Success and Failure of Adaptation-Diffusion Algorithms for Consensus in Multi-Agent Networks

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Abstract—This paper investigates the problem of distributed stochastic approximation in multi-agent systems. The algorithm under study consists of two steps: a local stochastic approximation step and a gossip step which drives the network to a consensus. The gossip step uses row-stochastic matrices to weight network exchanges.

We first prove the convergence of a distributed optimization algorithm, when the function to optimize may not be convex and the communication protocol in independent of the observations. In that case, we prove that the average estimate converges to a consensus; we also show that the set of limit points is not necessarily the set of the critical points of the function to optimize and is affected by the Perron eigenvector of a mean-matrix describing the communication protocol. Discussion about the success or failure of convergence to the minimizers of the function to optimize is also addressed. In a second part of the paper, we extend the convergence results to the more general context of distributed stochastic approximation.

I. INTRODUCTION

Distributed stochastic approximation has been recently proposed using different cooperative approaches. In the so-called incremental approach (see for instance [1], [2]) a message containing an estimate of the quantity of interest iteratively travels all over the network. This paper focuses on another cooperative approach based on average consensus techniques where the estimates computed locally by each agent are combined through the network. This idea traces back to [3] where the estimates computed locally by each agent are combined through the network. This idea traces back to [3] where a network of processors seeks to optimize some objective function known by all agents (possibly up to some additive noise).

In this paper, we consider the following cooperative approach. Let a network be composed by \(N\) agents, or nodes. Agents seek to find a consensus on some global parameter by means of local observations and peer-to-peer communications. Node \(i\) (\(i = 1, \ldots, N\)) generates a \(\mathbb{R}^d\)-valued stochastic process \((\theta_{n,i})_{n \geq 0}\), initialized at some arbitrary \(\theta_{0,i} \in \mathbb{R}^d\). Let \((\gamma_n)_{n \geq 1}\) be a deterministic positive step size sequence. At time \(n\), for all \(i = 1, \ldots, N\):

[Local step] Node \(i\) generates a temporary estimate \(\tilde{\theta}_{n,i}\)

\[
\tilde{\theta}_{n,i} := \theta_{n-1,i} + \gamma_n Y_{n,i},
\]

where the \(\mathbb{R}^d\)-valued random process \((Y_{n,i})_{n \geq 0}\) represents the observations made by agent \(i\).

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[Gossip step] Node \(i\) is able to observe the values \(\tilde{\theta}_{n,j}\) of some other nodes and computes the weighted average:

\[
\theta_{n,i} := \sum_{j=1}^{N} w_{n}(i,j) \tilde{\theta}_{n,j},
\]

where the \(w_{n}(i,j)\)'s are non-negative random coefficients such that \(\sum_{j=1}^{N} w_{n}(i,j) = 1\) for any \(i\). The random matrix \(W_n := [w_{n}(i,j)]_{i,j=1}^{N}\) represents the network connections between the nodes at time \(n\). One simply set \(w_{n}(i,j) = 0\) whenever nodes \(i\) and \(j\) are unable to communicate at time \(n\).

Application to distributed optimization. In many applications related to machine learning and sensor networks (we refer to [8] and [10]–[12] for more details) or smart grids [13], one seeks to minimize a sum of local private cost functions \(f_i\) of the agents:

\[
\min_{\theta \in \mathbb{R}^d} \sum_{i=1}^{N} f_i(\theta).
\]

In this context, the distributed Algorithm (1)-(2) reduces to a distributed stochastic gradient algorithm by letting

\[
Y_{n,i} = -\nabla f_i(\tilde{\theta}_{n-1,i}) + \xi_{n,i}
\]

where \(\nabla\) is the gradient operator and \(\xi_{n,i}\) represents some random perturbation which possibly occurs when observing the gradient. The function \(f_i\) is supposed to be unknown from the other agents \(j, j \neq i\). In this paper, we handle the case where functions \(f_i\) are not necessarily convex. Of course, in that case, there is generally no hope to ensure the convergence to a minimizer to (3). Instead, a more realistic objective is to achieve critical points of the objective function i.e., points \(\theta\) such that \(\sum_i \nabla f_i(\theta) = 0\).

Doubly and non-doubly stochastic matrices. In most works (see for instance [14], [15]), the matrices \((W_n)_{n \geq 1}\) are assumed doubly stochastic, meaning that \(W_n^T 1 = W_n 1 = 1\) where \(1\) is the \(N \times 1\) vector whose components are all equal to one and where \(T\) denotes transposition. Although row-stochasticity \((W_n 1 = 1)\) is rather easy to ensure in practice, column-stochasticity \((W_n^T 1 = 1)\) implies more stringent restrictions on the communication protocol. For instance, in [16], each one-way transmission from an agent \(i\) to another agent \(j\) requires at the same time a feedback link from \(j\) to \(i\). As a matter of fact, double stochasticity prevents from using natural broadcast schemes, in which a given node may transmit its local estimate to all neighbors without expecting any immediate feedback.

Remarkably, although generally assumed, double stochasticity of the matrices \(W_n\) is in fact not mandatory. A couple
of works (see e.g., [8], [18]) get rid of the column-stochasticity condition, but at the price of assumptions that may not always be satisfied in practice. Other works [7], [9] manage to circumvent the use of feedback links by coupling the gradient descent with the so-called push-sum protocol [19]. The latter however introduces an additional communication of weights in the network in order to keep track of some summary of the past transmissions.

The following questions remain unanswered. “What conditions on the sequence \((W_n)_{n \geq 0}\) are needed to ensure that Algorithm (1)-(2) drives all agents to a common critical point of \(f_i\)? What happens if these conditions are not satisfied?”.

Contributions.

1) Assuming that \((W_n)_{n \geq 0}\) forms an i.i.d. sequence of stochastic matrices, we prove under technical hypotheses that Algorithm (1)-(2) leads the agents to a consensus, which is characterized. It is shown that the latter consensus does not necessarily coincide with a critical point of \(\sum_i f_i\).

2) We provide sufficient conditions either on the communication protocol or on the functions \(f_i\) which ensure that limit points are the critical points of \(\sum_i f_i\). When such conditions are not satisfied, we also propose a simple modification of the algorithm which allows to recover the sought behavior.

3) We extend our results to a broader setting, assuming that the matrices \((W_n)_{n \geq 0}\) are no longer i.i.d., but are likely to depend on both the current observations and the past estimates. We also investigate a general stochastic approximation framework which goes beyond the model (4) and beyond the only problem of distributed optimization.

The paper is organized as follows. Section II addresses the convergence of a distributed stochastic algorithm. The convergence result is extended to the general setting of distributed stochastic approximation in Section III. The sketch of the proof is given in Section IV. The paper is finally concluded in Section V by numerical illustrations of our results.

Notation: Throughout the paper, the vectors are column vectors. For any vector \(x \in \mathbb{R}^d\), \(|x|\) represents the Euclidean norm of \(x\). \(I_N\) is the \(N \times N\) identity matrix. \(J := 11^T/N\) denotes the orthogonal projector onto the linear span of the all-one \(N \times 1\) vector 1, and \(J_\perp := I_N - J\).

The random variables \(W_n \in \mathbb{R}^{N \times N}\) and \(Y_n := (Y_{i,1},...,Y_{i,N})^T \in \mathbb{R}^{|\mathcal{I}|}\), \(n \geq 1\), are defined on the same measurable space equipped with a probability \(\mathbb{P}\); \(\mathcal{F}\) denotes the associated expectation. For any \(n \geq 1\), define the sigma-field \(\mathcal{F}_n := \sigma(\theta_0, W_1, ..., W_n, Y_1, ..., Y_n)\) where \(\theta_0\) is the (possibly random) initial point of the algorithm.

Throughout the paper, it is assumed that for any \(i \in 1, \ldots, N\), \((\theta_{n,i})_{n \geq 0}\) satisfies the update equations (1)-(2); and we set \(\theta_n := (\theta_{n,1}^T, ..., \theta_{n,N}^T)^T\).

II. DISTRIBUTED OPTIMIZATION

A. Framework

In this section, we consider the case when \(Y_{n,i}\) satisfies (4) with

**Assumption 1.**

1) \(f_i : \mathbb{R}^d \to \mathbb{R}\) is differentiable and \(\nabla f_i\) is locally Lipschitz-continuous.

2) For any Borel set \(A \subseteq \mathbb{R}^{dN}\), almost-surely

\[ \mathbb{P}[\mathcal{F}_{n+1} \in A | \mathcal{F}_n] = \nu_{\theta_n}(A)\]

where \((\nu_\theta)_{\theta \in \mathbb{R}^{dN}}\) is a family of probability measures such that

a) \(\int z \, d\nu_{\theta}(z) = 0\)

b) \(\sup_{\theta \in \mathbb{R}^{dN}} \int |z|^2 \, d\nu_{\theta}(z) < \infty\) for any compact set \(\mathcal{K} \subset \mathbb{R}^{dN}\).

We consider the following assumption on the communication matrix \(W_n\):

**Assumption 2.**

1) For any \(n \geq 0\), conditionally on \(\mathcal{F}_n\), \((W_{n+1}, Y_{n+1})\) are independent.

2) \((W_n)_{n \geq 1}\) is an independent and identically distributed (i.i.d.) sequence of row-stochastic matrices (i.e. \(W_n 1 = 1\) for any \(n\)) with non-negative entries.

3) The spectral radius of the matrix \(\mathbb{E}[W_1^T J_\perp W_1]\) is strictly lower than 1.

Assumption 2 implies that at each time \(n\), the communication matrix \(W_n\) is random and does not depend on the observations \((Y_1, ..., Y_n)\).

The row-stochasticity assumption is a rather mild condition. It claims that \(\sum_i w_{n}(i,j) = 1\) for any \(i\) and any \(j\). Our weaker framework addresses more general gossip protocols, usually less demanding in terms of scheduling and overall network coordination.

Assumption 2-3) is a contraction condition which is required to drive the network to a consensus.

We introduce some assumptions on the step-size sequence \(\gamma_n\) which are satisfied for example by polynomially decreasing sequences \(\gamma_n = \gamma_0/n^a\) for some \(a \in (1/2, 1]\) and \(\gamma_0 > 0\).

**Assumption 3.** The deterministic step size sequence \(\gamma_n\) satisfies \(\gamma_n > 0\) and:

1) \(\sum_n \gamma_n = +\infty\) \(\sum_n \gamma_n^{1-\lambda} < \infty\) for some \(\lambda \in (0, 1)\).

2) \(\lim_n \gamma_{n+1}/\gamma_n = 1\) and \(\sum_n \gamma_n/\gamma_{n+1} \gamma_n - 1 < \infty\).

Finally, we introduce a stability-like condition.

**Assumption 4.** Almost surely, there exists a compact set \(\mathcal{K}\) of \(\mathbb{R}^{dN}\) such that \(\theta_n \in \mathcal{K}\) for any \(n \geq 0\).

Assumption 4 claims that the sequence \((\theta_n)_{n \geq 0}\) remains in a compact set and this compact set may depend on the path. It is implied by the stronger assumption “there exists a compact
set $\mathcal{X}$ of $\mathbb{R}^{dN}$ such that with probability one, $\theta_n \in \mathcal{X}$ for any $n \geq 0$. Checking Assumption 4 is not always an easy task. As the main scope of this paper is the analysis of convergence rather than stability, it is taken for granted: we refer to [8] for sufficient conditions implying stability.

B. Main Result

The statement of our convergence result is prefaced with the following lemma, which shows that the matrix $W := \mathbb{E}[W_1]$

admits a unique left Perron eigenvector $v$; this vector will play a role in the characterization of the limiting points of the gossip algorithm (1)-(2).

Lemma 1. Under Assumption 2-3, the $\mathbb{R}^N$-valued vector $v$ defined by

$$v^T := \frac{1}{N} 1^T W (I_N - J_\perp W)^{-1} \tag{5}$$

is the unique non-negative vector satisfying $v^T = v^T W$ and $v^T 1 = 1$.

Proof: By the Jensen’s inequality, for any $x \in \mathbb{R}^N$,

$$x^T W J_\perp W x \leq x^T \mathbb{E}[W_1^T J_\perp W_1] x.$$  

Then, by Assumption 2-3, the spectral norm of $J_\perp W$ is strictly lower than one. Therefore, $I_N - J_\perp W$ is invertible.

The vector $v$ satisfies $v^T 1 = 1$ and $v^T W = v^T$; to that goal, observe that $(I_N - J_\perp W)^{-1} 1 = 1$. Let us prove that a vector satisfying these two properties is unique; let $w \in \mathbb{R}^N$ satisfying these properties. Then, $w^T W = w^T J_\perp W + 1^T W / N$ thus implying that $w^T = v^T$.

Since $W$ is a stochastic matrix, its spectral radius is one. By [21], there exists a non-negative vector $w$ such that $w^T W = w^T$ and $1^T w > 0$. We can therefore assume without loss of generality that $w^T 1 = 1$. The above discussion implies that $w = v$. This concludes the proof. □

Theorem 1. Let Assumptions 1, 2, 3 and 4 hold true. Define the function $V : \mathbb{R}^d \rightarrow \mathbb{R}$

$$V(\theta) := \sum_{i=1}^{N} v_i f_i(\theta) \tag{6}$$

where $v = (v_1, \ldots, v_N)$ is the vector defined in Lemma 1. Assume that the set $\mathcal{L} = \{ \theta \in \mathbb{R}^d \mid \nabla V = 0 \}$ of critical points of $V$ is nonempty, bounded, included in some level set $\{ \theta : V(\theta) \leq C \}$ and that $V(\mathcal{L})$ has an empty interior. Assume also that the level sets $\{ \theta : V(\theta) \leq C \}$ are either empty or compact. The following holds with probability one:

1. The algorithm converges to a consensus i.e., $\lim_{n \to \infty} \max_{i,j} |\theta_{n,i} - \theta_{n,j}| = 0$.

2. The sequence $(\theta_{n,1})_{n \geq 0}$ converges to $\mathcal{L}$ as $n \to \infty$.

C. Success and Failure of Convergence

Theorem 1 implies that the Algorithm (1)-(2) generally fails to converge towards a critical point of the problem (3). Instead, the algorithm converges to $\mathcal{L}$ which in general is not the set of the critical points of $\theta \to \sum f_i(\theta)$. We now discuss some examples of cases when the algorithm does converge to the sought points.

Scenario 1. All functions $f_i$ are strictly convex and admit a (unique) common minimizer $\theta_*$. 

This case is for instance investigated by [7] in the framework of statistical estimation in wireless sensor network. In this scenario, we may assume without loss of generality that $f_i(\theta) \geq f_i(\theta_*) = 0$ for all $i$ (note that the Algorithm (1)-(2) is not modified when $f_i$ is translated). Since $v_i \geq 0$, $V$ is a non-negative strictly convex function such that $V(\theta_*) = 0$. Therefore, the set of minimizers of $V$ is $\{ \theta_* \}$.

As the main scope of this paper is the analysis of convergence of statistical estimation in wireless sensor network. In this scenario, we may assume without loss of generality that $f_i(\theta) \geq f_i(\theta_*) = 0$ for all $i$ (note that the Algorithm (1)-(2) is not modified when $f_i$ is translated). Since $v_i \geq 0$, $V$ is a non-negative strictly convex function such that $V(\theta_*) = 0$. Therefore, the set of minimizers of $V$ is $\{ \theta_* \}$.

The same conclusion holds by relaxing the strict convexity assumption on the functions $f_i$; if the functions $f_i$ are convex with a common minimizer and $v_i > 0$ for any $i$, then $\mathcal{L}$ is formed by the minimizers of $f_i$. The proof is along the same lines and is omitted.

Scenario 2. $W$ is column-stochastic i.e., $1^T W = 1^T$.

In this case, $v$ given by Lemma 1 is the vector $\frac{1}{N} 1$. Consequently, $V = \frac{1}{N} \sum f_i$. Here, again, $\mathcal{L}$ is the set of minimizers of $f_i$. An example of random communication protocol satisfying $1^T W = 1^T$ is the following: at time $n$, a single node $i$ wakes up at random with probability $p_i$ and broadcasts its temporary update $\tilde{\theta}_{n,i}$ to all its neighbors $N_i$. Any neighbor $j$ computes the weighted average $\tilde{\theta}_{n,j} = \beta \tilde{\theta}_{n,i} + (1 - \beta) \tilde{\theta}_{n,j}$.

On the other hand, any node $k$ which does not belong to the neighborhood of $i$ (including $i$ itself) sets $\tilde{\theta}_{n,k} = \tilde{\theta}_{n,k}$. Then, given $i$ wakes up, the $(k, \ell)$th entry of $W_n$ is given by:

$$w_n(k, \ell) = \begin{cases} 1 & \text{if } k \notin N_i \text{ and } k = \ell, \\ \beta & \text{if } k \in N_i \text{ and } \ell = i, \\ 1 - \beta & \text{if } k \in N_i \text{ and } k = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

$W_n$ is not doubly stochastic. However, when nodes wake up according to the uniform distribution ($p_i = \frac{1}{N}$ for all $i$) it is easily seen that $1^T \mathbb{E}[W_n] = 1^T$.

Remark 1. We end up this section with a simple modification of the initial algorithm in the case where $v_i > 0$ for all $i$. Let us replace the local step (1) of the algorithm by

$$\tilde{\theta}_{n,i} := \theta_{n-1,i} + \gamma_n v_i^{-1} Y_{n,i} \tag{7}$$

where $Y_{n,i}$ is still given by (4). As an immediate Corollary of Theorem 1, the Algorithm (7)-(2) drives the agent to a consensus which coincides with the critical points of $\sum f_i$. Note that this modification requires for each node $i$ to have some prior knowledge of the communication protocol through the coefficients $v_i$ (questions related to the practical computation of $v_i$ are however beyond the scope of this paper).

III. A GENERAL ROBBINS-MONRO ALGORITHM

In this section, we consider the general setting described by Algorithm (1)-(2) with weaker conditions on the distribution of the observations $Y_n$. We also weaken the assumptions on
the conditional distribution of \((Y_{n+1}, W_{n+1})\) given the past behavior of the algorithm \(\mathcal{F}_n\); our general framework includes the case when the communication protocol is adapted at each time \(n\) and takes into account the network observations.

For a matrix \(A\), the spectral norm is denoted by \(\|A\|\). For any vector \(x \in \mathbb{R}^{dN}\) of the form \(x = (x_1, \ldots, x_N)^T\) where \(x_i \in \mathbb{R}^d\), we define the vector of \(\mathbb{R}^d\)

\[
\langle x \rangle := \frac{x_1 + \cdots + x_N}{N} = \frac{1}{N}(1^T \otimes I_d)x,
\]

where \(\otimes\) denotes the Kronecker product. Recall the standard formula

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
\]

We extend the notation to matrices \(X \in \mathbb{R}^{dN \times k}\) as \(\langle X \rangle := \frac{1}{N}(1^T \otimes I_d)X \in \mathbb{R}^{d \times k}\). Matrix \(\mathbf{J}\) represents the orthogonal projector onto the consensus space defined as the set of vectors \(x \in \mathbb{R}^{dN}\) whose \(d\)-dimensional blocks \(x_1, \ldots, x_N\) are all equal. As a consequence of (9), \(\mathbf{J}x = 1 \otimes \langle x \rangle\). Following this notation, we define the gossip matrix as \(W_n = W_n \otimes I_d\). The Algorithm (1)-(2) under study can be written as the following matrix form:

\[
\theta_n = W_n(\theta_{n-1} + \gamma_n Y_n).
\]

The following assumption describes the distribution of \((Y_{n+1}, W_{n+1})\) under the past \(\mathcal{F}_n\). We denote by \(M_1\) the set of \(N \times N\) non-negative row-stochastic matrices and endow \(M_1\) with its Borel \(\sigma\)-field.

**Assumption 5.**

1. There exists a collection of distributions \((\mu_{\theta})_{\theta \in \mathbb{R}^{dN}}\) on \(\mathbb{R}^{dN} \times M_1\) such that for any Borel set \(A\):

\[
P\{[Y_{n+1}, W_{n+1}] \in A | \mathcal{F}_n\} = \mu_{\theta_n}(A)
\]

almost-surely.

In addition, the application \(\theta \mapsto \mu_{\theta}(A)\) defined on \(\mathbb{R}^{dN}\) is measurable for any \(A\) in the Borel \(\sigma\)-field of \(\mathbb{R}^{dN} \times M_1\).

2. \(\sup \int |y|^2d\mu_{\theta}(y, w) < \infty\) for any compact set \(K \subset \mathbb{R}^{dN}\).

Assumption 5-1) means that the joint distribution of the r.v.’s \(Y_{n+1}\) and \(W_{n+1}\) depends on the past \(\mathcal{F}_n\) only through the last value \(\theta_n\) of the vector of estimates. It also implies that \(W_n\) is almost-surely (a.s.) non-negative and row-stochastic. Since the variables \((Y_{n+1}, W_{n+1})\) are not necessarily conditionally independent and \((W_n)_{n \geq 0}\) are no more i.i.d., the contraction condition on \(J_n W_1\) is replaced with the following condition:

**Assumption 6.**

For any compact set \(K \subset \mathbb{R}^{dN}\), there exists \(\rho_K \in (0, 1)\) such that for all \(\theta \in K\), \(\phi\) in \(\mathbb{R}^{dN}\) and \(A \in \mathbb{R}^{dN \times dN}\),

\[
\int (\phi + Ay)^T W^T J_{\perp} W (\phi + Ay) d\mu_\theta(y, w) \leq \rho_K \int |\phi + Ay|^2 d\mu_\theta(y, w),
\]

where \(W := (w \otimes I_d)\).

Assumption 6 implies that

\[
\sup_{\theta \in K} \sup_{x \mid x = 1} \left| \int J_{\perp} W x d\mu_\theta(y, w) \right|^2 \leq \sup_{\theta \in K} \left| \int W^T J_{\perp} W d\mu_\theta(y, w) \right| \leq \rho_K \cdot (11)
\]

Regularity conditions on the conditional distribution of the input variables \((Y_n, W_n)\) are also required.

**Assumption 7.**

For any compact set \(K \subset \mathbb{R}^{dN}\), there exists a constant \(C_K > 0\) such that for any \(\theta, \theta' \in K\),

\[
\left| \int w d\mu_\theta(y, w) - \int w d\mu_{\theta'}(y, w) \right| \leq C_K |\theta - \theta'|, (12)
\]

\[
\left| \int (W y) d\mu_\theta(y, w) - \int (W y) d\mu_{\theta'}(y, w) \right| \leq C_K |\theta - \theta'|, (13)
\]

\[
\left| \int J_{\perp} W y d\mu_\theta(y, w) - \int J_{\perp} W y d\mu_{\theta'}(y, w) \right| \leq C_K |\theta - \theta'|, (14)
\]

where \(W := w \otimes I_d\).

Let us introduce the following quantities for any \(\theta \in \mathbb{R}^{dN}\):

\[
\Omega_\theta := \int J_{\perp} W d\mu_\theta(y, w) \quad (15)
\]

\[
v_\theta := \int J_{\perp} W y d\mu_\theta(y, w) \quad (16)
\]

\[
m_\theta := (I_{dN} - \Omega_\theta)^{-1} v_\theta, (17)
\]

where \(W := w \otimes I_d\). Under Assumption 5-2), it is not difficult to show that for any compact \(K \subset \mathbb{R}^{dN}\), \(\sup_{\theta \in K} \|\Omega_\theta\| \leq \sqrt{\rho_K}\), which implies that \(m_\theta\) is well defined. Finally, define the so-called mean-field function \(h : \mathbb{R}^d \to \mathbb{R}^d\) by

\[
h(\bar{\theta}) = \int (W y + m_{1 \otimes \theta}) d\mu_{1 \otimes \theta}(y, w), (18)
\]

where \(W := w \otimes I_d\). We finally assume that there exists a Lyapunov function \(V\) for the mean field function \(h\).

**Assumption 8.**

1. \(h : \mathbb{R}^d \to \mathbb{R}^d\) is continuous.

2. There exists a continuously differentiable function \(V : \mathbb{R}^d \to \mathbb{R}^+\) such that

   a) there exists \(M > 0\) such that \(L := \{\bar{\theta} \in \mathbb{R}^d : \nabla V^T(\bar{\theta})h(\bar{\theta}) = 0\} \subset \{V \leq M\}\). In addition, \(V(L)\) has an empty interior.

   b) there exists \(M' > 0\) such that \(\{V \leq M'\}\) is a compact subset of \(\mathbb{R}^d\).

   c) for any \(\bar{\theta} \in \mathbb{R}^d \setminus L\), \(\nabla V^T(\bar{\theta})h(\bar{\theta}) < 0\).

Assumptions 5 and 6 imply that \(\bar{\theta} \mapsto m_{1 \otimes \theta}\) is continuous on \(\mathbb{R}^d\). Therefore, a sufficient condition for Assumption 8-1) is to strengthen the conditions (13-14) as follows

\[
\left| \int W y d\mu_\theta(y, w) - \int W y d\mu_{\theta'}(y, w) \right| \leq \rho_K |\theta - \theta'|.
\]

Observe that when \(V\) is a continuous coercive function \(i.e.,\), a continuous function such that \(\lim_{|\bar{\theta}| \to \infty} V(\bar{\theta}) = \infty\), then the level sets \(\{V \leq M\}\) are compact subsets of \(\mathbb{R}^d\).
Theorem 2. Let Assumptions 3, 4, 5, 6, 7 and 8 hold true for the algorithm defined by (10). The following properties hold with probability one:

1) The algorithm converges to a consensus i.e., \( \lim_{n \to \infty} \theta_n = \theta \);
2) The sequence \( (\theta_{n,1})_{n \geq 0} \) converges to a connected component of \( \mathcal{L} \), where \( \mathcal{L} \) is defined in Assumption 8-2).

Theorem 1 can be obtained as a special case of Theorem 2. Indeed, the assumptions of Theorem 1 imply the assumptions 5 to 8 as we now prove.

For any \( \theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^{dN} \) where \( \theta_i \in \mathbb{R}^d \), define the \( \mathbb{R}^{dN} \)-valued function \( g \) by

\[
g(\theta) := (-\nabla f_1(\theta_1)^T, \ldots, -\nabla f_N(\theta_N)^T)^T.
\]

Under Assumption 2-1) and Assumption 2-2), for any Borel set \( A \times B \) of \( \mathbb{R}^{dN} \times \mathbb{R}^\mathbb{N} \),

\[
P|Y_{n+1}, W_{n+1}) \in A \times B|\mathcal{F}_n = P(Y_{n+1} \in A|\mathcal{F}_n)P(W_{n+1} \in B).
\]

In addition, by Assumption 1 and Eq. (4)

\[
P|Y_{n+1}) = \int 1_A (g(\theta_n) + z) \ d\nu_\theta(z),
\]

where \( 1_A \) denotes the indicator function of a set \( A \). This above discussion provides the expression of \( \mu_\theta \) in Assumption 5. In addition, under Assumption 1-2), for any compact set \( \mathcal{K} \) of \( \mathbb{R}^{dN} \),

\[
\sup_{\theta \in \mathcal{K}} \int |y|^2 d\mu_\theta(y, w) = \sup_{\theta \in \mathcal{K}} \left( |g(\theta)|^2 + \int |z|^2 d\nu_\theta(z) \right) < \infty
\]

which proves Assumption 5-2). The above expression of \( \mu_\theta \) implies that

\[
\int (\phi + Ay)^T W^T \mathbb{1} Y W (\phi + Ay) d\mu_\theta(y, w) = \int (\phi + A(g(\theta)+z))^T \mathbb{E}[W^T \mathbb{1} W] (\phi + A(g(\theta)+z)) d\nu_\theta(z).
\]

Therefore, Assumption 6 easily follows from Assumption 2-3). The regularity conditions of Assumption 7 are satisfied by Assumption 1 as the left hand side of (13) is zero and (12) and (14) are true as long as \( \nabla f_i \) are local Lipschitz-continuous.

Again, the expression of \( \mu_\theta \) implies that

\[
\Omega_\theta = \mathbb{1} E|W_1|, \quad v_\theta = \mathbb{1} E[W_1] g(\theta).
\]

Therefore, the mean field vector \( h \) defined by (18) gets into \( h(\theta) = \langle E[W_1] A g(1 \otimes \theta) \rangle \) where

\[
A := (I_{dN} + (I - \mathbb{1} E[W_1])^{-1} \mathbb{1} E[W_1])
\]

Using the Woodbury matrix identity (see [21]), we have

\[
h(\theta) = (v^T \otimes I_d) g(1 \otimes \theta) = -\sum_{i=1}^N v_i \nabla f_i(\theta)
\]

where \( v = (v_1, \ldots, v_N) \) is given by Lemma 1. Set \( \bar{V} := \exp(V) \) where \( V \) is defined by (6). Upon noting that \( \nabla \bar{V} = -h \bar{V} \), it is easily seen that under the assumptions of Theorem 1, Assumption 8 holds.

IV. SKETCH OF THE PROOF OF THEOREM 2

We provide here the sketch of the proof of Theorem 2. The detailed proof will be provided in an extended version of this paper.

By using (8), we write

\[
\theta_n = 1 \otimes (\theta_n) + \mathbb{1} \theta_n.
\]

We define the normalized disagreement vector \( (\phi_n)_{n \geq 0} \) by

\[
\phi_n = \gamma_{n+1}^{-1} \mathbb{1} \theta_n \quad \text{where} \quad \alpha_n = \gamma_n/\gamma_{n+1}.
\]

The following lemma establishes the dynamics of the consensus sequence \( (\theta_n)_{n \geq 0} \) and of the normalized disagreement sequence \( (\phi_n)_{n \geq 0} \).

Lemma 2. Let \( (\theta_n)_{n \geq 0} \) be the sequence given by (10). Assume that \( (W_n)_{n \geq 0} \) are row-stochastic matrices. It holds

\[
\theta_n = \theta_{n-1} + \gamma_n (W_n(Y_n + \phi_{n-1})) \quad \text{and} \quad \phi_n = \alpha_n \mathbb{1} W_n(\phi_{n-1} + Y_n).
\]

The next step is to prove that the square norm of the normalized disagreement sequence is uniformly bounded, conditionally on the event that the past values of the sequence \( (\theta_i)_{i \geq 0} \) remain in a compact set. A main ingredient for the proof of this lemma is the contraction property Assumption 6.

Lemma 3. Let Assumptions 5 and 6 hold true. For any compact set \( \mathcal{K} \subset \mathbb{R}^{dN} \),

\[
\sup_n \mathbb{E} [\phi_n^2 | \cap_{j=1}^n | \theta_n \in \mathcal{K}] < \infty.
\]

Since \( \sum_n \gamma_n^{1-\lambda} < \infty \) for some \( \lambda \in (0, 1) \) under Assumption 3, this lemma implies the convergence result on the disagreement vector:

Proposition 1 (Agreement). Let Assumptions 3-1), 4, 5 and 6 hold true. Then almost-surely, \( \lim_{n \to \infty} \mathbb{1} \theta_n = 0 \).

The second step is to address the long-time behavior of the average estimate \( (\theta_n) \). To that goal, we write the update rule (20) as a stochastic approximation algorithm. We have

\[
\theta_n = \theta_{n-1} + \gamma_n E[y_n|\mathcal{F}_{n-1}] + \gamma_n (\eta_n - E[\eta_n|\mathcal{F}_{n-1}])
\]

where \( \eta_n := (W_n(Y_n + \phi_{n-1})) \). Under the stated assumptions on the conditional distribution of \( (Y_{n+1}, W_{n+1}) \) given the past \( \mathcal{F}_n \), \( E[\eta_n|\mathcal{F}_{n-1}] = \int (W(y + \phi_{n-1})) d\mu_{\theta_{n-1}}(y, w) \) where \( W := w \otimes I_d \). By Proposition 1, \( \theta_{n-1} \) and 1 \( \otimes (\theta_{n-1}) \) are close when \( n \) is large so

\[
E[\eta_n|\mathcal{F}_{n-1}] = \int (W(y + \phi_{n-1})) d\mu_{\theta_{n-1}}(y, w) + \Xi_{n-1}^{(1)}
\]

where \( \Xi_{n-1}^{(1)} \) is a remainder term in an appropriate sense. In addition, (21) shows that \( (\phi_n)_{n \geq 1} \) is a controlled Markov chain:

\[
P(\phi_n \in A|\mathcal{F}_{n-1}) = P_{\theta_{n-1}, \alpha_n}(\phi_{n-1}, A)
\]

where \( P_{\theta, \alpha}(x, A) := \int 1_A (\alpha \mathbb{1} W(y + x)) d\mu_\theta(y, w) \). We show that under our assumptions, the transition kernel \( P_{\theta, \alpha} \)
possess a unique invariant distribution $\pi_{\theta, \alpha}$. When $n$ is large, $\alpha_n \sim 1$ and $\lim_{n} \theta_n = 1 \otimes \theta_0$ is almost-surely the solution intuition is that in (22), $\theta_n$ can be replaced with the expectation of $\theta_n \otimes \theta_{n-1}$; this expectation is $\alpha_{n-1}$ given by (17). Combining these successive approximations yields $E[\theta_n | \theta_{n-1}] = h(\theta_{n-1}) + \Xi_{n-1}^{(1)} + \Xi_{n-1}^{(2)}$ where $h$ is given by (18) and $(\Xi_{n-1}^{(2)})_n$ is small in some appropriate sense. We then establish the convergence results by verifying the sufficient conditions of [22, Theorem 2.2 and 2.3] for the convergence of stochastic approximation algorithms.

V. NUMERICAL RESULTS

Consider a network of $N = 5$ agents and for any $i = 1, \ldots, 5$, we define $f_i : \mathbb{R} \to \mathbb{R}$ by $f_i(\theta) = \frac{1}{2}(\theta - \lambda_i)^2$ where $\lambda_i = (1, 4, -2, 2, 0)$. The minimizer of $\sum_i f_i$ is $\theta_f = 1$. Consider the graph with vertices $\{1, \ldots, N\}$ and edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{3, 5\}$. We choose $\theta_{0,i} = 0$ for all agent $i$, $\gamma_n = 1/n$. Figure 1 represents a realization of the sequence $(\theta_n)_{n \geq 0}$ as a function of $n$ in 3 different scenarios. The plain line curve with square markers corresponds to the standard algorithm (10) when $Y_{n,i}$ is defined as in (4): $(\xi_{n,i})_{n,i}$ is a i.i.d. sequence with Gaussian distribution $N(0, 1)$ and $W_{n,i}$ is assumed fixed and deterministic ($W_n = W_1$); we select $W_1$ in such a way that each agent computes the average of the temporary estimates in its neighborhood. This is equivalent to set $W_i = (D + I)^{-1}(A + I)$, where $A$ is the adjacency matrix of the graph and where $D$ is the diagonal matrix containing the degrees. Note that $1^T W_1 = \frac{1}{2}(10, 13, 15, 11, 11) \neq 1^T$. Computing the left Perron eigenvector (see (5)) yields $\frac{1}{\sqrt{5}} 1^T = (\frac{2}{5}, 1, 1, 1, 1)$. The minimizer of $V = \sum_i v_i f_i$ is $\theta_V = 0.8$. Figure 1 shows that the sequence $(\theta_n)_{n}$ converges to $\theta_V$. The plain line curve with circle markers on Figure 1 represents the same algorithm when $Y_{n,i}$ is replaced by $v_i^{-1}(-\nabla f_i(\theta_{n-1,i}) + \xi_{n,i})$ as in Remark 1. We refer to this algorithm as weighted. As predicted by the Theorem 1-1), the algorithm now converges to the sought value $\theta_f = 1$. Finally, the plain line curve represents the case where the broadcast gossip protocol depicted in Scenario 2 is used with uniform node selection $p_i = \frac{1}{N}$ and with $\beta = \frac{1}{2}$. The algorithm converges to the sought value $\theta_f$ as discussed in Section II-C.

Figure 2 represents the norm of the scaled disagreement vector as a function of $n$ for the same algorithms. As expected from Theorem 1-2), consensus is asymptotically achieved.

REFERENCES