TELECOM ParisTech

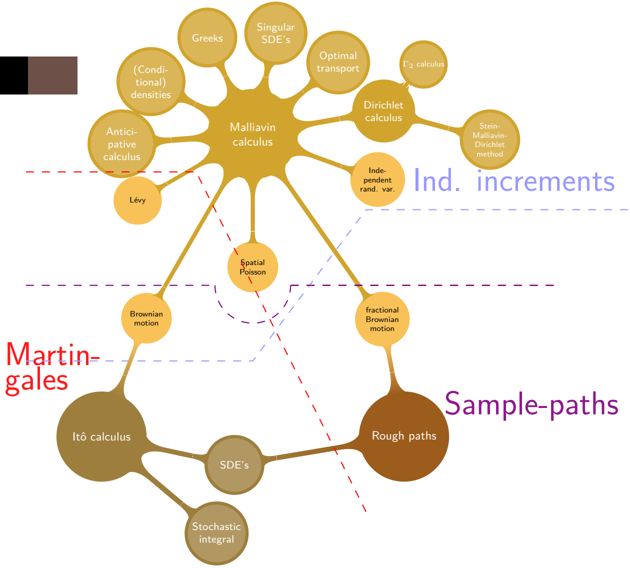
Institut
Mines-Télécom

Malliavin calculus

How far can you go with integration by parts?
L. Gross

L.Decreusefond

université
PARIS-SACLAY


Greeks
Singular SDE's
Optimal transport
Γ_2 calculus
(Conditional) densities
Dirichlet calculus
Malliavin calculus
Stein-Malliavin-Dirichlet method
Anticipative calculus
Independent rand. var.
Ind. increments
Lévy
Spatial Poisson
Brownian motion
fractional Brownian motion
Martingales
Sample-paths
Itô calculus
Rough paths
SDE's
Stochastic integral

Invariance yields IBP

Invariance with respect to translation

$$\frac{d}{d\tau} \int_{\mathbf{R}} f(x+\tau)g(x+\tau)\ \mathrm{d}x \Big|_{\tau=0} = \int_{\mathbf{R}} f(x)g(x)\ \mathrm{d}x 0$$

Integration by parts

$$\int_{\mathbf{R}} f'(x)g(x)\ \mathrm{d}x + \int_{\mathbf{R}} f(x)g'(x)\ \mathrm{d}x = 0$$

Brownian motion

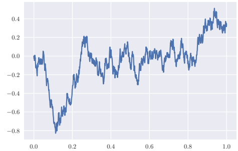
Definition

The Brownian motion is the unique centered Gaussian process such that

$$\mathbf{E}\left[B(t)B(s)\right] = \mathbf{min}(t,s) = \frac{1}{2}(t+s-|t-s|)$$

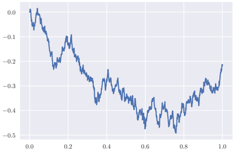
- Independent and stationary increments
- Hölder $(1/2-\varepsilon)$ sample-paths
- Finite quadratic variation

Construction



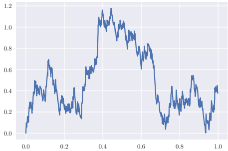
Donsker

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} X_j$$



Karhunen-Loève

$$\sqrt{2} \sum Y_n \frac{\sin(\pi(n+\frac{1}{2})t)}{\pi(n+\frac{1}{2})}$$



Poisson

$$\frac{N_\lambda(t)-\lambda t}{\sqrt{\lambda}}$$

Standard Wiener measure

Distribution of B supported by

- $\mathcal{C}_0([0,1];\mathbf{R})$
- $\mathbf{Hol}(1/2-\varepsilon)$
- $W_{\eta,p}$ for $0<\eta-1/p<1/2$

$$\|f\|^p_{W_{\eta,p}} = \iint_{[0,1]^2} \frac{|f(t)-f(s)|^p}{|t-s|^{1+\eta p}}\, \mathrm{d}s\, \mathrm{d}t$$

How to characterize a Gaussian measure on these Banach spaces ?

RKHS

Definition (Reproducing Kernel Hilbert Space)

$$\mathcal{H} = \overline{\textsf{span}}\Big\{t \wedge ., \; t \in [0,1]\Big\}$$

for the norm induced by the scalar product

$$\langle t \wedge ., \; s \wedge . \rangle_{\mathcal{H}} = t \wedge s$$

Identification

Since

$$t \wedge s = \int_0^s \mathbf{1}_{[0,t]}(u)\, \mathrm{d}u = I^1_{0^+}(\mathbf{1}_{[0,t]})(s) \Rightarrow t \wedge . \in I^1_{0^+}\Big(L^2([0,1])\Big)$$

$$\text{and } t \wedge s = \int_0^1 \mathbf{1}_{[0,t]}(u)\mathbf{1}_{[0,s]}(u)\, \mathrm{d}u = \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{L^2}$$

$\mathcal{H}$ as a Besov-Liouville space

$$\mathcal{H} = \overline{\operatorname{span}}\Big\{ t \wedge ., \ t \in [0,1] \Big\} \simeq I^1_{0^+}\Big(L^2([0,1])\Big)$$

with

$$\| I^1_{0^+}(\dot{h}) \|_{\mathcal{H}} = \| \dot{h} \|_{L^2([0,1])}$$

Abstract Wiener space

Theorem

For any choice of W among $\mathcal{C}$, $\mathbf{Hol}(1/2-\varepsilon)$ or $W_{\eta,p}$, the triplet $\mathbf{emb} : \mathcal{H} \longrightarrow W$ is an AWS: $\mathcal{H}$ is dense in W, $\mathbf{emb}$ is radonifying and the standard Gaussian measure is characterized by

$$\mathbf{E}\left[\mathbf{exp}(-\langle \zeta, \omega\rangle_{W^*,W})\right] = \mathbf{exp}(-\frac{1}{2}\| \mathbf{emb}^* \zeta\|_{\mathcal{H}}^2)$$

where

$$W^* \xrightarrow{\mathbf{emb}^*} \mathcal{H}^* \simeq \mathcal{H} \xrightarrow{\mathbf{emb}} W$$

Wiener integral

$$W^* \xrightarrow{\textsf{emb}^*} \mathcal{H}^* \simeq \mathcal{H} \xrightarrow{\textsf{emb}} W$$

Definition

The Wiener integral is the isometric extension of the map

$$\begin{aligned}\delta \,:\, \textsf{emb}^*(W^*) \subseteq \mathcal{H} &\longrightarrow L^2(\mathbf{P}_{1/2})\\ \textsf{emb}^*(\zeta) &\longmapsto \langle \zeta,\, \omega\rangle_{W^*,W}.\end{aligned}$$

Identification

$$W^* \xrightarrow{\mathbf{emb}^*} \mathcal{H}^* \simeq \mathcal{H} \xrightarrow{\mathbf{emb}} W \text{ and } \mathcal{H} = I_{1,2}$$

$h_t = \mathbf{emb}^*\, \varepsilon_t$ must satisfy for $\omega \in \mathcal{H} \subset W$ and differentiable

$$\omega(t) = \langle \varepsilon_t, \mathbf{emb}\, \omega\rangle_{W^*,W} = \langle \mathbf{emb}^*\, \varepsilon_t, \omega\rangle_{\mathcal{H}} = \int_0^1 \dot{h}_t(u)\dot{\omega}(u)\, \mathrm{d}u$$

$$\Rightarrow \dot{h}_t(u) = \mathbf{1}_{[0,t]}(u) \Rightarrow h_t(u) = t \wedge u$$

Consequence

$$\delta \,:\, \mathbf{emb}^*(W^*) \subseteq I_{1,2} \longrightarrow L^2(\mathbf{P}_{1/2})$$
$$\mathbf{emb}^*(\zeta) \longmapsto \langle \zeta,\, \omega\rangle_{W^*,W}$$

$$\delta(t \wedge .) = \delta(\mathbf{emb}^*(\zeta)) = B(t)$$

Fractional Brownian motion

Definition

For any H in $(0,1)$, $\{B_H(t);\ t \geq 0\}$ is the centered Gaussian process whose covariance kernel is given by

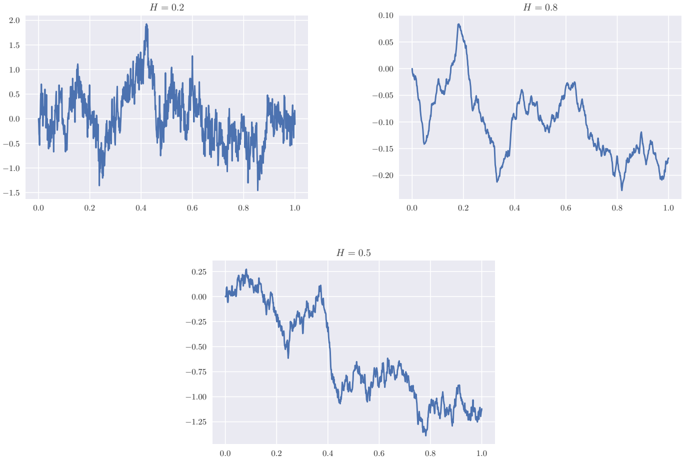
$$R_H(s,t) = \mathbf{E}\left[B_H(s)B_H(t)\right] = \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

- stationary increments
- Hölder $(H-\varepsilon)$ sample-paths
- Finite $1/H$-variation

Some realizations



…r $H = 0.2$, $H = 0.5$ (middle) and $H = 0.8$.

Besov-Liouville spaces

Definition (Riemann-Liouville fractional spaces)

For $\alpha > 0$, for $f \in L^p([0,1])$,

$$I^\alpha_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds.$$

The space $I_{\alpha,p}$ is the set $I^\alpha_{0+}(L^p[0,1])$ equipped with the scalar product

$$\langle I^\alpha_{0+} f, \, I^\alpha_{0+} g \rangle_{I_{\alpha,p}} = \langle f, \, g \rangle_{L^p} = \int_0^1 f(s) g(s) \, ds.$$

Sobolev embeddings

For $a > b > c$ and p, $c - 1/p > 0$

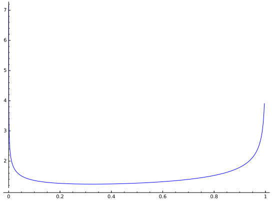
$$W_{a,p} \subset I_{b,p} \subset W_{c,p} \subset \mathbf{Hol}(c - 1/p)$$

Key lemma

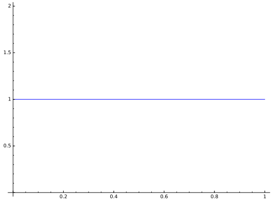
Square root of R_H

There exists K_H such that

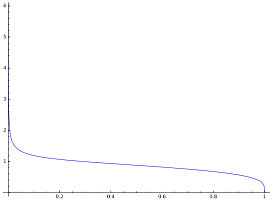
$$R_H(t,s) = \int_0^1 K_H(t,r) K_H(s,r)\, \mathrm{d}r$$



$K_{0.25}(1,s)$



$K_{0.5}(1,s)$



$K_{0.75}(1,s)$

Properties

Properties

$$K_H(t,s) = t^{H-1/2}\, K_H(1, \frac{s}{t})$$

$$K_H(t,s) = 0 \text{ if } s > t$$

$$K_{1/2}(t,s) = \mathbf{1}_{[0,t]}(s)$$

$$K_H(t,s) = r^{-|H-1/2|}\ (t-s)^{H-1/2} \times \mathcal{C}^\infty\!-\!\mathit{function}$$

Warning !

Singularity of K_H increases as $H \uparrow$

Regularity

$$K_H \,:\, L^p \longrightarrow I_{H+1/2,p}$$
$$f \longmapsto K_H f(t) = \int_0^t K_H(t,s) f(s)\, \mathrm{d}s$$

is continuous.

Essential remarks

$$R_H(t,s) = \int_0^1 K_H(t,r) K_H(s,r)\, \mathrm{d}r \Rightarrow R_H(t,s) = K_H\Big(K_H(t,.)\Big)(s)$$

$$K_{1/2} = I^1_{0+}$$

RKHS

Identification

$$\mathsf{RKHS}(B_H) = \overline{\mathsf{span}}\{R_H(t,.), t \in [0,1]\}$$
$$= K_H(L^2) = I_{H+1/2,2}$$

with scalar product

$$\langle K_H\dot{h},\ K_H\dot{g}\rangle_{K_H(L^2)} = \int_0^1 \dot{h}(s)\dot{g}(s)\ \mathrm{d}s$$

Identification

Abstract Wiener space for fBm

$$W^* \xrightarrow{\textbf{emb}^*} I^*_{H+1/2,2} \simeq I_{H+1/2,2} \xrightarrow{\textbf{emb}} W$$

For $h = K_H(\dot{h}) \in I_{H+1/2,2} \subset W$

$$\begin{aligned} \langle \varepsilon_t, \textbf{emb}\, h\rangle_{W^*,W} &= \langle \textbf{emb}^*\, \varepsilon_t, h\rangle_{I_{H+1/2,2}} = h(t) \\ &= \int_0^1 K_H(t,s)\dot{h}(s)\, \mathrm{d}s = \langle K_H(t,.),\, \dot{h}\rangle_{L^2} = \langle K_H(K_H(t,.)),\, h\rangle_{I_{H+1/2,2}} \end{aligned}$$

hence

$$\textbf{emb}^*(\varepsilon_t) = K_H(K_H(t,.)) = R_H(t,.)$$

Consequences

Definition

The Wiener integral is the isometric extension of the map

$$\begin{aligned} \delta \,:\, \mathbf{emb}^*(W^*) \subseteq I_{1,2} &\longrightarrow L^2(\mu_W) \\ \mathbf{emb}^*(\zeta) &\longmapsto \langle \zeta, \omega \rangle_{W^*,W} \end{aligned}$$

Lemma

$$\delta(R_H(t,.)) = B_H(t)$$

$(\delta(K_H(\mathbf{1}_{[0,t]})),\ t \in [0,1])$ *is an ordinary BM*

$$\mathbf{E}\left[\delta(K_H(\mathbf{1}_{[0,t]}))^2\right] = \|K_H(\mathbf{1}_{[0,t]}))\|^2_{I_{H+1/2,2}} = \|\mathbf{1}_{[0,t]}\|^2_{L^2} = t$$

In brief

Embeddings and identifications

$$\begin{array}{ccccc} W^* & \xrightarrow{\textbf{emb}^*} & \mathcal{H}^* = (I_{H+1/2,2})^* & & \\ & & \Big\downarrow \simeq & & \\ L^2 & \xrightarrow{K_H} & \mathcal{H} = I_{H+1/2,2} & \xrightarrow{\textbf{emb}} & W \end{array}$$

for $W = \mathcal{C}$, or $W_{\eta,p}$ with $0 < \eta - 1/p < H$

$$\mathbf{E}\left[\exp(-\langle \zeta, B_H \rangle_{W^*,W})\right] = \exp\left(-\frac{1}{2}\| \textbf{emb}^* \zeta\|^2_{\mathcal{H}}\right)$$

Quasi invariance

Theorem (Cameron-Martin)

For any $h \in \mathcal{H}$,

$$\mathbf{E}\left[F\Big(B_H + \mathbf{emb}(h)\Big)\right] = \mathbf{E}\left[F(B_H)\ \mathbf{exp}\Big(\delta h - \frac{1}{2}\|h\|_{\mathcal{H}}^2\Big)\right]$$

Girsanov theorem

Theorem

For $u = K_H \dot{u}$ *adapted,*

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}_H}\right|_{\mathcal{F}_t} = \mathbf{exp}\left(\int_0^t \dot{u}(s)\ dB(s) - \frac{1}{2}\int_0^t \dot{u}(s)^2\ ds \right)$$

then

$$Law_{\mathbf{Q}}\left(B_H(t) - \int_0^t K_H(t,s)\dot{u}(s)\ ds \right) = Law_{\mathbf{P}_H}(B_H)$$

Differentiation on W

Obstacles

- Why not consider the Fréchet derivative

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F(\omega + \varepsilon\omega') - F(\omega) \right) ?$$

- The Itô map is not continuous on W, so not Fréchet differentiable
- A random variable F is defined up to a negligeable set. We must ensure that for any admissible direction of differentiation h

$$F = G \text{ a.s.} \Rightarrow F(B_H + h) = G(B_H + h) \text{ a.s.}$$

- The Cameron-Martin theorem ensures that this is true for $h \in \mathcal{H}$

Cylindrical functionals

Definition

A function F is said to be cylindrical if there exists $f \in \mathcal{S}(\mathbf{R}^n)$, $(h_1, \cdots, h_n) \in \mathcal{H}^n$ such that

$$F(\omega) = f(\delta h_1, \cdots, \delta h_n).$$

The set of such functionals is denoted by Cyl.

Gross-Sobolev derivative

Definition

For $F(\omega) = f(\delta h_1, \cdots, \delta h_n)$. Set

$$\nabla F = \sum_{j=1}^{n} \partial_j F(\delta h_1, \cdots, \delta h_n)\ h_j,$$

so that

$$\langle \nabla F, h \rangle_{\mathcal{H}} = \sum_{j=1}^{n} \partial_j F(\delta h_1, \cdots, \delta h_n)\ \langle h_j,\ h \rangle_{\mathcal{H}}.$$

$$\langle \nabla F, h \rangle_{\mathcal{H}} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(F(B_H + \varepsilon h) - F(B_H) \right)$$

Examples

$$\nabla(B_H(t))(s) = \nabla\Big(\delta\big(R_H(t,.)\big)\Big)(s) = R_H(t,s)$$

$$\dot{\nabla}_s B_H(t) := K_H^{-1}(\nabla(B_H(t)))(s) = K_H(t,s)$$

Recall

$$L^2 \longrightarrow \mathcal{H} = I_{H+1/2,2} \longrightarrow W$$

$$\forall h \in \mathcal{H},\ \dot{h} = K_H^{-1} h$$

$$\dot{\nabla} = K_H^{-1}\nabla$$

For $H = 1/2$

$$\nabla_s B(t) = t \wedge s$$

$$\dot{\nabla}_s B(t) = \mathbf{1}_{[0,t]}(s)$$

Classical formulas

Derivation and chain rule

$$\nabla(FG) = F\nabla G + G\nabla F$$
$$\nabla\varphi(F) = \varphi'(F)\nabla F$$

Extension to a larger set

Closability

$$\left(F_n \in \text{Cyl} \longrightarrow 0 \text{ and } \nabla F_n \longrightarrow \xi\right) \Rightarrow \xi = 0 \ ?$$

Theorem (Integration by parts)

$$\mathbf{E}\left[G\langle \nabla F, h\rangle_{\mathcal{H}}\right] = \mathbf{E}\left[FG\,\delta h\right] - \mathbf{E}\left[F\,\langle \nabla G, h\rangle_{\mathcal{H}}\right]$$

$$\begin{aligned}
\mathbf{E}\left[F(\omega+\tau h)G(\omega)\right] &= \mathbf{E}\left[F\circ T_h\ G\circ T_{-h}\circ T_h\right]\\
&= \mathbf{E}\left[F\ G\circ T_{-h}\,\mathbf{exp}\left(\delta h - \frac{1}{2}\|h\|_H^2\right)\right]\\
&= \mathbf{E}\left[F(\omega)G(\omega-\tau h)\,\mathbf{exp}\left(\tau\delta h(\omega) - \frac{1\tau^2}{2}\|h\|_H^2\right)\right]
\end{aligned}$$

where T_h …

Gross-Sobolev spaces

Definition

$\mathbb{D}_{2,1}$ is the closure of Cyl for the norm

$$\begin{aligned}\|F\|_{2,1}^2 &= \mathbf{E}\left[F^2\right] + \mathbf{E}\left[\|\nabla F\|_{\mathcal{H}}^2\right] \\ &= \mathbf{E}\left[F^2\right] + \mathbf{E}\left[\int_0^1 (\dot{\nabla}_s F)^2 \,\mathrm{d}s\right]\end{aligned}$$

where $\dot{\nabla} = K_H^{-1}\nabla$.

Support of the derivative

Theorem

$$F \in \mathcal{F}_t \iff \dot{\nabla}_s F = 0 \text{ for } s > t$$

$$\mathbf{E}\left[f\Big(B(t_1), \cdots, B(t_n + s)\Big) \mid \mathcal{F}_{t_n}\right] = \mathbf{E}\left[f\Big(B(t_1), \cdots, B(t_n) + \mathcal{N}(0, s)\Big)\right]$$

and

$$\dot{\nabla}_r f(B(t_1), \cdots, B(t_n))) = \sum_{j=1}^{n} \partial_j f(\ldots) \mathbf{1}_{[0,t_j]}(s) = 0 \text{ for } s > t_n$$

Iteration

An example

$$s \longmapsto \nabla_s B_H(t)^2 = 2B_H(t)\,\nabla_s B_H(t) = 2B_H(t)\,R_H(t,.s) \in \mathcal{H}$$

$$\nabla_r\Big(\nabla_s B_H(t)^2\Big) = 2R_H(t,r)\,R_H(t,s)$$

meaning that

$$\nabla^{(2)}_{r,s}\Big(B_H(t)^2\Big) = 2R_H(t,r)\,R_H(t,s) \in \mathcal{H}\otimes\mathcal{H}$$

Higher order derivative

Definition

$\mathbb{D}_{2,k}$ is the closure of Cyl for the norm

$$\|F\|_{2,k}^2 = \sum_{j=0}^{k} \mathbf{E}\left[\|\nabla^{(j)}F\|_{\mathcal{H}^{\otimes j}}^2\right]$$

$$= \mathbf{E}\left[F^2\right] + \sum_{j=1}^{k} \mathbf{E}\left[\int_{[0,1]^j} \dot{\nabla}^{(j)}_{s_1,\cdots,s_j} F^2 \, \mathrm{d}s_1 \ldots \, \mathrm{d}s_j\right]$$

Divergence

Theorem

$$\nabla \ : \mathbb{D}_{2,1} \subset L^2(W;\mathsf{P}_H) \longmapsto L^2(W \otimes \mathcal{H})$$

is continuous.

Consequence

$$\nabla^* \ : \mathbf{dom}\, \nabla^* \subset L^2(W \otimes \mathcal{H})^* \longrightarrow L^2(W;\mathsf{P}_H)^*$$

by identification of the Hilbert spaces and their dual

Divergence

Definition

$$\mathbf{dom}\,\delta = \Big\{ U(\omega, s) \in L^2(W \otimes \mathcal{H}), \exists c > 0, \forall F \in \mathsf{Cyl}, \\ |\mathbf{E}\left[\langle \nabla F, U\rangle_{\mathcal{H}}\right]| \le c\|F\|_{L^2(\mathbf{P}_H)} \Big\}$$

Then,

$$\mathbf{E}\left[\langle \nabla F, U\rangle_{\mathcal{H}}\right] = \mathbf{E}\left[F\ \nabla^* U\right]$$

If U is deterministic,

$$\mathbf{E}\left[\langle \nabla F, U\rangle_{\mathcal{H}}\right] = \mathbf{E}\left[F\ \delta U\right] \Rightarrow \nabla^* = \delta \text{ on } \mathcal{H}$$

A key formula

Theorem

a a random variable, U a process

$$\delta(aU) = a\delta U - \langle \nabla a, U\rangle_{\mathcal{H}}$$

$$\begin{aligned}
\mathbf{E}\left[\delta(aU)\,\psi\right] &= \mathbf{E}\left[a\langle U, \nabla\psi\rangle_{\mathcal{H}}\right] \\
&= \mathbf{E}\left[\langle U, \nabla(a\psi) - \psi\nabla a\rangle_{\mathcal{H}}\right] \\
&= \mathbf{E}\left[a\delta U\,\psi\right] - \mathbf{E}\left[\langle \nabla a, U\rangle_{\mathcal{H}}\psi\right]
\end{aligned}$$

Divergence extends Itô integral

Corollary ($H = 1/2$)

If $U \in \textbf{dom}\,\delta$ and $\dot{U}$ is adapted,

$$\delta U = \int_0^1 \dot{U}(s)\ dB(s)$$

$$\dot{U}(s) = \sum U_{t_i} \mathbf{1}_{(t_i,t_{i+1}]}(s) \Longleftrightarrow U(t) = \sum U_{t_i} I^1_{0^+}(\mathbf{1}_{(t_i,t_{i+1}]})(t)$$

$$\begin{aligned}\delta(U_{t_i} I^1_{0^+}(\mathbf{1}_{(t_i,t_{i+1}]})) &= U_{t_i}\ \delta\Big(t_{i+1} \wedge . - t_i \wedge .\Big) - \int_0^1 \dot{\nabla}_s U(t_i)\ \mathbf{1}_{(t_i,t_{i+1}]}(s)\ \mathrm{d}s \\ &= U_{t_i}\ B(t_{i+1} - B(t_i)) - 0\end{aligned}$$

Extension of the Itô isometry formula

Theorem

$$\mathbf{E}\left[\delta U^2\right] = \mathbf{E}\left[\int_0^1 \dot{U}(s)^2\ ds\right] + \mathbf{E}\left[\iint_0^1 \dot{\nabla}_r \dot{U}_s\ \dot{\nabla}_s \dot{U}_r\ ds\ dr\right]$$

For the sake of notations

- Recall that $\delta(K_H \mathbf{1}_{[0,t]})$ is a B.M.
- Recall that δ extends Itô integral on adapted processes

$$\int_0^1 \dot{U}(s)\ \delta B(s) = \delta(K_H \dot{U})$$

$$B(t) = \int_0^1 \mathbf{1}_{[0,t]}(s)\ \delta B(s)$$

$$B_H(t) = \int_0^1 K_H(t,s)\ \delta B(s)$$

Inversion formula

Theorem ($I^{\alpha} \circ I^{\beta} = I^{\alpha+\beta}$ and $K_H \simeq I^{H+1/2}$)

$$\mathcal{K} = K_H \circ K_{1/2}^{-1} : I_{1/2,p} \xrightarrow[\text{bijective}]{\text{continuous}} I_{H,p}$$

Formally, $B_H = K_H(\dot{B}) = K_H \circ K_{1/2}^{-1}(B)$

Theorem

For any H, we have

$$B = \mathcal{K}^{-1}(B_H),\ \mathbf{P}_H - a.s.$$

Remark

$$\big(t \mapsto B_H(t)\big) \in I_{H,p} \Rightarrow \big(t \mapsto \mathcal{K}^{-1}(B_H)(t)\big) \in I_{1/2,p} \subset \mathbf{Hol}(1/2 - 1/p)$$

Proof

If for any $\dot{U} \in L^2(W \times [0,1])$

$$\mathbf{E}\left[\int_0^1 B(t)\ \dot{U}(t)\ \mathrm{d}t\right] = \mathbf{E}\left[\int_0^1 \mathcal{K}^{-1}(B_H)(t)\ \dot{U}(t)\ \mathrm{d}t\right]$$

then

$$B(t) = \mathcal{K}^{-1}(B_H)(t)\ \mathbf{P}_H \otimes\ \mathrm{d}t \text{ a.s.}$$

Continuity argument and the proof holds.
It suffices to consider $\dot{U}(\omega, s) = \psi(\omega)g(s)$.
Integration by parts + adjunction + Fubini

Itô formula

Theorem (LD)

For $f \in \mathcal{C}^2_b$,

$$f(B_H(t)) = f(0) + \int_0^t \mathcal{K}_t^*\big(f' \circ B_H\big)(s)\ \delta B(s) \\ + H\, V_H \int_0^1 f''\big(B_H(s)\big) s^{2H-1}\ ds.$$

where $\mathcal{K}_1^*$ is the adjoint of $\mathcal{K}$ in $L^2([0,1])$ and

$$\mathcal{K}_t^* f = \mathcal{K}_1^*(f\ \mathbf{1}_{[0,t]})$$

The usual way for $H = 1/2$

$$
\begin{aligned}
f(B(t)) - f(0) &= \sum f(B(t_{i+1})) - f(B(t_i)) \\
&= \sum f'(B(t_i))(B(t_{i+1}) - B(t_i)) \\
&\quad + \frac{1}{2}\sum f''(B(s_i))\Big(B(t_{i+1}) - B(t_i)\Big)^2
\end{aligned}
$$

- First term: Riemann approximations converge in probability to stochastic integrals
- Second term: Converge in probability to the integral of $f''(B_s)$ with respect to the square bracket of the BM

By parts, integrate you shall

Lemma

$$f(B_H(t)) - f(B_H(s))$$

$$= \sum_{j=0}^{n} \frac{2^{-2j}}{(2j+1)!} (B_H(t) - B_H(s))^{2j+1} \; f^{(2j+1)}\left(\frac{B_H(t) + B_H(s)}{2}\right)$$

$$+ \frac{\Big(B_H(t) - B_H(s)\Big)^{2(n+1)}}{2} \int_0^1 \lambda^{2n+1} f^{(2(n+1))}\Big(\lambda B_H(t) + (1-\lambda) B_H(s)\Big) \, d\lambda$$

$$\mathbf{E}\left[|B_H(t) - B_H(s)|^{2(n+1)}\right] \leq c|t-s|^{2(n+1)H} \Rightarrow 2(n+1)H > 1$$

Fundamental theorem of calculus

$$\mathbf{E}\left[\psi\, f\big(B_H(t)\big)\right] - \mathbf{E}\left[\psi\, f\big(B_H(0)\big)\right] = \int_0^1 \frac{d}{d\varepsilon}\mathbf{E}\left[\psi\, f\big(B_H(t+\varepsilon)\big)\right]\, \mathrm{d}\varepsilon$$

If $H > 1/2$

$$\begin{aligned}&\mathbf{E}\left[\psi\, f\big(B_H(t+\varepsilon)\big)\right] - \mathbf{E}\left[\psi\, f\big(B_H(t)\big)\right]\\ &= \mathbf{E}\left[\big(B_H(t+\varepsilon) - B_H(t)\big)\ f'\left(\frac{B_H(t) + B_H(t+\varepsilon)}{2}\right)\ \psi\right] + O(\varepsilon^{2H})\end{aligned}$$

First order term

$$\mathbf{E}\left[\left(B_H(t+\varepsilon)-B_H(t)\right)\ f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \color{red}{\psi}\right]$$

$$=\mathbf{E}\left[\int_0^1\left(K_H(t+\varepsilon,s)-K_H(t,s)\right)\delta B(s)\ f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right]$$

$$=\mathbf{E}\left[\int_0^1\left(K_H(t+\varepsilon,s)-K_H(t,s)\right)\ \dot{\nabla}_s\left(f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\ \psi\right)\ \mathrm{d}s\right]$$

Proof (cont'd)

$$A_0 = \mathbf{E}\left[f'\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\dot{\nabla}_s\psi \; \mathrm{d}s\right]$$
$$+\,\mathbf{E}\left[\psi\, f''\left(\frac{B_H(t)+B_H(t+\varepsilon)}{2}\right)\right.$$
$$\left.\times \int_0^1 \big(K_H(t+\varepsilon,s)-K_H(t,s)\big)\big(K_H(t+\varepsilon,s)+K_H(t,s)\big)\; \mathrm{d}s\right]$$
$$= B_1 + B_2.$$

Proof (cont'd)

$$\varepsilon^{-1}\, B_1 \xrightarrow{\varepsilon \to 0} \mathbf{E}\left[f'(B_H(t))\ \mathcal{K}\dot{\nabla}\psi(t) \right].$$

$$\begin{aligned} \varepsilon^{-1}\, B_2 \xrightarrow{\varepsilon \to 0} \mathbf{E}\left[\psi\, f''(B_H(t)) \right]\ \frac{d}{dt} R_H(t,t) \\ = H V_H t^{2H-1}\ \mathbf{E}\left[\psi\, f''\left(B_H(t)\right) \right] \end{aligned}$$

The final step: Fundamental theorem of calculus

$$\mathbf{E}\left[\psi\, f\big(B_H(t)\big)\right] - \mathbf{E}\left[\psi\, f\big(B_H(0)\big)\right] = \mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot{\nabla}\psi(s)\ \mathrm{d}s\right] + H\,V_H\ \mathbf{E}\left[\psi\int_0^t f''\big(B_H(s)\big)\,s^{2H-1}\ \mathrm{d}s\right].$$

+ Integration by parts

$$\mathbf{E}\left[\int_0^t f'\big(B_H(s)\big)\,\mathcal{K}\dot{\nabla}\psi(s)\ \mathrm{d}s\right] = \mathbf{E}\left[\int_0^1 f'\big(B_H(s)\big)\,\mathbf{1}_{[0,t]}(s)\,\mathcal{K}\dot{\nabla}\psi(s)\ \mathrm{d}s\right]$$

$$= \mathbf{E}\left[\int_0^1 \mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)\,\dot{\nabla}_s\psi\ \mathrm{d}s\right] = \mathbf{E}\left[\psi\int_0^1 \mathcal{K}_1^*\big(f'\circ B_H\,\mathbf{1}_{[0,t]}\big)(s)\ \delta B(s)\right]$$

Stochastic integrals: Discretize B_H

Stratonovitch type integrals

$$\mathrm{RS}_\pi(u) = \sum_{t_i \in \pi} u(t_i)\Big(B_H(t_{i+1}) - B_H(t_i)\Big) \text{ or}$$

$$\mathrm{SS}_\pi(u) = \sum_{t_i \in \pi} \frac{1}{\theta_i} \int_{t_i}^{t_{i+1}} u(s)\, \mathrm{d}s \, \Big(B_H(t_{i+1}) - B_H(t_i)\Big)$$

Discretize B and use $B_H = K_H \dot{B}$

$$B^\pi(t) = B(t_i) + \frac{1}{\theta_i} \Delta B_i \, (t - t_i) \text{ for } t \in [t_i, t_{i+1}),$$

$$\begin{aligned} B_H^\pi(t) &= \sum_{t_i \in \pi} \frac{1}{\theta_i} \int_{t_i}^{t_{i+1}} K(t,s) \, ds \; \Delta B_i \\ &= \sum_{t_i \in \pi} \frac{1}{\theta_i} K(\mathbf{1}_{[t_i, t_{i+1}]})(t) \Delta B_i \end{aligned}$$

Riemann-Stratonovitch integral

$$R_T^\pi(u) := \sum_{t_i \in \pi} \frac{1}{\theta_i} \left\{ \int_0^T u(t) \frac{d}{dt} K(\mathbf{1}_{[t_i, t_{i+1}]})(t) \, \mathrm{d}t \right\} \Delta B_i$$

$a\delta U = \delta(aU) + \langle \nabla a, U \rangle$

$$R_T^{\pi}(u) = \delta\Big(\sum_{t_i \in \pi} \frac{1}{\theta_i} \int_{t_i}^{t_{i+1}} \mathcal{K}_T^* u(t) \, \mathrm{d}t \Big) + \sum_{t_i \in \pi} \frac{1}{\theta_i} \iint_{[t_i, t_{i+1}]^2} \mathcal{K}_T^* (\dot{\nabla}_r u)(t) \, \mathrm{d}t \, \mathrm{d}r$$

Theorem

Under technical conditions on u,

$$R_T^{\pi}(u) \longrightarrow \delta(\mathcal{K}_T^* u) + \int_0^t \dot{\nabla}(\mathcal{K}_T^* u)(s) \, ds$$

Back to B_H discretization

$$\mathsf{SS}_\pi(u) \longrightarrow \delta(\mathcal{K}_T^* u) + \int_0^t (\mathcal{K}\dot{\nabla})_s u(s) \, \mathrm{d}s$$

Why is $H < 1/2$ so hard ?

$\mathcal{K}_H$ behaves as $I^{H-1/2}$

Ornstein-Uhlenbeck semi-group

Definition

$$\forall F \in L^1(\mathbf{P}_H),\ P_t F(\omega) = \int_W F(e^{-t}\omega + \sqrt{1-e^{-2t}}\zeta)\ d\mathbf{P}_H(\zeta)$$

Theorem

- *Invariance of Gaussian measures with respect to rotations imply*

$$P_{t+s} = P_t \circ P_s$$

- $P_t F(\omega) \xrightarrow{t\to\infty} \int_W F\ d\mathbf{P}_H$
- $\mathbf{P}_H$ *is the stationary measure of* P_t
- *If* $F \in L^1$, $P_t F \in \bigcap_{k=1}^{\infty} \mathbb{D}_{2,k}$ *for any* $t > 0$

Generator

Theorem (A consequence of IBP)

For any $F \in L^1(\mathbf{P}_H) \cap \mathbf{Lip}(W)$,

$$\frac{d}{dt} P_t F = L P_t F$$

where

$$LF(\omega) = -\langle \nabla F,\, \omega \rangle_{W^*,W} + \sum_{j=1}^{\infty} \langle \nabla^{(2)} F,\, h_j \otimes h_j \rangle_{I_{1,2}}$$

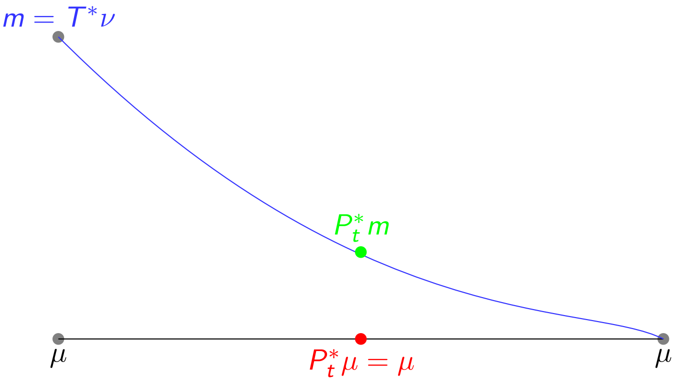
$$= -\delta \nabla F$$

with $(h_j, j \geq 1)$ *a CONB of* $I_{1,2}$

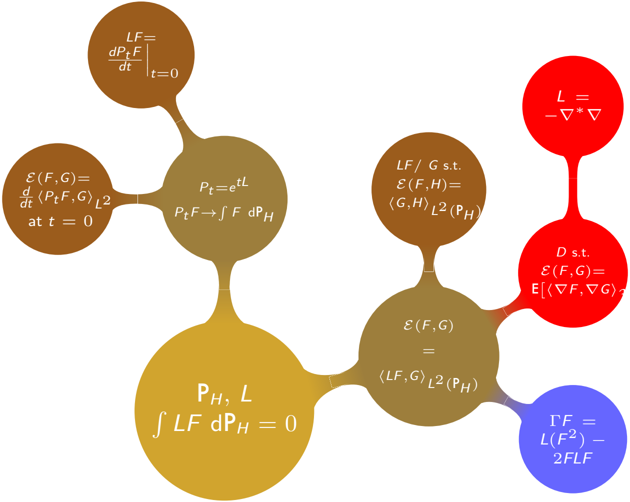
Stein-Malliavin representation formula

$$P_r F(\omega) \int_W F \, \mathrm{d}\mathbf{P}_H - \int_W P_{s0} F(\omega) \, \mathrm{d}\mathbf{Q}(\omega) = \int_W \int_{s0}^{r\infty} \frac{d}{dt} L P_t F(\omega) \, \mathrm{d}t \, \mathrm{d}\mathbf{Q}(\omega)$$

Stein method in one picture



Dirichlet-Malliavin structure



Malliavin calculus for Poisson point processes

Configuration space

Definition

A configuration is a locally finite set of points of a set E. $\mathbb{N}_E$ = the set of configurations of E.

Definition

Let μ σ-finite Radon measure on a Polish space E. The Poisson process with intensity μ is such that for any function $f : E \to \mathbf{R}^+$,

$$\mathbf{E}\left[\mathbf{exp}(-\int f \, \mathrm{d}N)\right] = \mathbf{exp}\left(-\int_E (1 - e^{-f(s)}) \, \mathrm{d}\mu(s)\right).$$

where $\int f \, \mathrm{d}N = \sum_{x \in N} f(x)$

Campbell-Mecke formula

Theorem

$$\mathbf{E}\left[\int_E F(N \oplus x,\, x)\ d\mu(x)\right] = \mathbf{E}\left[\int_E F(N,\, x)\ dN(x)\right]$$

$$\frac{d}{dt}\mathbf{E}\left[\mathbf{exp}(-\int_E (f + t\,\mathbf{1}_B)\ \mathrm{d}N)\right]\Big|_{t=0} = \frac{d}{dt}\mathbf{exp}\left(-\int_E (1 - e^{-(f(s)+t\,\mathbf{1}_B(s))})\ \mathrm{d}\mu(s)\right)\Big|_{t=0}$$

yields CM formula for $F(N,x) = \mathbf{exp}(-\int f\ \mathrm{d}N)\, g(x)$

Gradient

Definition (Discrete gradient)

$$\textbf{dom}\, D = \left\{ F \,:\, \mathbb{N}_E \longrightarrow \mathbf{R},\ \mathbf{E}\left[\int_E |F(N \oplus x) - F(N)|^2 \, \mathrm{d}\mu(x) \right] < \infty \right\}$$

For $F \in \textbf{dom}\, D$, we set

$$D_x F(N) = F(N \oplus x) - F(N).$$

Divergence

Definition (Poisson divergence)

We denote by $\mathbf{dom}\,\delta$, the set of vector fields such that

$$\mathbf{dom}\,\delta = \Big\{ U : \mathbb{N}_E \times E \to \mathbf{R}, \\ \mathbf{E}\left[\left(\int_E U(N \ominus x, x)(\,\mathrm{d}N(x) - \,\mathrm{d}\mu(x))\right)^2\right] < \infty \Big\}$$

Then,

$$\delta U(N) = \int_E U(N \ominus x, x)(\,\mathrm{d}N(x) - \,\mathrm{d}\mu(x)).$$

Campbell-Mecke formula is equivalent to IBP

Theorem (Integration by parts for Poisson process)

For $F \in \mathbf{dom}\, D$ and any $U \in \mathbf{dom}\, \delta$,

$$\mathbf{E}\left[\int_E D_x F(N)\; U(N,x)\; d\mu(x)\right] = \mathbf{E}\left[F(N)\; \delta U(N)\right].$$

Corollary (Skorohod isometry)

For any $U \in \mathbf{dom}\, \delta$,

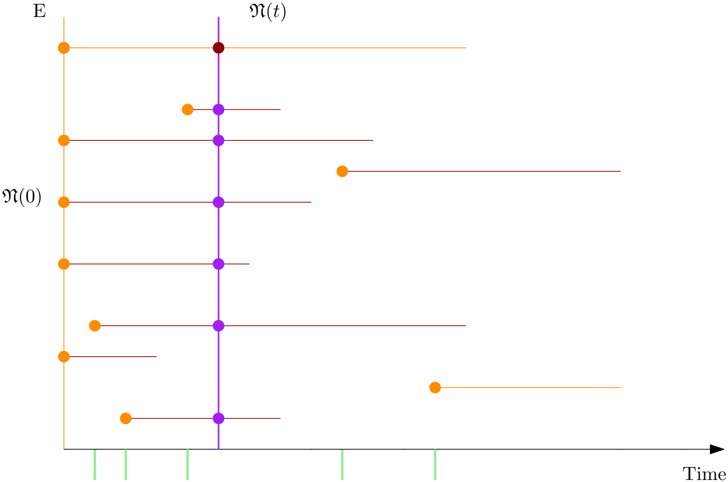
$$\mathbf{E}\left[\delta U^2\right] = \mathbf{E}\left[\int_E U(N,x)^2\; d\mu(x)\right] + \mathbf{E}\left[\int_E \int_E D_x U(N,y)\, D_y U(N,x)\; d\mu(x)\; d\mu(y)\right].$$

Glauber process

Construction

- $\mathfrak{N}(0) = \eta \in \mathbb{N}_E$,
- Each atom of η has a life duration, independent of that of the other atoms, exponentially distributed with parameter 1.
- Atoms are born at moments following a Poisson process with intensity $\mu(\Lambda)$. On its appearance, each atom is localised independently from all the others according to $\mu/\mu(\Lambda)$. It is also assigned in an independent manner, a life duration exponentially distributed with parameter 1.

Glauber process



Generator

Theorem

The infinitesimal generator of $\mathfrak{N}$ is given by

$$-L_{\mathfrak{N}}F(N) = (\delta D)F(N)$$
$$= \mu(E)\int_E (F(N\oplus x) - F(N))\, d\mu(x)\frac{d\mu(x)}{\mu(E)}$$
$$+ N(E)\int (F(N-\delta_x) - F(N))\frac{1}{N(E)}\, dN(x)$$

for F bounded from $\mathbb{N}_E$ into $\mathbf{R}$.

Definition

$$P_tF(N) = \mathbf{E}\left[F(\mathfrak{N}(t))\,|\,\mathfrak{N}(0) = N\right]$$

Similarities and difference

Brownian

$\|\delta U\|_{L^2} = \|U\|^2_{L^2} + \mathbf{E}\left[\mathrm{tr}(\nabla U \circ \nabla U)\right]$

$F = \mathbf{E}\left[F\right] + \sum_{n\ge 1} \delta^n(E[\nabla^{(n)} F])$

$\mathrm{var}(F) \le \|\nabla F\|^2_{L^2(W;\mathcal{H})}$

Poisson

$\|\delta U\|_{L^2} = \|U\|^2_{L^2} + \mathbf{E}\left[\mathrm{tr}(DU \circ DU)\right]$

$F = \mathbf{E}\left[F\right] + \sum_{n\ge 1} \delta^n(E[D^{(n)} F])$

$\mathrm{var}(F) \le \|DF\|^2_{L^2(W\times E)}$

Brownian

$\nabla(FG) = F\,\nabla G + G\,\nabla F$

$P_t F \in \cap_{k\ge 1}\mathbb{D}_{2,k}$

$\mathrm{Ent}(F) \le \mathbf{E}\left[\langle \nabla F, \nabla \log F\rangle_{\mathcal{H}}\right]$

Poisson

$D(FG) = F\,DG + G\,DF + DF\,DG$

No regularizing property

$\mathrm{Ent}(F) \le \mathbf{E}[\int_E \min(F^{-1}|D_x F|^2, D_x F\, D_x \log F)\, \mathrm{d}\mu(x)]$

Independent random variables

Definition (LD, H. Halconruy)

For $X_A = (X_a, a \in A)$ independent random variables, $X_a \in E_a$ a Polish space.

$$D_a F(X_A) = F(X_A) - \mathbf{E}\left[F \mid X_b, b \neq a\right]$$

$$\delta U = \sum_{a \in A} D_a U_a$$

Then,

$$\mathbf{E}\left[\sum_{a \in A} D_a F \, U_a\right] = \mathbf{E}\left[F \ \delta U\right]$$

Similarities and difference

Brownian

$\|\delta U\|_{L^2} = \|U\|_{L^2}^2 + \mathbf{E}\left[\mathrm{tr}(\nabla U \circ \nabla U)\right]$

$F = \mathbf{E}[F] + \sum_{n\ge 1} \delta^n(E[\nabla^{(n)} F])$

$\mathrm{var}(F) \le \|\nabla F\|^2_{L^2(W;\mathcal{H})}$

$\nabla(FG) = F\,\nabla G + G\,\nabla F$

$P_t F \in \cap_{k\ge 1} \mathbb{D}_{2,k}$

$\mathrm{Ent}(F) \le \mathbf{E}\left[\langle \nabla F, \nabla \log F\rangle_{\mathcal{H}}\right]$

Independent r.v.

$\|\delta U\|_{L^2} = \mathbf{E}\left[\mathrm{tr}(DU \circ DU)\right]$

No chaos decomposition

$\mathrm{var}(F) \le \|DF\|^2_{L^2(W\times E)}$

$D_a(FG) = \ldots + D_a F\, D_a G - \mathbf{E}\left[FG|\mathcal{G}_a\right] + \mathbf{E}\left[F|\mathcal{G}_a\right]\mathbf{E}\left[F|\mathcal{G}_a\right]$

No regularizing property

$\mathrm{Ent}(F) \le \sum_{a\in A} \mathbf{E}\left[D_k G^2 / \mathbf{E}\left[G|\mathcal{G}_k\right]\right]$

Clark representation formula

Brownian motion

$$F = \mathbf{E}[F] + \int_0^1 \mathbf{E}\left[\dot{\nabla} F \,|\, \mathcal{F}_s\right] \,\mathrm{d}B(s)$$

Poisson process on $\mathbf{R}^+$

$$F = \mathbf{E}[F] + \int_0^1 \mathbf{E}[D_s F \,|\, \mathcal{F}_s]\,\mathrm{d}(N-\mu)(s)$$

Bernoulli random variables

$$F = \mathbf{E}[F] + \sum^{\infty} \mathbf{E}[D_k F \,|\, \mathcal{F}_{k-1}]\, X_k$$

Independent random variables

$$F = \mathbf{E}[F] + \sum_{k=1}^{\infty} D_k\, \mathbf{E}[F \,|\, \mathcal{F}_k]$$

Clark-Hoeffding formula

$$F = \mathbf{E}[F] + \sum_{B \subset A} \binom{|A|}{|B|}^{-1} \frac{1}{|B|} \sum_{b \in B} D_b\, \mathbf{E}[F \,|\, X_B]$$

Mallows distribution of random permutations

$$
\begin{aligned}
E_{\mathbf{N}} &= \times_{j=1}^{\infty}\{1, \cdots, j\} \\
\mathsf{P}_j^t(k) &= \frac{1}{t+j-1} \text{ if } k \neq j \\
&= \frac{t}{t+j-1} \text{ for } k = j \\
\mathsf{P} &= \bigotimes_{j=1}^{\infty} \mathsf{P}_j \\
X_j &= \text{the i-th coordinate of law } \mathsf{P}_j
\end{aligned}
$$

Construction

- Start with $\sigma = (1)$
- If $X_2 = 2$ then $\sigma = (1, 2)$ else $\sigma = (2, 1)$
- If $X_k = k$ then concatenate k to the current permutations
- If $X_k = j$, then $\sigma \leftarrow (k, j) \circ \sigma$
- Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & {\color{red}3}3 & 4 & 2 \end{pmatrix}$$

If $X_5 = 3$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & {\color{blue}5} \\ 1 & {\color{blue}5} & 4 & 2 & {\color{blue}3} \end{pmatrix}$$

Mallows distribution

Theorem (Kerov et al.)

For any $\sigma \in \mathfrak{S}_N$,

$$\mathsf{P}^t(\{\sigma\}) = \frac{t^{\mathsf{cyc}(\sigma)}}{(t+1)(t+2)\times\cdots\times(t+N-1)},$$

where $\mathsf{cyc}(\sigma)$ is the number of cycles of σ.

- $t = 1$: *uniform distribution*
- $t \Rightarrow \infty$: *σ close to identity*
- $t \Rightarrow 0$: *almost a unique cycle*

A new representation of the number of fixed points

Number of cycles

$$C_1 = \sum_{k=1}^{N} \mathbf{1}_{(I_k=k)} \mathbf{1}_{(I_m \neq k,\ m \in \{k+1,\cdots,N\})}$$

Theorem (LD, HH)

$$C_1 = t\left(1 - \frac{t-1}{N+t-1}\right) + \sum_{l=1}^{N}\left(\mathbf{1}_{(I_l=l)} - \frac{t}{t+l-1}\right)\prod_{m=l+1}^{N}\mathbf{1}_{(I_m \neq l)}$$
$$- \sum_{l=2}^{N-1}\frac{t}{t+l-2}\sum_{k=1}^{l-1}\left(\mathbf{1}_{(I_l=k)} - \frac{1}{t+l-1}\right)\prod_{m=l+1}^{N}\mathbf{1}_{(I_m \neq k)}.$$

Variance of C_1

Theorem (LD, HH)

For any $t \in \mathbf{R}$, we get

$$var[\tilde{C}_1] = \frac{Nt}{t+N-1}\left(\frac{t}{t+N-1} + 1 - \frac{2t^2}{N}\sum_{k=1}^{N}\frac{1}{t+k-1}\right).$$

Dirichlet forms

$$(E_N, \otimes_{j=1}^N \mu_j) \xrightarrow{U_N} (W, \mathbf{P}) \xrightarrow{F} \mathbf{R}$$

$$\mathcal{E}_N(F) = \mathbf{E}\left[\langle D_N(F \circ U_N),\ D_N(F \circ U_N)\rangle_{L^2(E_N)}\right]$$

is a family of Dirichlet forms on W

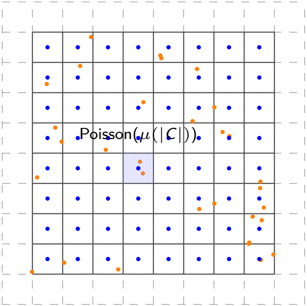
Convergence of Dirichlet forms

Definition

Let $(\mathcal{E}_n, n \geq 1)$ be a sequence of Dirichlet forms on W. It converges in the Dirichlet sense to the Dirichlet form $\mathcal{E}$ if for $F \in \textbf{dom}\,\mathcal{E} \cap \textbf{Lip}$,

$$\mathcal{E}_n(F, F) = \mathbf{E}\left[\langle \nabla_n F, \nabla_n F \rangle\right] \xrightarrow{n \to \infty} \mathcal{E}(F, F) = \mathbf{E}\left[\langle \nabla F, \nabla F \rangle\right]$$

Poisson point process as limit of discrete random variables



- A=set of cells ($|\text{mesh}| \to 0$)
- X_a=Poisson($\mu(|C_a|)$)
- Choose $\zeta_a \in C_a$ (deterministically)
- $N_A = \sum_{a \in A} X_a \, \varepsilon_{\zeta_a}$
- If $|F(\omega) - F(\eta)| \le d_{TV}(\omega, \eta)$ then $\mathcal{E}^A(F) \to \mathcal{E}^{\text{Pois}}(F) = \mathbf{E}\left[\|DF\|^2_{L^2(\mu)}\right]$

Brownian motion as a limit of a random walk

$$e_k^N(t) = \sqrt{N}\ \mathbf{1}_{[(k-1)/N,\, k/N)}(t) \text{ and } h_k^N(t) = \int_0^t e_k^N(s)\ ds.$$

$$\omega^N(t) = \sum_{k=1}^N M_k\, h_k^N(t), \text{ for all } t \in [0,1],$$

where $(M_k, k = 1, \cdots, N)$ are iid standard Gaussian r.v.

Theorem (LD, HH)

$$\mathcal{E}^{U_N}(F) = \sum_{k=1}^N \mathbf{E}\left[\left(F(\omega^N) - \mathbf{E}'\left[F(\omega^N_{(k)} + M'_k\, h_k^N)\right]\right)^2\right],$$

where $\omega^N_{(k)} = \omega^N - M_k\, h_k^N$ *and* M'_k *is an independent copy of* M_k*. For* $F \in \mathbf{Lip}(\mathcal{H})$

$$\mathcal{E}^{U_N}(F) \xrightarrow{N\to\infty} \mathcal{E}(F) = \mathbf{E}\left[\langle \nabla F, \nabla F\rangle_{\mathcal{H}}\right]$$

Process convergence

Barbour'90

$$\sup_{\|F\|_{\mathcal{C}_b^3(\mathbb{D};\,\mathbf{R})}\le 1} \left| \mathbf{E}\left[F\left(\frac{N^n - n}{\sqrt{n}}\right)\right] - \mathbf{E}\left[F(B)\right]\right| \le c\, \frac{\log n}{\sqrt{n}}$$

Common space

Brownian motion

$$(t \mapsto B(t)) \in \mathbf{Hol}(1/2-\varepsilon) \subset I_{0^+}^{1/2-\varepsilon'}(L^\infty) \subset I_{0^+}^{1/2-\varepsilon'}(L^2)$$

Poisson process

$$N(t) = \sum_j \mathbf{1}_{[T_j,\,1]}(t) \in I_{0^+}^{1/2-\varepsilon}(L^2),\ \forall \epsilon > 0$$

Common space

$$I_{0^+}^{1/2-\varepsilon}(L^2) \text{ for arbitrary } \varepsilon > 0$$

Wiener measure on $I_{0+}^{\beta}(L^2)$

Itô-Nisio theorem

$$B(t) := \sum_{n\geq 1} X_n \; I_{0+}^{1}(e_n)(t) \text{ a.s.}$$

with $(e_n,\; n \geq 1)$ CONB of L^2 and $(X_n,\; n \geq 1)$ independent $\mathcal{N}(0,1)$

Convergence in $L^2(\Omega;\; I_{0+}^{\beta}(L^2))$

$$\sum_{n\geq 1} \| I_{0+}^{1} e_n \|_{I_{\beta,\,2}}^2 = \sum_{n\geq 1} \| I_{0+}^{1-\beta} e_n \|_{L^2}^2 = \| I_{0+}^{1-\beta} \|_{\text{HS}} < \infty$$

implies

$$\beta < 1/2$$

Wiener measure

$$\mathsf{E}_{\mu_\beta}\left[\exp(i\langle \eta,\, \omega\rangle_{I_{\beta,\,2}})\right] = \mathsf{E}_{\mathsf{P}_{1/2}}\left[\exp(i\sum_{n\geq 1}\int_0^1 (I_{1-}^{1-\beta}\circ I_{0+}^{-\beta})\eta(s)\ e_n(s)\,ds\ X_n)\right]$$
$$= \exp(-\frac{1}{2}\sum_{n\geq 1}\left(\int_0^1 (I_{1-}^{1-\beta}\circ I_{0+}^{-\beta})\eta(s)\ e_n(s)\,ds\right)^2)$$
$$= \exp(-\frac{1}{2}\|(I_{1-}^{1-\beta}\circ I_{0+}^{-\beta})\eta\|^2_{L^2([0,\,1])})$$
$$= \exp(-\frac{1}{2}\int_0^1 (I_{0+}^{1-\beta}\circ I_{1-}^{1-\beta})\dot\eta(s)\ \dot\eta(s)\,ds),$$

Conclusion

$$\mathsf{E}_{\mu_\beta}\left[\exp(i\langle \eta,\, \omega\rangle_{I_{\beta,\,2}})\right] = \exp(-\frac{1}{2}\langle V_\beta\eta,\, \eta\rangle_{I_{\beta,\,2}})$$

where

Gaussian structure on $l^2(\mathbf{N})$

Hilbert spaces isometry

$$\begin{aligned}\mathfrak{J}_\beta \ : \ I^\beta_{0+}(L^2) &\longrightarrow l^2(\mathbf{N}) \\ f &\longmapsto \left(\langle e_n,\ I^{-\beta}_{0+} f \rangle_{L^2([0,1])},\ n \ge 1 \right)\end{aligned}$$

Commutative diagram

$$\begin{array}{ccc} I^\beta_{0+}(L^2) & \xrightarrow{\mathfrak{J}_\beta} & l^2(\mathbf{N}) \\ V_\beta \Big\downarrow & & \Big\downarrow S_\beta := \mathfrak{J}_\beta \circ V_\beta \circ \mathfrak{J}_\beta^{-1} \\ I^\beta_{0+}(L^2) & \xrightarrow{\mathfrak{J}_\beta} & l^2(\mathbf{N}) \end{array}$$

[redacted] $= m_\beta$, where m_β is the Gaussian measure on $l^2(\mathbf{N})$ such that for any $v \in l^2(\mathbf{N})$,

$$\int_{l^2(\mathbf{N})} \mathbf{exp}(i\, v.u)\ dm_\beta(u) = \mathbf{exp}(-\frac{1}{2} S_\beta v.v)$$

Expression of S_β

$$S_\beta\ :\ l^2(\mathbf{N}) \longrightarrow l^2(\mathbf{N})$$
$$u = (u_n,\ n \in \mathbf{N}) \longmapsto (\sum_{j\geq 1} \langle h_n,\ h_j\rangle_{L^2}\ u_j,\ \ n \in \mathbf{N})$$

$$\text{where } h_n = I_{0^+}^{1-\beta}(e_n)$$

Stein equation

Ornstein-Uhlenbeck process

$$P_t^\beta F(x) = \int_{l^2(\mathbf{N})} F(e^{-t}x + \sqrt{1-e^{-2t}}\, v)\ dm_\beta(v)$$

Infinitesimal generator

$$\frac{d}{dt} P_f^\beta F(x) = A^\beta P_t^\beta F(x) \text{ where}$$

$$A^\beta F(x) = \langle x, \nabla F(x)\rangle_{l^2} - \mathbf{trace}(S_\beta \circ \nabla^{(2)} F(x)),\ x \in l^2(\mathbf{N})$$

Stein equation

$$\int_{l^2(\mathbf{N})} F(x)\ dm_\beta(x) - F(x) = \int_0^\infty A^\beta P_t^\beta F(x)\ dt,\ m\text{-a.s.}$$

Embedding of the Poisson process

Poisson process

$$N^{\lambda}(t) = \frac{1}{\sqrt{\lambda}}\left(N_{\lambda}(t) - \lambda t\right)$$

$$\mathfrak{J}_{\beta} N^{\lambda} = \left(\frac{1}{\sqrt{\lambda}} \int_0^1 h_n(s)(\ dN_{\lambda}(s) - \lambda\ ds),\ n \in \mathbf{N} \right)$$

where

$$h_n = I_{0^+}^{1-\beta}(e_n)$$

Integration by parts

$$\mathbf{E}\left[F(N^{\lambda}) \int_0^1 g(\tau)(\ dN_{\lambda}(\tau) - \lambda\ d\tau) \right] = \lambda\ \mathbf{E}\left[\int_0^1 D_{\tau} F(N^{\lambda})\ g(\tau)\ d\tau \right]$$

Convergence in $l_{0^+}^{\beta}(L^2)$

Theorem

$$\sup_{\|F\|_{\mathcal{C}_b^3(l^2(\mathbf{N});\,\mathbf{R})}\le 1} \left(\mathbf{E}\left[F(N^\lambda) \right] - \int F\ dm_\beta \right) \le \frac{c_\beta}{\sqrt{\lambda}}$$

Proof I

$$H_\lambda = \lambda^{-1/2} \sum_{n \geq 1} h_n^{1-\beta} \otimes x_n = \lambda^{-1/2} H_1$$

$$\begin{aligned}
\mathbf{E}\left[\mathfrak{J}_\beta N_\lambda . G(\mathfrak{J}_\beta N_\lambda)\right] &= \frac{1}{\sqrt{\lambda}} \sum_{n \geq 1} \mathbf{E}\left[\delta^\lambda(h_n^{1-\beta}) F(\mathfrak{J}_\beta N_\lambda)\right] \; x_n . x \\
&= \frac{1}{\sqrt{\lambda}} \sum_{n \geq 1} \mathbf{E}\left[\int_0^1 h_n^{1-\beta}(\tau) D_\tau F(\mathfrak{J}_\beta N_\lambda) \lambda \, \mathrm{d}\tau\right] \; x_n . x \\
&= \mathbf{E}\left[\int_0^1 D_\tau F(\mathfrak{J}_\beta N_\lambda) . H_\lambda(\tau) \, \mathrm{d}\tau\right].
\end{aligned}$$

Proof II

According to the Taylor formula,

$$D_\tau F(\mathfrak{I}_\beta N_\lambda) = F(\mathfrak{I}_\beta N_\lambda + H_\lambda(\tau)) - F(\mathfrak{I}_\beta N_\lambda)$$
$$= \frac{1}{\sqrt{\lambda}} \nabla F(\mathfrak{I}_\beta N_\lambda).H_1(\tau) + \frac{1}{\lambda} \int_0^1 (1-r)\, \nabla^2 F(\mathfrak{I}_\beta N_\lambda + H_\lambda(\tau)).H_1(\tau)^{\otimes(2)} \, \mathrm{d}r.$$

For the rest, see Coutin and Decreusefond, "Stein's Method for Brownian Approximations".

Bibliography I

Reference books Privault, *Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales*

Nualart, *The Malliavin Calculus and Related Topics*

Üstünel, "Analysis on Wiener Space and Applications"

fBm Decreusefond, "Stochastic Calculus with Respect to Volterra Processes"

Coutin and Decreusefond, "Volterra Stochastic Differential Equations with Singular Kernels"

Bibliography II

Other approaches for fBm: Feyel and de La Pradelle, "On Fractional Brownian Processes"
Zähle, "On the link between fractional and stochastic calculus"
Russo and Vallois, "Elements of stochastic calculus via regularization"

Dirichlet forms and Malliavin calculus: Bouleau, "Théorème de Donsker et formes de Dirichlet"
Bouleau and Hirsch, *Dirichlet forms and analysis on Wiener space*

Independent random variables: Decreusefond and Halconruy, "Malliavin and Dirichlet structures for independent random variables"

Bibliography III

Bernoulli random variables Privault, *Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales*
Nourdin, Peccati, and Reinert, "Stein's Method and Stochastic Analysis of Rademacher Functionals"

Bouleau, Nicolas. “Théorème de Donsker et formes de Dirichlet”. In: n *des Sciences Mathématiques* 129.5 (2005), pp. 369–380. ISSN: 0007-4497. DOI: 10.1016/j.bulsci.2004.09.005.

Bouleau, Nicolas and Francis Hirsch. *Dirichlet forms and analysis on Wiener space*. Vol. 14. De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, 1991, pp. x+325. ISBN: 3-11-012919-1. DOI: 10.1515/9783110858389.

Coutin, L. and L. Decreusefond. “Stein’s Method for Brownian Approximations”. In: *Communications on Stochastic Analysis* 7.3 (Sept. 2013). 00000, pp. 349–372.

– .“Volterra Stochastic Differential Equations with Singular Kernels”. In: *Stochastic Analysis and Mathematical Physics*. Progress in Probability 50 (2001). Ed. by A.B. Cruzeiro and J.-C. Zambrini, pp. 39–50.

Decreusefond, L. “Stochastic Calculus with Respect to Volterra Processes”. In: *Annales de l’Institut Henri Poincaré (B) Probability and Statistics* 41 (2005). 00000, pp. 123–149.

- Decreusefond, L. and Hélène Halconruy. "Malliavin and Dirichlet structures for independent random variables". In: (July 25, 2017). arXiv: 1707.07915v1 [math.PR].
- Feyel, D. and A. de La Pradelle. "On Fractional Brownian Processes". In: *Potential Anal.* 10.3 (1999), pp. 273–288. ISSN: 0926-2601.
- Nourdin, Ivan, Giovanni Peccati, and Gesine Reinert. "Stein's Method and Stochastic Analysis of Rademacher Functionals". In: *Electronic Journal of Probability* 15 (2010), no. 55, 1703–1742. ISSN: 1083-6489. DOI: 10.1214/EJP.v15-843.
- Nualart, D. *The Malliavin Calculus and Related Topics*. 00000. Springer–Verlag, 1995.
- Privault, Nicolas. *Stochastic Analysis in Discrete and Continuous Settings with Normal Martingales*. Vol. 1982. Lecture Notes in Mathematics. Berlin: Springer-Verlag, 2009. ISBN: 978-3-642-02379-8.

Russo, Francesco and Pierre Vallois. “Elements of stochastic calculus via regularization”. In: *Séminaire de Probabilités XL*. Vol. 1899. Lecture Notes in Math. Springer, Berlin, 2007, pp. 147–185. DOI: 10.1007/978-3-540-71189-6_7. URL: https://doi.org/10.1007/978-3-540-71189-6_7.

Üstünel, A. S. “Analysis on Wiener Space and Applications”. In: *arXiv:1003.1649* (Mar. 2010). arXiv: 1003.1649.

Zähle, Martina. “On the link between fractional and stochastic calculus”. In: *Stochastic dynamics (Bremen, 1997)*. Springer, New York, 1999, pp. 305–325. DOI: 10.1007/0-387-22655-9_13. URL: https://doi.org/10.1007/0-387-22655-9_13.