N \in \mathbb{N}$

$$N'_0 = N$$

$$N'_0 = 3 N'_1 + a_0 \quad a_0 \in A$$

$$N'_1 = 3 N'_2 + a_1 \quad a_1 \in A$$

...

$$N = \sum_0^k a_i 3^i$$

$$\langle N \rangle_3 = a_k a_{k-1} \dots a_1 a_0$$

The base 3 number system – another look

$$V = \{v_i = (3)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

“Euclide” algorithm $17 \in \mathbb{N}$

$$N'_0 = 17$$

$$17 = N'_0 = 3 \cdot 5 + 2$$

$$a_0 = 2 \in A$$

$$5 = N'_1 = 3 \cdot 1 + 2$$

$$a_1 = 2 \in A$$

$$1 = N'_2 = 3 \cdot 0 + 1$$

$$a_2 = 1 \in A$$

$$17 = ((1) \cdot 3 + 2) \cdot 3 + 2$$

$$\langle 17 \rangle_3 = 122$$

The base $\frac{3}{2}$ number system – second version

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2} \right)^i \mid i \in \mathbb{N} \right\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Tuned “Euclide” algorithm $N \in \mathbb{N}$

$$N_0 = N$$

$$2N_0 = 3N_1 + a_0 \quad a_0 \in A$$

$$2N_1 = 3N_2 + a_1 \quad a_1 \in A$$

...

$$N = \sum_0^k a_i \frac{1}{2} \left(\frac{3}{2} \right)^i$$

$$\langle N \rangle_{\frac{3}{2}} = a_k a_{k-1} \dots a_1 a_0$$

The base $\frac{3}{2}$ number system – second version

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2} \right)^i \mid i \in \mathbb{N} \right\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Tuned “Euclide” algorithm $5 \in \mathbb{N}$

$$N_0 = 5$$

$$2N_0 = 2 \cdot 5 = 3 \cdot 3 + 1 \quad 1 \in A$$

$$2N_1 = 2 \cdot 3 = 3 \cdot 2 + 0 \quad 0 \in A$$

$$2N_2 = 2 \cdot 2 = 3 \cdot 1 + 1 \quad 1 \in A$$

$$2N_3 = 2 \cdot 1 = 3 \cdot 0 + 2 \quad 2 \in A$$

$$5 = \frac{1}{2} \left[\left(\left(\left((2) \cdot \frac{3}{2} + 1 \right) \cdot \frac{3}{2} + 0 \right) \cdot \frac{3}{2} + 1 \right) \right] \quad \langle 5 \rangle_{\frac{3}{2}} = 2101$$

The base $\frac{3}{2}$ number system

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2} \right)^i \mid i \in \mathbb{N} \right\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

The base $\frac{3}{2}$ number system

$$U = \left\{ u_i = \frac{1}{2} \left(\frac{3}{2} \right)^i \mid i \in \mathbb{N} \right\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Theorem

Every N in \mathbb{N} has an *integer* representation in the $\frac{3}{2}$ -system.

It is the *unique finite* $\frac{3}{2}$ -representation of N .

The base $\frac{3}{2}$ number system

$$U = \{u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Theorem

Every N in \mathbb{N} has an *integer* representation in the $\frac{3}{2}$ -system.

It is the *unique finite* $\frac{3}{2}$ -representation of N .

We call this representation the $\frac{3}{2}$ -*expansion* of N
and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

The base $\frac{3}{2}$ number system

$$U = \{u_i = \frac{1}{2} \left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Theorem

Every N in \mathbb{N} has an *integer* representation in the $\frac{3}{2}$ -system.

It is the *unique finite* $\frac{3}{2}$ -representation of N .

We call this representation the $\frac{3}{2}$ -*expansion* of N
and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

$$L_{\frac{3}{2}} = \{ \langle N \rangle_{\frac{3}{2}} \mid N \in \mathbb{N} \} = \text{????}$$

	0	212211	17
2	1	2101100	18
21	2	2101102	19
210	3	2101121	20
212	4	2120010	21
2101	5	2120012	22
2120	6	2120201	23
2122	7	2120220	24
21011	8	2120222	25
21200	9	2122111	26
21202	10	21011000	27
21221	11	21011002	28
210110	12	21011021	29
210112	13	21011210	30
212001	14	21011212	31
212020	15	21200101	32
212022	16	21200120	33

The set $L_{\frac{3}{2}}$ of $\frac{3}{2}$ -expansions

The restriction to $L_{\frac{3}{2}}$ of the coarsest right regular equivalence that saturates $L_{\frac{3}{2}}$ is the identity.

$\implies L_{\frac{3}{2}}$ is not rational (regular).

The set $L_{\frac{3}{2}}$ of $\frac{3}{2}$ -expansions

The restriction to $L_{\frac{3}{2}}$ of the coarsest right regular equivalence that saturates $L_{\frac{3}{2}}$ is the identity.

$\implies L_{\frac{3}{2}}$ is not rational (regular).

$$\forall m, n \in \mathbb{N}, \forall t \in A^* \left. \begin{array}{l} \langle m \rangle_{\frac{3}{2}} t \in L_{\frac{3}{2}} \\ \langle n \rangle_{\frac{3}{2}} t \in L_{\frac{3}{2}} \end{array} \right\} \iff m \equiv n \pmod{2^{|t|}}$$

The set $L_{\frac{3}{2}}$ of $\frac{3}{2}$ -expansions

The restriction to $L_{\frac{3}{2}}$ of the coarsest right regular equivalence that saturates $L_{\frac{3}{2}}$ is the identity.

$\implies L_{\frac{3}{2}}$ is not rational (regular).

Every word of A^* is a suffix of a word of $L_{\frac{3}{2}}$.

$$\forall w \in A^k \quad \exists! n \in \mathbb{N} \quad 0 \leq n < 3^k \quad \exists v \in A^*$$

$$\langle n \rangle_{\frac{3}{2}} = v w$$

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

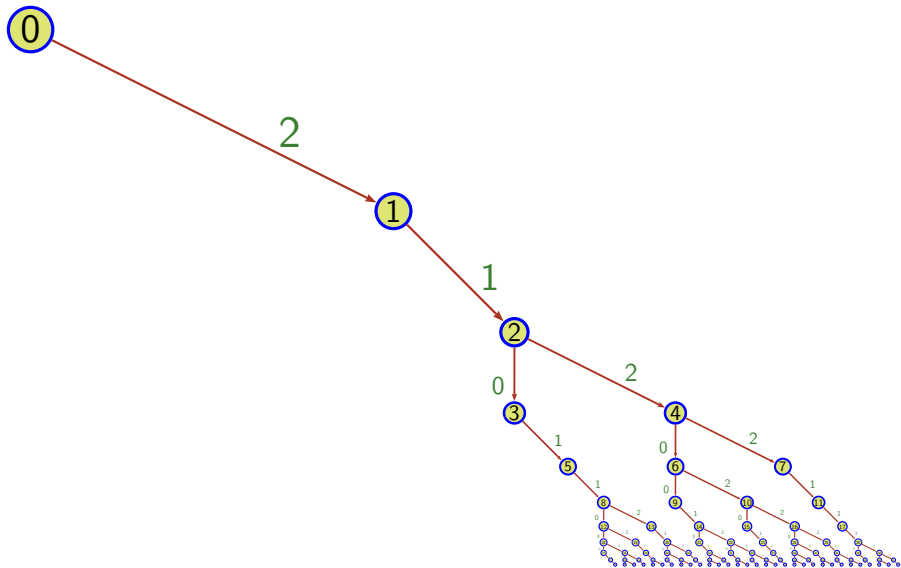
00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

00000000	0	00212211	17
00000002	1	02101100	18
00000021	2	02101102	19
00000210	3	02101121	20
00000212	4	02120010	21
00002101	5	02120012	22
00002120	6	02120201	23
00002122	7	02120220	24
00021011	8	02120222	25
00021200	9	02122111	26
00021202	10	21011000	27
00021221	11	21011002	28
00210110	12	21011021	29
00210112	13	21011210	30
00212001	14	21011212	31
00212020	15	21200101	32
00212022	16	21200120	33

The tree $I_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$L_{\frac{3}{2}}$ prefix-closed $\implies L_{\frac{3}{2}}$ spans the edges
of a subtree $I_{\frac{3}{2}}$ of the full 3-ary tree.

The tree $I_{\frac{3}{2}}$



The tree $I_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$L_{\frac{3}{2}}$ prefix-closed $\implies L_{\frac{3}{2}}$ spans the edges
of a subtree $I_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $I_{\frac{3}{2}}$ are labeled by the integers.

The tree $L_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$L_{\frac{3}{2}}$ prefix-closed $\implies L_{\frac{3}{2}}$ spans the edges
of a subtree $L_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $L_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented
by the label of the path from the root to that node.

The tree $I_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$L_{\frac{3}{2}}$ prefix-closed $\implies L_{\frac{3}{2}}$ spans the edges
of a subtree $I_{\frac{3}{2}}$ of the full 3-ary tree.

The nodes of $I_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented
by the label of the path from the root to that node.

These labels give *the radix order* in $I_{\frac{3}{2}}$.

The tree $I_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$L_{\frac{3}{2}}$ prefix-closed $\implies L_{\frac{3}{2}}$ spans the edges
of a subtree $I_{\frac{3}{2}}$ of the full 3-ary tree.

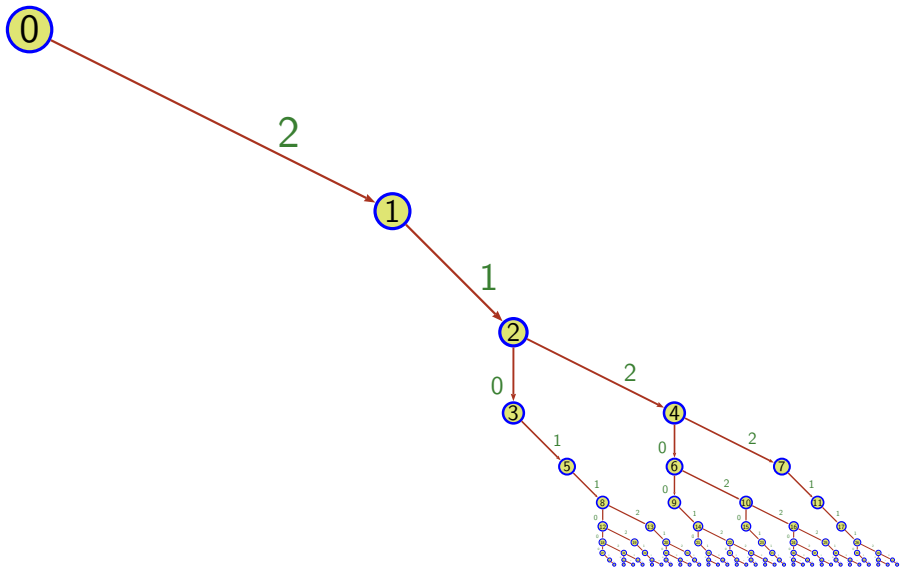
The nodes of $I_{\frac{3}{2}}$ are labeled by the integers.

The label of a node is the integer represented
by the label of the path from the root to that node.

These labels give *the radix order* in $I_{\frac{3}{2}}$.

Any two distinct subtrees of $I_{\frac{3}{2}}$ are not isomorphic.

The tree $I_{\frac{3}{2}}$



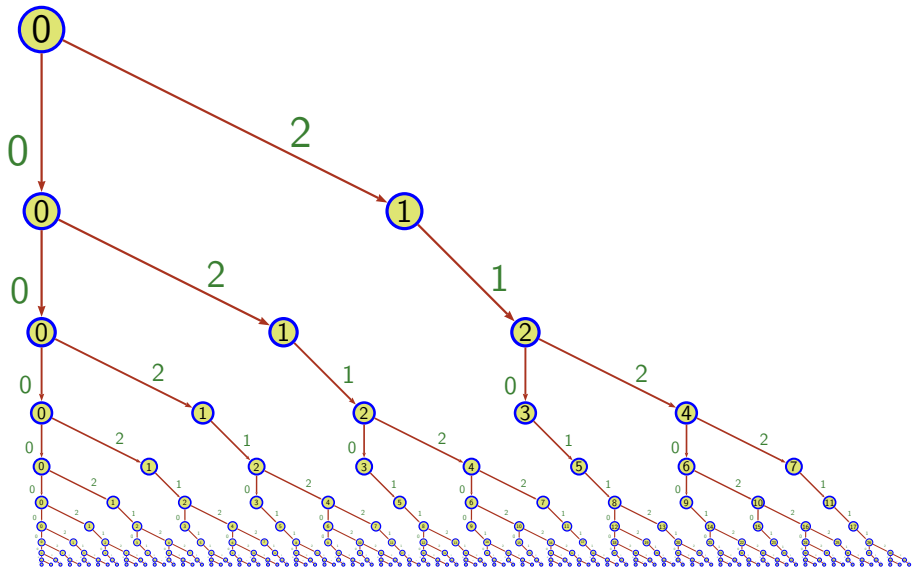
The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$ -expansions

$I_{\frac{3}{2}}$ can be “completed” into a tree $T_{\frac{3}{2}}$.

There exists a sequence of integers M_k such that
the nodes of depth k in $T_{\frac{3}{2}}$
are labeled by integers from 0 to M_k .

These labels give the lexicographic ordering
on the nodes of depth k .

The tree T_{2^3}



Digit conversion

D finite digit alphabet, that contains A .

$$\chi_D: D^* \rightarrow A^* \quad \forall w \in D^* \quad \pi(\chi_D(w)) = \pi(w) .$$

Proposition

For every D , χ_D is realised
by a letter-to letter sequential right transducer.

Digit conversion

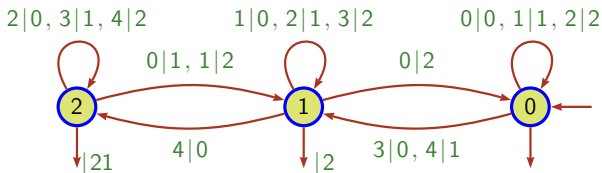
D finite digit alphabet, that contains A .

$$\chi_D: D^* \rightarrow A^* \quad \forall w \in D^* \quad \pi(\chi_D(w)) = \pi(w) .$$

Proposition

For every D , χ_D is realised

by a letter-to letter sequential right transducer.

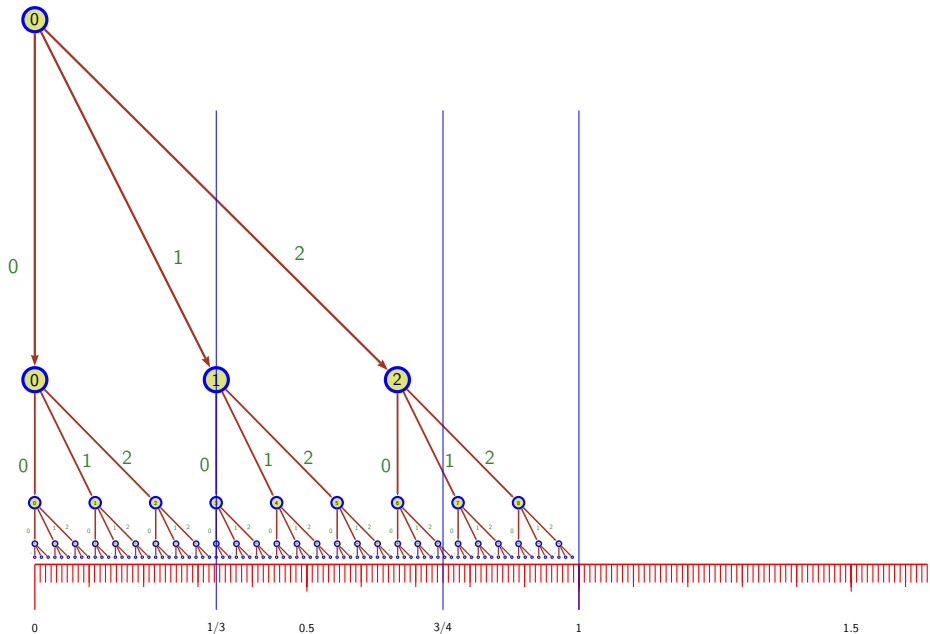


Part IV

Representation of reals in a rational base

Representation of reals in base 3 : the tree T_3

Representation of reals in base 3 : the tree T_3



Representation of reals in base 3 : the tree T_3

$A^{\mathbb{N}}$ = labels of the *infinite paths* in T_3 $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an *expansion in base 3* of the real $x \in [0, 1]$ defined by:

$$x = \pi(\cdot \mathbf{a}) = \sum_{i \geq 1} a_i \left(\frac{1}{3}\right)^i .$$

Representation of reals in base 3 : the tree T_3

$A^{\mathbb{N}}$ = labels of the *infinite paths* in T_3 $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an *expansion in base 3* of the real $x \in [0, 1]$ defined by:

$$x = \pi(\cdot \mathbf{a}) = \sum_{i \geq 1} a_i \left(\frac{1}{3}\right)^i .$$

Every real in $[0, 1]$ has (at least) one expansion in base 3 .

Representation of reals in base 3 : the tree T_3

$A^{\mathbb{N}}$ = labels of the *infinite paths* in T_3 $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an *expansion in base 3* of the real $x \in [0, 1]$ defined by:

$$x = \pi(\cdot \mathbf{a}) = \sum_{i \geq 1} a_i \left(\frac{1}{3}\right)^i .$$

Every real in $[0, 1]$ has (at least) one expansion in base 3 .

Every real in $[0, 1]$ has at most **two** expansions in base 3 .

Representation of reals in base 3 : the tree T_3

$A^{\mathbb{N}}$ = labels of the *infinite paths* in T_3 $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an *expansion in base 3* of the real $x \in [0, 1]$ defined by:

$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \left(\frac{1}{3}\right)^i .$$

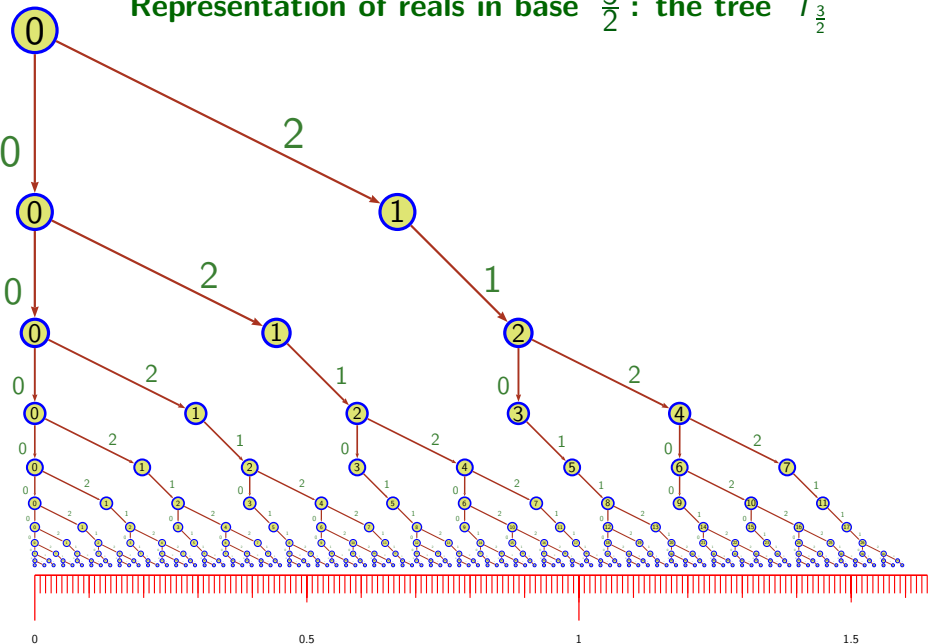
Every real in $[0, 1]$ has (at least) one expansion in base 3 .

Every real in $[0, 1]$ has at most **two** expansions in base 3 .

The set of reals in $[0, 1]$ which have two expansions
is infinite countable.

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) **iff** $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an \mathbf{a} is a $\frac{3}{2}$ -**expansion** of **the** real x defined by:

$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i.$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an \mathbf{a} is a $\frac{3}{2}$ -**expansion** of the real x defined by:

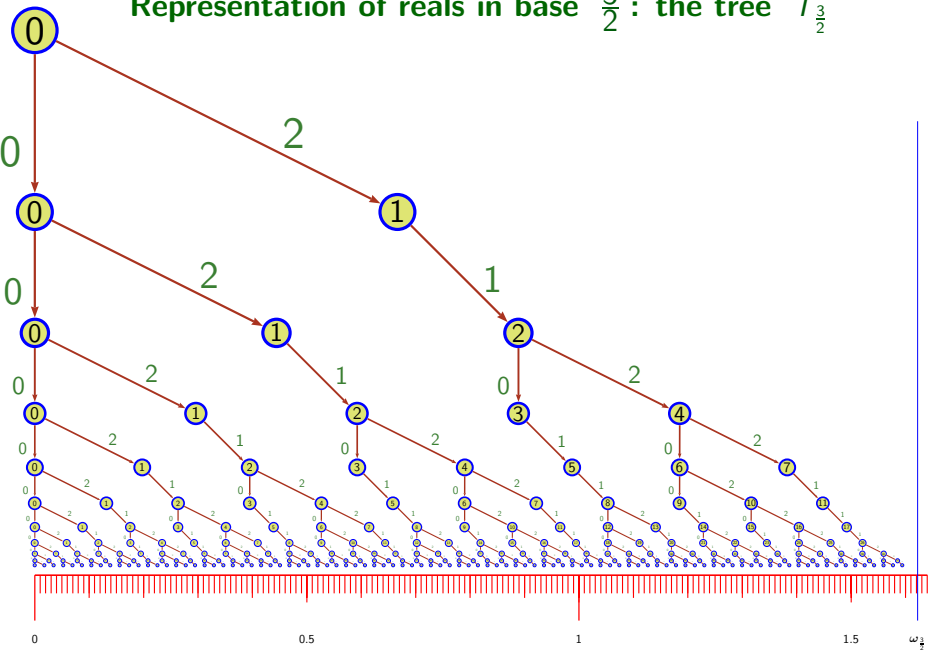
$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i.$$

$W_{\frac{3}{2}}$ contains a **maximal** word. $\mathbf{t}_{\frac{3}{2}}$ $\omega_{\frac{3}{2}} = \pi(\mathbf{t}_{\frac{3}{2}})$

$$\mathbf{t}_{\frac{3}{2}} = 212211122121122121211221 \dots$$

$$\langle \omega_{\frac{3}{2}} \rangle_{10} = 1.622270502884767315956950982 \dots$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) **iff** $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an \mathbf{a} is a $\frac{3}{2}$ -**expansion** of **the** real x defined by:

$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i.$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) **iff** $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an \mathbf{a} is a $\frac{3}{2}$ -**expansion** of **the** real x defined by:

$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i.$$

Theorem (A.-F.-S. 05)

Every real of $[0, \omega_{\frac{3}{2}}]$ has **(at least) one** $\frac{3}{2}$ -**expansion**.

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Such an \mathbf{a} is a $\frac{3}{2}$ -**expansion** of **the** real x defined by:

$$x = \pi(\mathbf{a}) = \sum_{i \geq 1} a_i \frac{1}{2} \left(\frac{2}{3}\right)^i.$$

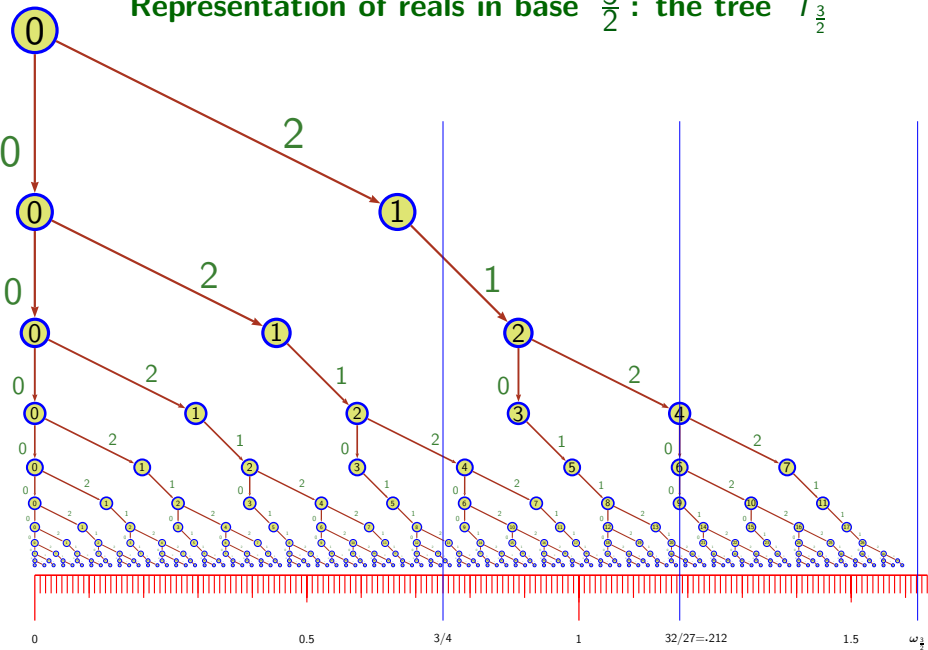
Theorem (A.-F.-S. 05)

Every real of $[0, \omega_{\frac{3}{2}}]$ has **(at least) one** $\frac{3}{2}$ -**expansion**.

$$X_{\frac{3}{2}} = \pi(W_{\frac{3}{2}})$$

$$X_{\frac{3}{2}} = [0, \omega_{\frac{3}{2}}].$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$



Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

$$X_{\frac{3}{2}} = \pi(W_{\frac{3}{2}})$$

$$X_{\frac{3}{2}} = [0, \omega_{\frac{3}{2}}].$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Theorem (A.-F.-S. 05)

Every real of $[0, \omega_{\frac{3}{2}}]$ has **(at least) one** $\frac{3}{2}$ -expansion.

$$X_{\frac{3}{2}} = \pi(W_{\frac{3}{2}})$$

$$X_{\frac{3}{2}} = [0, \omega_{\frac{3}{2}}] .$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Theorem (A.-F.-S. 05)

Every real of $[0, \omega_{\frac{3}{2}}]$ has **(at least) one** $\frac{3}{2}$ -expansion.

$$X_{\frac{3}{2}} = \pi(W_{\frac{3}{2}})$$

$$X_{\frac{3}{2}} = [0, \omega_{\frac{3}{2}}] .$$

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$

$W_{\frac{3}{2}}$ = labels of the *infinite paths* in $T_{\frac{3}{2}}$ $\mathbf{a} = \{a_i\}_{i \geq 1} \in A^{\mathbb{N}}$

Definition

\mathbf{a} is an **expansion** in base $\frac{3}{2}$ (of a real x) iff $\mathbf{a} \in W_{\frac{3}{2}}$.

Theorem (A.-F.-S. 05)

Every real of $[0, \omega_{\frac{3}{2}}]$ has **(at least) one** $\frac{3}{2}$ -expansion.

$$X_{\frac{3}{2}} = \pi(W_{\frac{3}{2}}) \qquad X_{\frac{3}{2}} = [0, \omega_{\frac{3}{2}}].$$

$W_{\frac{3}{2}}$ is closed in $A^{\mathbb{N}}$ compact $\Rightarrow W_{\frac{3}{2}}$ is compact.

$\pi: W_{\frac{3}{2}} \rightarrow X_{\frac{3}{2}}$ continuous $\Rightarrow X_{\frac{3}{2}}$ is closed

π preserves the order + property of $T_{\frac{3}{2}} \Rightarrow$

$[0, \omega_{\frac{3}{2}}] \setminus X_{\frac{3}{2}}$ cannot contain a non empty open interval.

Multiples $\frac{3}{2}$ -expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have **more than one $\frac{3}{2}$ -expansion** is infinite countable.

Multiples $\frac{3}{2}$ -expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have **more than one $\frac{3}{2}$ -expansion** is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω , is eventually periodic.

Multiples $\frac{3}{2}$ -expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have **more than one** $\frac{3}{2}$ -expansion
is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω , is eventually periodic.

The finite prefixes of a $\frac{3}{2}$ -expansion, completed with 0's,
are not $\frac{3}{2}$ -expansions.

Multiples $\frac{3}{2}$ -expansions

The set of reals of $[0, \omega_{\frac{3}{2}}]$ that have **more than one** $\frac{3}{2}$ -expansion is infinite countable.

No element of $W_{\frac{3}{2}}$, but 0^ω , is eventually periodic.

The finite prefixes of a $\frac{3}{2}$ -expansion, completed with 0's, **are not** $\frac{3}{2}$ -expansions.

Every real in $[0, \omega_{\frac{3}{2}}]$ has **at most two** $\frac{3}{2}$ -expansions.

The search for small I with non empty $Z_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$Z_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right]\right)$ is non empty.

Theorem (A.-F.-S. 05)

Let $p \geq 2q - 1$. There exists $Y_{\frac{p}{q}} \subset [0, 1[$ of measure $\frac{q}{p}$
such that $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ is (countable) infinite.

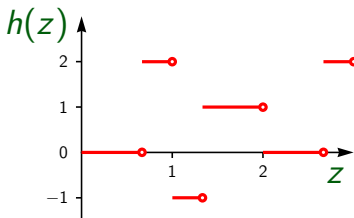
Indeed $Z_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions}\}$.

Part V

The companion representation

The companion $\frac{3}{2}$ -representation

$h: \mathbb{R}_+ \rightarrow \mathbb{Z}$ function defined by: $h(z) = 2 \lfloor (\frac{3}{2})z \rfloor - 3 \lfloor z \rfloor$



Proposition

h is periodic of period 2 and

$$h(z) \in C = \{-1, 0, 1, 2\}, \quad \forall z \in \mathbb{R}_+$$

The companion $\frac{3}{2}$ -representation

$$h_n(z) = h\left(\left(\frac{3}{2}\right)^{n-1}z\right) = c_n$$

$$\varphi(z): \mathbb{R}_+ \rightarrow \mathbb{C}^{\mathbb{N}} \quad \varphi(z) = \mathbf{c} = .c_1 c_2 \cdots c_n \cdots .$$

Proposition

$\forall z \in \mathbb{R}_+, \varphi(z)$ is a $\frac{3}{2}$ -representation of $\{z\} = z - \lfloor z \rfloor$.

$\forall k \in \mathbb{N}, .c_k c_{k+1} c_{k+2} \cdots$ is a $\frac{3}{2}$ -representation of $\left\{\left(\frac{3}{2}\right)^{k-1}z\right\}$.

The companion $\frac{3}{2}$ -representation

$$h_n(z) = h\left(\left(\frac{3}{2}\right)^{n-1}z\right) = c_n$$

$$\varphi(z): \mathbb{R}_+ \rightarrow \mathbb{C}^{\mathbb{N}} \quad \varphi(z) = \mathbf{c} = .c_1c_2 \cdots c_n \cdots .$$

Proposition

$\forall z \in \mathbb{R}_+, \varphi(z)$ is a $\frac{3}{2}$ -representation of $\{z\} = z - \lfloor z \rfloor$.

$\forall k \in \mathbb{N}, .c_k c_{k+1} c_{k+2} \cdots$ is a $\frac{3}{2}$ -representation of $\left\{\left(\frac{3}{2}\right)^{k-1}z\right\}$.

Proposition

$$\forall k \in \mathbb{N}, h_k(z) = 0 \implies \left\{\left(\frac{3}{2}\right)^{k-1}z\right\} \in [0, \frac{1}{3}[.$$

$$\forall k \in \mathbb{N}, h_k(z) = 1 \implies \left\{\left(\frac{3}{2}\right)^{k-1}z\right\} \in [\frac{2}{3}, 1[.$$

The **right** converter from C^* to A^*

$C = \{-1, 0, 1, 2\}$ contains A .

$$\chi_C: C^* \rightarrow A^* \quad \forall w \in C^* \quad \pi(\chi_C(w)) = \pi(w) .$$

Proposition

χ_C is realised by a letter-to letter sequential right transducer.

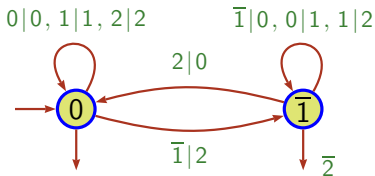
The **right** converter from C^* to A^*

$C = \{-1, 0, 1, 2\}$ contains A .

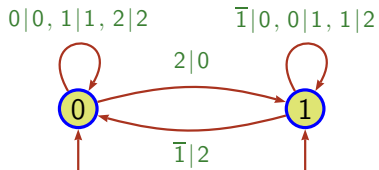
$$\chi_C: C^* \rightarrow A^* \quad \forall w \in C^* \quad \pi(\chi_C(w)) = \pi(w) .$$

Proposition

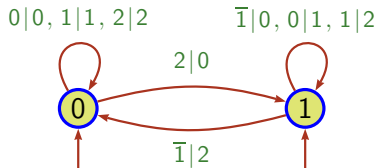
χ_C is realised by a letter-to letter sequential right transducer.



The **left** converter from C^* to A^*



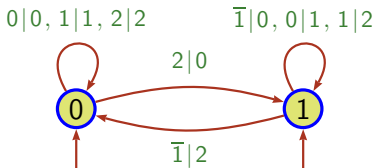
The left converter from C^* to A^*



Proposition

If $p \geq 2q - 1$, then the left converter has only two states.

The left converter from C^* to A^*



Proposition

Let $z \in [0, \omega_{\frac{3}{2}}]$ and \mathbf{c} its companion representation.

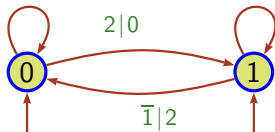
Then \mathbf{a} is a $\frac{3}{2}$ -expansion of z iff

(\mathbf{c}, \mathbf{a}) is an infinite path in the left converter.

Squaring the left converter

$0|0, 1|1, 2|2$

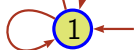
$\bar{1}|0, 0|1, 1|2$



$0|0, 1|1, 2|2$



$\bar{1}|0, 0|1, 1|2$

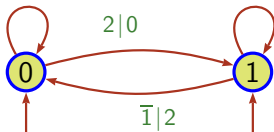


$0|(1,0), 1|(2,1)$

Squaring the left converter

$0|0, 1|1, 2|2$

$\bar{1}|0, 0|1, 1|2$



$0|0, 1|1, 2|2$



$\bar{1}|0, 0|1, 1|2$



$2|(2,0)$



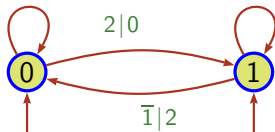
$2|(0,2)$

$\bar{1}|(0,2)$

Squaring the left converter

$0|0, 1|1, 2|2$

$\bar{1}|0, 0|1, 1|2$



$0|0, 1|1, 2|2$

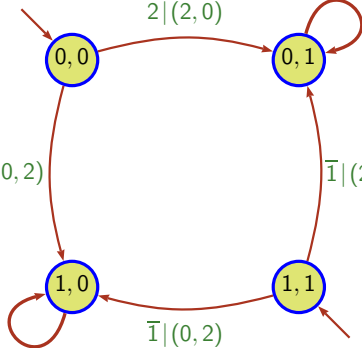


$2|0$

$\bar{1}|2$

$\bar{1}|0, 0|1, 1|2$

$2|(2,0)$



$2|(0,2)$

$\bar{1}|(2,0)$

$\bar{1}|(0,2)$

$0|(1,0), 1|(2,1)$