Central Limit Theorems for Multiple Skorokhod Integrals

Ivan Nourdin · David Nualart

Abstract In this paper, we prove a central limit theorem for a sequence of multiple Skorokhod integrals using the techniques of Malliavin calculus. The convergence is stable, and the limit is a conditionally Gaussian random variable. Some applications to sequences of multiple stochastic integrals, and renormalized weighted Hermite variations of the fractional Brownian motion are discussed.

Keywords Central limit theorem · Fractional Brownian motion · Malliavin calculus

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1 Introduction

Consider a sequence of random variables $\{F_n, n \geq 1\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Suppose that the σ-field $\mathcal{F}$ is generated by an isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$ on a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. This just means that X is a centered Gaussian family of random variables indexed by the elements of $\mathfrak{H}$, and such that, for every $h, g \in \mathfrak{H}$,

$$E\big[X(h)X(g)\big] = \langle h, g \rangle_{\mathfrak{H}}. \tag{1.1}$$

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Suppose that the sequence $\{F_n, n \geq 1\}$ is normalized, that is, $E(F_n) = 0$ and $\lim_{n \to \infty} E(F_n^2) = 1$. A natural problem is to find suitable conditions ensuring that F_n converges in law towards a given distribution. When the random variables F_n belong to the qth Wiener chaos of X (for a fixed $q \geq 2$), then it turns out that the following conditions are equivalent:

  (i) F_n converges in law to $N(0, 1)$;
 (ii) $\lim_{n \to \infty} E[F_n^4] = 3$;
(iii) $\lim_{n \to \infty} \|DF_n\|_{\mathfrak{H}}^2 = q$ in $L^2(\Omega)$.

Here, D stands for the derivative operator in the sense of Malliavin calculus (see Sect. 2 below for more details). More precisely, the following bound is in order, where N denotes a standard Gaussian random variable:

$$\sup_{z \in \mathbb{R}} \left| P(F_n \leq z) - P(N \leq z) \right| \leq \sqrt{E\left[\left(1 - \frac{1}{q}\|DF_n\|_{\mathfrak{H}}^2\right)^2\right]} \tag{1.2}$$

$$\leq \sqrt{\frac{q-1}{3q}}\sqrt{\left|E\left(F_n^4\right) - 3\right|}. \tag{1.3}$$

The equivalence between conditions (i) and (ii) was proved in Nualart and Peccati [22] by means of the Dambis, Dubins, and Schwarz theorem. It implies that the convergence in distribution of a sequence of multiple stochastic integrals towards a Gaussian random variable is completely determined by the asymptotic behavior of their second and fourth moments, which represents a drastic simplification of the classical "method of moments and diagrams" (see, for instance, the survey by Peccati and Taqqu [26], as well as the references therein). The equivalence with condition (iii) was proved later by Nualart and Ortiz-Latorre [21] using tools of Malliavin calculus. Finally, the Berry–Esseen-type bound (1.2) is taken from Nourdin and Peccati [16], while (1.3) was shown in Nourdin, Peccati, and Reinert [14].

Peccati and Tudor [27] also obtained a multidimensional version of the equivalence between (i) and (ii). In particular, they proved that, given a sequence $\{F_n, n \geq 1\}$ of d-dimensional random vectors such that F_n^i belongs to the q_ith Wiener chaos for $i = 1, \ldots, d$, where $1 \leq q_1 \leq \cdots \leq q_d$, then if the covariance matrix of F_n converges to the $d \times d$ identity matrix I_d, the convergence in distribution to each component towards the law $N(0, 1)$ implies the convergence in distribution of the whole sequence F_n towards the standard centered Gaussian law $N(0, I_d)$.

Recent examples of application of these results are, among others, the study of p-variations of fractional stochastic integrals (Corcuera et al. [4]), quadratic functionals of bivariate Gaussian processes (Deheuvels et al. [6]), self-intersection local times of fractional Brownian motion (Hu and Nualart [7]), approximation schemes for scalar fractional differential equations (Neuenkirch and Nourdin [12]), high-frequency CLTs for random fields on homogeneous spaces (Marinucci and Peccati [10, 11] and Peccati [23]), needlets analysis on the sphere (Baldi et al. [1]), estimation of self-similarity orders (Tudor and Viens [31]), weighted power variations of iterated Brownian motion (Nourdin and Peccati [17]), or bipower variations of Gaussian processes with stationary increments (Barndorff-Nielsen et al. [2]).

Since the works by Nualart and Peccati [22] and Peccati and Tudor [27], great efforts have been made to find similar statements in the case where the limit is not necessarily Gaussian. In the references [25] and [24], Peccati and Taqqu propose sufficient conditions ensuring that a given sequence of multiple Wiener–Itô integrals converges stably towards mixtures of Gaussian random variables. In another direction, Nourdin and Peccati [15] proved an extension of the above equivalence (i)–(iii) for a sequence of random variables $\{F_n, n \geq 1\}$ in a fixed qth Wiener chaos, $q \geq 2$, where the limit law is $2\,G_{\nu/2} - \nu$, $G_{\nu/2}$ being the Gamma distribution with parameter $\nu/2$.

The purpose of the present paper is to study the convergence in distribution of a sequence of random variables of the form $F_n = \delta^q(u_n)$, where u_n are random variables with values in $\mathfrak{H}^{\otimes q}$ (the qth tensor product of $\mathfrak{H}$), and δ^q denotes the multiple Skorokhod integral (that is, $\delta^2(u) = \delta(\delta(u))$, $\delta^3(u) = \delta(\delta(\delta(u)))$, and so on), towards a mixture of Gaussian random variables. Our main abstract result, Theorem 3.1, roughly says that under some technical conditions, if $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in $L^1(\Omega)$ to a nonnegative random variable S^2, then the sequence F_n converges stably to a random variable F with conditional characteristic function $E(e^{i\lambda F}|X) = E(e^{-\frac{\lambda^2}{2}S^2})$. Notice that if u_n is deterministic, then F_n belongs to the qth Wiener chaos, and we have a sequence of the type considered above. In particular, if S^2 is also deterministic, we recover the fact that condition (iii) above implies the convergence in distribution to the law $N(0, 1)$.

We develop some particular applications of Theorem 3.1 in the following directions. First, we consider a sequence of random variables in a fixed Wiener chaos, and we derive new criteria for the convergence to a mixture of Gaussian laws. Second, we show the convergence in law of the sequence $\delta^q(u_n)$, where $q \geq 2$, and u_n is a q-parameter process of the form

$$u_n = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \mathbf{1}_{(k/n,(k+1)/n]^q},$$

towards the random variable $\sigma_{H,q} \int_0^1 f(B_s)\,dW_s$, where B is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4q}, \frac{1}{2})$, W is a standard Brownian motion independent of B, and $\sigma_{H,q}$ denotes some positive constant. This convergence allows us to establish a new asymptotic result for the behavior of the weighted qth Hermite variation of the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4q}, \frac{1}{2})$, which complements and provides a new perspective to the results proved by Nourdin [13], Nourdin, Nualart and Tudor [19], and Nourdin and Réveillac [18]. The reader is referred to Sect. 5 for a detailed description of these results.

The paper is organized as follows. In Sect. 2, we present some preliminary results about Malliavin calculus. Section 3 contains the statement and the proof of the main abstract result. In Sect. 4, we apply it to sequences of multiple stochastic integrals, while Sect. 5 focuses on the applications to the weighted Hermite variations of the fractional Brownian motion.

2 Preliminaries

Let $\mathfrak{H}$ be a real separable infinite-dimensional Hilbert space. For any integer $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ be the qth tensor product of $\mathfrak{H}$. Also, we denote by $\mathfrak{H}^{\odot q}$ the qth symmetric tensor product.

Suppose that $X = \{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process on $\mathfrak{H}$, defined on some probability space $(\Omega, \mathcal{F}, P)$. Recall that this means that the covariance of X is given in terms of the scalar product of $\mathfrak{H}$ by (1.1). Assume from now on that $\mathcal{F}$ is generated by X.

For every integer $q \geq 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables

$$\big\{ H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1 \big\},$$

where H_q is the qth Hermite polynomial defined by

$$H_q(x) = \frac{(-1)^q}{q!} e^{x^2/2} \frac{d^q}{dx^q} \big(e^{-x^2/2} \big).$$

We denote by $\mathcal{H}_0$ the space of constant random variables. For any $q \geq 1$, the mapping $I_q(h^{\otimes q}) = q! H_q(X(h))$ provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!} \| \cdot \|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$ (equipped with the $L^2(\Omega)$ norm). For $q = 0$, by convention $\mathcal{H}_0 = \mathbb{R}$, and I_0 is the identity map.

It is well known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q$. That is, any square-integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.1}$$

where $f_0 = E[F]$, and the $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F. For every $q \geq 0$, we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.1), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k, k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \ldots, p \wedge q\}$, the rth contraction of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1, \ldots, i_r = 1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.2}$$

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g, while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$.

In the particular case $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space and μ is a σ-finite and nonatomic measure, one has that $\mathfrak{H}^{\odot q} = L_s^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$ is the space of symmetric and square-integrable functions on A^q. Moreover, for every $f \in \mathfrak{H}^{\odot q}$, $I_q(f)$ coincides with the multiple Wiener–Itô integral of order q of f with

respect to X (introduced by Itô in [8]), and (2.2) can be written as

$$(f \otimes_r g)(t_1, \ldots, t_{p+q-2r})$$
$$= \int_{A^r} f(t_1, \ldots, t_{p-r}, s_1, \ldots, s_r)$$
$$\times g(t_{p-r+1}, \ldots, t_{p+q-2r}, s_1, \ldots, s_r) \, d\mu(s_1) \ldots d\mu(s_r).$$

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process X. We refer the reader to Nualart [20] for a more detailed presentation of these notions. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\big(X(\phi_1), \ldots, X(\phi_n)\big), \tag{2.3}$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}\big(X(\phi_1), \ldots, X(\phi_n)\big)\phi_i.$$

By iteration, one can define the qth derivative $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$.

For $q \geq 1$ and $p \geq 1$, $\mathbb{D}^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\| \cdot \|_{\mathbb{D}^{q,p}}$ defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p = E\big[|F|^p\big] + \sum_{i=1}^q E\big(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\big).$$

The Malliavin derivative D verifies the following chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \ldots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We denote by δ the adjoint of the operator D, also called the divergence operator. The operator δ is also called the Skorokhod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorokhod in [30]. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, denoted $\mathrm{Dom}\,\delta$, if and only if it verifies

$$\big|E\big(\langle DF, u\rangle_{\mathfrak{H}}\big)\big| \leq c_u \sqrt{E(F^2)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}\,\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (called "integration by parts formula")

$$E\big(F\delta(u)\big) = E\big(\langle DF, u\rangle_{\mathfrak{H}}\big), \tag{2.4}$$

which holds for every $F \in \mathbb{D}^{1,2}$. Formula (2.4) extends to the multiple Skorokhod integral δ^q, and we have

$$E\big(F\delta^q(u)\big) = E\big(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}}\big) \tag{2.5}$$

for any element u in the domain of δ^q and any random variable $F \in \mathbb{D}^{q,2}$. Moreover, $\delta^q(h) = I_q(h)$ for any $h \in \mathfrak{H}^{\odot q}$.

The following property will be extensively used in the paper.

Lemma 2.1 *Let $q \geq 1$ be an integer. Suppose that $F \in \mathbb{D}^{q,2}$, and let u be a symmetric element in $\mathrm{Dom}\,\delta^q$. Assume that, for any $0 \leq r + j \leq q$, $\langle D^r F, \delta^j(u) \rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q - r - j})$. Then, for any $r = 0, \ldots, q-1$, $\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}$ belongs to the domain of δ^{q-r}, and we have*

$$F\delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\big(\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}\big). \tag{2.6}$$

(We use the convention that $\delta^0(v) = v$, $v \in \mathbb{R}$, and $D^0 F = F$, $F \in L^2(\Omega)$.)

Proof We prove this lemma by induction on q. For $q = 1$, it reads $F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathfrak{H}}$, and this formula is well known, see, e.g., [20, Proposition 1.3.3]. Suppose the result is true for q. Then, if u belongs to the domain of δ^{q+1}, by the induction hypothesis applied to $\delta(u)$,

$$F\delta^{q+1}(u) = F\delta^q\big(\delta(u)\big) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r}\big(\langle D^r F, \delta(u) \rangle_{\mathfrak{H}^{\otimes r}}\big). \tag{2.7}$$

On the other hand,

$$\langle D^r F, \delta(u) \rangle_{\mathfrak{H}^{\otimes r}} = \delta\big(\langle D^r F, u \rangle_{\mathfrak{H}^{\otimes r}}\big) + \langle D^{r+1} F, u \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.8}$$

Finally, substituting (2.8) into (2.7) yields the desired result.   □

For any Hilbert space V, we denote by $\mathbb{D}^{k,p}(V)$ the corresponding Sobolev space of V-valued random variables (see [20, page 31]). The operator δ^q is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$ for any $p > 1$ and any integers $k \geq q \geq 1$, that is, we have

$$\big\|\delta^q(u)\big\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})} \tag{2.9}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ and some constant $c_{k,p} > 0$. These estimates are consequences of Meyer inequalities (see [20, Proposition 1.5.7]). In particular, these estimates imply that $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\,\delta^q$ for any integer $q \geq 1$.

We will also use the following commutation relationship between the Malliavin derivative and the Skorokhod integral (see [20, Proposition 1.3.2]):

$$D\delta(u) = u + \delta(Du) \tag{2.10}$$

for any $u \in \mathbb{D}^{2,2}(\mathfrak{H})$. By induction we can show the following formula for any symmetric element u in $\mathbb{D}^{j+k,2}(\mathfrak{H}^{\otimes j})$:

$$D^k \delta^j(u) = \sum_{i=0}^{j \wedge k} \binom{k}{i}\binom{j}{i} i! \delta^{j-i}(D^{k-i}u). \tag{2.11}$$

We will make use of the following formula for the variance of a multiple Skorokhod integral. Let $u, v \in \mathbb{D}^{2q,2}(\mathfrak{H}^{\otimes q}) \subset \text{Dom}\,\delta^q$ be two symmetric functions. Then

$$\begin{aligned} E\big(\delta^q(u)\delta^q(v)\big) &= E\big(\langle u, D^q\big(\delta^q(v)\big)\rangle_{\mathfrak{H}^{\otimes q}}\big) \\ &= \sum_{i=0}^q \binom{q}{i}^2 i! E\big(\langle u, \delta^{q-i}\big(D^{q-i}v\big)\rangle_{\mathfrak{H}^{\otimes q}}\big) \\ &= \sum_{i=0}^q \binom{q}{i}^2 i! E\big(\langle D^{q-i}u, D^{q-i}v\rangle_{\mathfrak{H}^{\otimes(2q-i)}}\big). \end{aligned} \tag{2.12}$$

The operator L is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -q J_q$$

and is called the infinitesimal generator of the Ornstein–Uhlenbeck semigroup. The domain of this operator in $L^2(\Omega)$ is the set

$$\text{Dom}\, L = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|^2_{L^2(\Omega)} < \infty \right\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators D, δ, and L (see [20, Proposition 1.4.3]). A random variable F belongs to the domain of L if and only if $F \in \text{Dom}(\delta D)$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\,\delta$), and in this case

$$\delta D F = -LF. \tag{2.13}$$

Note also that a random variable F as in (2.1) is in $\mathbb{D}^{1,2}$ if and only if

$$\sum_{q=1}^{\infty} q q! \|f_q\|^2_{\mathfrak{H}^{\otimes q}} < \infty,$$

and, in this case, $E(\|DF\|^2_{\mathfrak{H}}) = \sum_{q \geq 1} q q! \|f_q\|^2_{\mathfrak{H}^{\otimes q}}$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with μ nonatomic), then the derivative of a random variable F as in (2.1) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}\big(f_q(\cdot, a)\big), \quad a \in A. \tag{2.14}$$

Finally, we need the definition of stable convergence (see, for instance, the original paper [29] or the book [9] for an exhaustive discussion of stable convergence).

Definition 2.2 Let F_n be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, and suppose that F is a random variable defined on an enlarged probability space $(\Omega, \mathcal{G}, P)$ with $\mathcal{F} \subseteq \mathcal{G}$. We say that F_n converges $\mathcal{G}$-stably to F (or only stably when the context is clear) if, for any continuous and bounded function $f : \mathbb{R} \to \mathbb{R}$ and any bounded $\mathcal{F}$-measurable random variable Z, we have $E[f(F_n)Z] \to E[f(F)Z]$ as n tends to infinity.

3 Convergence in Law of Multiple Skorokhod Integrals

As in the previous section, $X = \{X(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process associated with a real separable infinite-dimensional Hilbert space $\mathfrak{H}$. The next theorem is the main abstract result of the present paper.

Theorem 3.1 *Fix an integer $q \geq 1$, and suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$ for some symmetric functions u_n in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Suppose moreover that the sequence F_n is bounded in $L^1(\Omega)$ and that:*

(i) $\langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathfrak{H}^{\otimes q}}$ *converges in $L^1(\Omega)$ to zero for all integers $r, k_1, \ldots, k_{q-1} \geq 0$ such that*

$$k_1 + 2k_2 + \cdots + (q-1)k_{q-1} + r = q$$

and all $h \in \mathfrak{H}^{\otimes r}$;

(ii) $\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ *converges in $L^1(\Omega)$ to a nonnegative random variable S^2.*

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X.

Remark 3.2 When $q = 1$, condition (i) of the theorem is that $\langle u_n, h \rangle_{\mathfrak{H}}$ converges to zero in $L^1(\Omega)$ for each $h \in \mathfrak{H}$. When $q = 2$, condition (i) means that $\langle u_n, h \otimes g \rangle_{\mathfrak{H}^{\otimes 2}}$, $\langle u_n, DF_n \otimes h \rangle_{\mathfrak{H}^{\otimes 2}}$, and $\langle u_n, DF_n \otimes DF_n \rangle_{\mathfrak{H}^{\otimes 2}}$ converge to zero in $L^1(\Omega)$ for all h, $g \in \mathfrak{H}$. And so on.

Proof of Theorem 3.1 Taking into account Definition 2.2, it suffices to show that for any $h_1, \ldots, h_m \in \mathfrak{H}$, the sequence

$$\xi_n = (F_n, X(h_1), \ldots, X(h_m))$$

converges in distribution to a vector $(F_\infty, X(h_1), \ldots, X(h_m))$, where F_∞ satisfies, for any $\lambda \in \mathbb{R}$,

$$E\left(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)\right) = e^{-\frac{\lambda^2}{2}S^2}. \tag{3.1}$$

Since the sequence F_n is bounded in $L^1(\Omega)$, the sequence ξ_n is tight. Assume that $(F_\infty, X(h_1), \ldots, X(h_m))$ denotes the limit in law of a certain subsequence of ξ_n, denoted again by ξ_n.

Let $Y = \phi(X(h_1), \ldots, X(h_m))$ with $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$ (ϕ is infinitely differentiable, bounded, with bounded partial derivatives of all orders), and consider $\phi_n(\lambda) = E(e^{i\lambda F_n} Y)$ for $\lambda \in \mathbb{R}$. The convergence in law of ξ_n, together with the fact that F_n is bounded in $L^1(\Omega)$, implies that

$$\lim_{n \to \infty} \phi_n'(\lambda) = \lim_{n \to \infty} i E(F_n e^{i\lambda F_n} Y) = i E(F_\infty e^{i\lambda F_\infty} Y). \tag{3.2}$$

On the other hand, by (2.5) and the Leibnitz rule for D^q, we obtain

$$
\begin{aligned}
\phi_n'(\lambda) &= i E(F_n e^{i\lambda F_n} Y) = i E(\delta^q(u_n) e^{i\lambda F_n} Y) \\
&= i E(\langle u_n, D^q(e^{i\lambda F_n} Y) \rangle_{\mathfrak{H}^{\otimes q}}) \\
&= i \sum_{a=0}^q \binom{q}{a} E(\langle u_n, D^a(e^{i\lambda F_n}) \widetilde{\otimes} D^{q-a} Y \rangle_{\mathfrak{H}^{\otimes q}}) \\
&= i \sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \cdots k_a!} (i\lambda)^{k_1 + \cdots + k_a} \\
&\quad \times E(e^{i\lambda F_n} \langle u_n, (DF_n)^{\otimes k_1} \widetilde{\otimes} \cdots \widetilde{\otimes} (D^a F_n)^{\otimes k_a} \widetilde{\otimes} D^{q-a} Y \rangle_{\mathfrak{H}^{\otimes q}}) \\
&= i \sum_{a=0}^q \binom{q}{a} \sum \frac{a!}{k_1! \cdots k_a!} (i\lambda)^{k_1 + \cdots + k_a} \\
&\quad \times E(e^{i\lambda F_n} \langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^a F_n)^{\otimes k_a} \otimes D^{q-a} Y \rangle_{\mathfrak{H}^{\otimes q}}),
\end{aligned}
$$

where the second sum in the two last equalities runs over all sequences of integers $(k_1, \ldots, k_a)$ such that $k_1 + 2k_2 + \cdots + ak_a = a$, due to the Faá di Bruno's formula. By condition (i), this yields that

$$\phi_n'(\lambda) = -\lambda E(e^{i\lambda F_n} \langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}} Y) + R_n$$

with R_n converging to zero as $n \to \infty$. Using condition (ii) and (3.2), we obtain that

$$i E(F_\infty e^{i\lambda F_\infty} Y) = -\lambda E(e^{i\lambda F_\infty} S^2 Y).$$

Since S^2 is defined through condition (ii), it is in particular measurable with respect to X. Thus, the following linear differential equation verified by the conditional characteristic function of F_∞ holds:

$$\frac{\partial}{\partial \lambda} E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)) = -\lambda S^2 E(e^{i\lambda F_\infty} | X(h_1), \ldots, X(h_m)).$$

By solving it, we obtain (3.1), which yields the desired conclusion. $\qquad\square$

The next corollary provides stronger but easier conditions for the stable convergence.

Corollary 3.3 *For a fixed $q \geq 1$, suppose that F_n is a sequence of random variables of the form $F_n = \delta^q(u_n)$ for some symmetric functions u_n in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Suppose moreover that the sequence F_n is bounded in $\mathbb{D}^{q,p}$ for all $p \geq 2$ and that:*

(i′) *$\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ for all $h \in \mathfrak{H}^{\otimes q}$, and $u_n \otimes_l D^l F_n$ converges to zero in $L^2(\Omega; \mathfrak{H}^{\otimes(q-l)})$ for all $l = 1, \ldots, q-1$;*

(ii) *$\langle u_n, D^q F_n \rangle_{\mathfrak{H}^{\otimes q}}$ converges in $L^1(\Omega)$ to a nonnegative random variable S^2.*

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X.

Proof It suffices to show that condition (i′) implies condition (i) in Theorem 3.1. When $k_a \neq 0$ for $1 \leq a \leq q-1$, we have, for all $h \in \mathfrak{H}^{\otimes r}$ (with $r = q - k_1 - 2k_2 - \cdots - ak_a$),

$$
\begin{aligned}
&\left| \left\langle u_n, \left(DF_n\right)^{\otimes k_1} \otimes \cdots \otimes (D^a F_n)^{\otimes k_a} \otimes h \right\rangle_{\mathfrak{H}^{\otimes q}} \right| \\
&= \Big| \big\langle u_n \otimes_a D^a F_n, \\
&\qquad (DF_n)^{\otimes k_1} \otimes \cdots \otimes \left(D^{a-1} F_n\right)^{\otimes k_{a-1}} \otimes \left(D^a F_n\right)^{\otimes(k_a-1)} \otimes h \big\rangle_{\mathfrak{H}^{\otimes(q-a)}} \Big| \\
&\leq \left\| u_n \otimes_a D^a F_n \right\|_{\mathfrak{H}^{\otimes(q-a)}} \\
&\qquad \times \left\| (DF_n)^{\otimes k_1} \otimes \cdots \otimes \left(D^{a-1} F_n\right)^{\otimes k_{a-1}} \otimes \left(D^a F_n\right)^{\otimes(k_a-1)} \otimes h \right\|_{\mathfrak{H}^{\otimes(q-a)}}.
\end{aligned}
$$

The second factor is bounded in $L^2(\Omega)$, and the first factor converges to zero in $L^2(\Omega)$ for all $a = 1, \ldots, q-1$. In the case $a = 0$, we have that $\langle u_n, h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ for all $h \in \mathfrak{H}^{\otimes q}$, by condition (i′). This completes the proof. $\qquad \square$

4 Multiple Stochastic Integrals

Suppose that $\mathfrak{H}$ is a Hilbert space $L^2(A, \mathcal{A}, \mu)$, where $(A, \mathcal{A})$ is a measurable space, and μ is a σ-finite and nonatomic measure.

Fix an integer $m \geq 2$, and consider a sequence of multiple stochastic integrals $\{F_n = I_m(g_n), n \geq 1\}$ with $g_n \in \mathfrak{H}^{\odot m}$. We would like to apply Theorem 3.1 with $q = 1$ to the sequence F_n. To do this, we represent each F_n as

$$
F_n = \delta(u_n) \quad \text{with } u_n = I_{m-1}(\widehat{g}_n)
$$

for some function $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$ which is symmetric in the first $m-1$ variables.

Notice that from (2.14) we have $DF_n = m I_{m-1}(g_n)$. Hence, since $F_n = -\frac{1}{m} L F_n = \frac{1}{m}\delta(DF_n)$ by (2.13), g_n is always a possible choice for $\widehat{g}_n$. (In this case, $\widehat{g}_n$ is symmetric in all the variables.) However, as observed, for instance, in Example 4.2 below, the choice $\widehat{g}_n = g_n$ does not allow us to conclude in general.

Proposition 4.1 *For a fixed integer $m \geq 2$, let F_n be a sequence of random variables of the form $F_n = I_m(g_n)$ with $g_n \in \mathfrak{H}^{\odot m}$. Suppose moreover that F_n is bounded in $L^2(\Omega)$ and that $F_n = \delta(u_n)$, where $u_n = I_{m-1}(\widehat{g}_n)$ for some function $\widehat{g}_n \in \mathfrak{H}^{\otimes m}$ which is symmetric in the first $m-1$ variables. Finally, assume that:*

(a) $\langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}}$ converges to zero for all $h \in \mathfrak{H}$;

(b) $\langle u_n, DF_n \rangle_{\mathfrak{H}}$ converges in $L^1(\Omega)$ to a nonnegative random variable S^2.

Then, F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given X.

Proof It suffices to apply Theorem 3.1 to $u_n = I_{m-1}(\widehat{g}_n)$ and $q = 1$. Indeed, we have

$$E\big(\langle u_n, h \rangle_{\mathfrak{H}}^2\big) = E\big(\langle I_{m-1}(\widehat{g}_n), h \rangle_{\mathfrak{H}}^2\big) = E\big(I_{m-1}\big(\widehat{g}_n \otimes_1 h\big)^2\big)$$

$$= (m-1)! \big\|\widehat{g}_n \otimes_1 h\big\|_{\mathfrak{H}^{\otimes(m-1)}}^2$$

$$= (m-1)! \langle \widehat{g}_n \otimes_{m-1} \widehat{g}_n, h^{\otimes 2} \rangle_{\mathfrak{H}^{\otimes 2}} \to 0,$$

which implies condition (i) in Theorem 3.1, see also Remark 3.2. Condition (ii) in Theorem 3.1 follows from (b). $\qquad\square$

Example 4.2 (See also [28, Proposition 2.1] or [25, Proposition 18] for two different proofs using other techniques) Suppose that $\{W_t, \ t \in [0, 1]\}$ is a standard Brownian motion. (This corresponds to $A = [0, 1]$ and μ the Lebesgue measure.) Assume that $m = 2$ and take $g_n(s, t) = \frac{1}{2}\sqrt{n}(s \vee t)^n$. Then

$$F_n = I_2(g_n) = \sqrt{n} \int_0^1 t^n W_t \, dW_t$$

and

$$D_s F_n = \sqrt{n} s^n W_s + \sqrt{n} \int_s^1 t^n W_t \, dW_t.$$

We can take $u_n(t) = \sqrt{n} t^n W_t$, that is, $\widehat{g}_n(s, t) = \sqrt{n} t^n \mathbf{1}_{[0,t]}(s)$. In this case,

$$\big(\widehat{g}_n \otimes_1 \widehat{g}_n\big)(s, t) = n s^n t^n (s \wedge t),$$

which converges to zero weakly in $L^2(\Omega)$, and

$$\langle u_n, DF_n \rangle_{\mathfrak{H}} = \int_0^1 n t^{2n} W_t^2 \, dt + n \int_0^1 t^n W_t \left(\int_0^t s^n W_s \, dW_s \right) dt,$$

which converges in $L^2(\Omega)$ to $\frac{1}{2}W_1^2$. Therefore, conditions (a) and (b) of Proposition 4.1 are satisfied with $S^2 = \frac{1}{2}W_1^2$, and F_n converges in distribution to $\frac{1}{\sqrt{2}}W_1 \times N$, with $N \sim N(0, 1)$. One easily sees on this particular example that the choice $\widehat{g}_n = g_n$ does not allow us to conclude in general (except when S^2 is deterministic); indeed, one can check here that $\langle u_n, DF_n \rangle_{\mathfrak{H}} = \frac{1}{m}\|DF_n\|_{\mathfrak{H}}^2$ does not converge in $L^1(\Omega)$.

If we take $\widehat{g}_n = g_n$ and $S^2 = 1$, then condition (b) coincides with condition (iii) in the introduction. In this case, Nualart and Peccati criterion combined with Lemma 6 in [21] tells us that, if the sequence of variances converges to one, then condition (a) is automatically satisfied.

On the other hand, we can also apply Theorem 3.1 with $u_n = g_n$. In this way, applying Corollary 3.3, we obtain that the following conditions imply that F_n converges to a normal random variable $N(0, 1)$ independent of X:

(α) g_n converges weakly to zero;
(β) $\|g_n \otimes_l g_n\|_{\mathfrak{H}^{\otimes 2(q-l)}}$ converges to zero for all $l = 1, \ldots, q-1$;
(γ) $q! \|g_n\|^2_{\mathfrak{H}^{\otimes q}}$ converges to 1.

Indeed, notice first that if g_n is bounded in $\mathfrak{H}^{\odot q}$, then F_n is bounded in all the Sobolev spaces $\mathbb{D}^{q,p}$, $p \geq 2$. Then, condition (ii) in Corollary 3.3 follows from (γ) and the equality $D^q(I_q(g_n)) = q! g_n$. On the other hand, condition (i$'$) in Corollary 3.3 follows from (ii) and

$$
\begin{aligned}
E\big[\big\|g_n \otimes_l D^l F_n\big\|^2_{\mathfrak{H}^{\otimes(q-l)}}\big] &= \frac{q!^2}{(q-l)!^2} E\big[\big\|g_n \otimes_l I_{q-l}(g_n)\big\|^2_{\mathfrak{H}^{\otimes(q-l)}}\big] \\
&= \frac{q!^2}{(q-l)!^2} E\big[\big\|I_{q-l}(g_n \otimes_l g_n)\big\|^2_{\mathfrak{H}^{\otimes(q-l)}}\big] \\
&= \frac{q!^2}{(q-l)!} \big\|g_n \widetilde{\otimes}_l g_n\big\|^2_{\mathfrak{H}^{\otimes 2(q-l)}} \\
&\leq \frac{q!^2}{(q-l)!} \big\|g_n \otimes_l g_n\big\|^2_{\mathfrak{H}^{\otimes 2(q-l)}}.
\end{aligned}
$$

In this way, we recover the fact that condition (iii) in the introduction implies the normal convergence.

5 Weighted Hermite Variations of the Fractional Brownian Motion

5.1 Description of the Results

The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function

$$
E(B_s B_t) = R_H(s, t) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t - s|^{2H}\big). \tag{5.1}
$$

From (5.1) it follows that $E|B_t - B_s|^2 = (t - s)^{2H}$ for all $0 \leq s < t$ and that, for each $a > 0$, the process $\{a^{-H} B_{at}, t \geq 0\}$ is also an fBm with Hurst parameter H (self-similarity property). As a consequence, the sequence $\{B_j - B_{j-1}, j = 1, 2, \ldots\}$ is stationary, Gaussian, and ergodic, with correlation given by

$$
\rho_H(n) = \frac{1}{2}\big[|n + 1|^{2H} - 2|n|^{2H} + |n - 1|^{2H}\big], \tag{5.2}
$$

which behaves as $H(2H - 1)|n|^{2H-2}$ as n tends to infinity.

Set $\Delta B_{k/n} = B_{(k+1)/n} - B_{k/n}$, where $k = 0, 1, \ldots, n$, and $n \geq 1$. The ergodic theorem combined with the self-similarity property implies that the sequence

$n^{2H-1} \sum_{k=0}^{n-1} (\Delta B_{k/n})^2$ converges, almost surely and in $L^1(\Omega)$, to $E(B_1^2) = 1$. More-over, it is well known (see, e.g., [3]) that, provided $H \in (0, \frac{3}{4})$, a central limit theorem holds: the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(n^{2H} (\Delta B_{k/n})^2 - 1 \right) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_2 \left(n^H \Delta B_{k/n} \right) \tag{5.3}$$

converges in law to $N(0, \sigma_H^2)$ as $n \to \infty$ for some constant $\sigma_H > 0$. (Notice also that, by normalizing with $\sqrt{n \log n}$ instead of $\sqrt{n}$, the central limit theorem continues to hold in the critical case $H = \frac{3}{4}$.) When $H > \frac{3}{4}$, the situation is very different. Indeed, we have in contrast that

$$n^{1-2H} \sum_{k=0}^{n-1} \left(n^{2H} (\Delta B_{k/n})^2 - 1 \right) = n^{1-2H} \sum_{k=0}^{n-1} H_2 \left(n^H \Delta B_{k/n} \right)$$

converges in $L^2(\Omega)$. More generally, consider an integer $q \geq 2$. If $H < 1 - \frac{1}{2q}$, then the sequence

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} H_q \left(n^H \Delta B_{k/n} \right) \tag{5.4}$$

converges in law to $N(0, \sigma_{q,H}^2)$ (for some constant $\sigma_{q,H} > 0$), whereas, if $H > 1 - \frac{1}{2q}$, then the sequence

$$n^{q-qH-1} \sum_{k=0}^{n-1} H_q \left(n^H \Delta B_{k/n} \right)$$

converges in $L^2(\Omega)$.

Some unexpected results happen when we introduce a weight of the form $f(B_{k/n})$ in (5.4). In fact, a new critical value ($H = \frac{1}{2q}$) plays an important role. More precisely, consider the following sequence of random variables:

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}) H_q \left(n^H \Delta B_{k/n} \right). \tag{5.5}$$

Here, the integer $q \geq 2$ is fixed, and the function $f : \mathbb{R} \to \mathbb{R}$ is supposed to satisfy some suitable regularity and growth conditions. In [13, 19], the following conver-gences as $n \to \infty$ are shown:

- If $H < \frac{1}{2q}$, then

$$n^{qH-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) \, ds. \tag{5.6}$$

- If $\frac{1}{2q} < H < 1 - \frac{1}{2q}$, then

$$G_n \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) \, dW_s, \qquad (5.7)$$

where W is a Brownian motion independent of B, and

$$\sigma_{H,q}^2 = q! \sum_{r \in \mathbb{Z}} \rho_H(r)^q < \infty. \qquad (5.8)$$

- If $H = 1 - \frac{1}{2q}$, then

$$\frac{G_n}{\sqrt{\log n}} \xrightarrow{\text{stably}} \sqrt{\frac{2}{q!} \left(1 - \frac{1}{2q}\right)^{q/2} \left(1 - \frac{1}{q}\right)^{q/2}} \int_0^1 f(B_s) \, dW_s,$$

where W is a Brownian motion independent of B.
- If $H > 1 - \frac{1}{2q}$, then

$$n^{q(1-H)-\frac{1}{2}} G_n \xrightarrow{L^2(\Omega)} \int_0^1 f(B_s) \, dZ_s^{(q)},$$

where $Z^{(q)}$ denotes the Hermite process of order q canonically constructed from B (see [19] for the details).

In addition, when $q = 2$ and $H = \frac{1}{4}$, it was shown in [18] that G_n converges stably to a linear combination of the limits in (5.7) and (5.6). (The proof of this last result follows an approach similar to the proof of our Theorem 3.1 and allows one to derive a change of variable formula for the fBm of Hurst index $\frac{1}{4}$, with a correction term that is an ordinary Itô integral with respect to a Brownian motion that is independent of B.) But the convergence of G_n in the critical case $H = \frac{1}{2q}$, $q \geq 3$, was open till now.

In the present paper, we are going to show that Theorem 3.1 provides a proof of the following new result, valid for any integer $q \geq 2$ and any index $H \in (\frac{1}{4q}, \frac{1}{2})$:

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \xrightarrow{\text{stably}} \sigma_{H,q} \int_0^1 f(B_s) \, dW_s. \qquad (5.9)$$

(See Theorem 5.3 below for a precise statement.) Notice that (5.9) provides a new proof of (5.7) in the case $H \in (\frac{1}{2q}, \frac{1}{2})$ (without considering two different levels of discretization $n \leq m$, as in [19]). More importantly, in the critical case $H = \frac{1}{2q}$, convergence (5.9) yields:

$$G_n \xrightarrow{\text{stably}} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B_s) \, ds + \sigma_{1/(2q),q} \int_0^1 f(B_s) \, dW_s.$$

Hence, the understanding of the asymptotic behavior of the weighted Hermite variations of the fBm is now complete (indeed, the case $H = \frac{1}{2q}$, $q \geq 3$, was the only remaining case, as mentioned in the discussion above).

The main idea of the proof of (5.9) is a decomposition of the random variable G_n using (2.6). The term with $r = 0$ is a multiple Skorokhod integral of order q and, by Theorem 5.2 below, it converges in law for any $H \in (\frac{1}{4q}, \frac{1}{2})$. The term with $r = q$ behaves as $-n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n} f^{(q)}(B_{k/n})$. The remaining terms $(1 \leq r \leq q - 1)$ converge to zero in $L^2(\Omega)$.

5.2 Some Preliminaries on the Fractional Brownian Motion

Before proving (5.9), we need some preliminaries on the Malliavin calculus associated with the fBm and some technical results (see [20, Chap. 5]).

In the following, we assume $H \in (0, \frac{1}{2})$. We denote by $\mathcal{E}$ the set of step functions on $[0, 1]$. Let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s) = \frac{1}{2} \left(s^H + t^H - |t - s|^H \right).$$

The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to a linear isometry between the Hilbert space $\mathfrak{H}$ and the Gaussian space spanned by B. We denote this isometry by $\phi \to B(\phi)$. In this way, $\{B(\phi), \phi \in \mathfrak{H}\}$ is an isonormal Gaussian space. (In fact, we know that the space $\mathfrak{H}$ coincides with $I_{0+}^{H-\frac{1}{2}}(L^2[0, 1])$, where

$$I_{0+}^{H-\frac{1}{2}} f(x) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_0^x (x - y)^{H-\frac{3}{2}} f(y) \, dy$$

is the left-sided Liouville fractional integral of order $H - \frac{1}{2}$, see [5].)

From now on, we will make use of the notation

$$\varepsilon_t = \mathbf{1}_{[0,t]},$$

$$\partial_{k/n} = \varepsilon_{(k+1)/n} - \varepsilon_{k/n} = \mathbf{1}_{(k/n,(k+1)/n]},$$

for $t \in [0, 1]$, $n \geq 1$, and $k = 0, \ldots, n - 1$. Notice that $H_q(n^H \Delta B_{k/n}) = n^{qH} I_q(\partial_{k/n}^{\otimes q})$.

We need the following technical lemma.

Lemma 5.1 *Recall that* $H < \frac{1}{2}$. *Let* $n \geq 1$ *and* $k = 0, \ldots, n - 1$. *We have*

(a) $|E(B_r(B_t - B_s))| \leq (t - s)^{2H}$ *for any* $r \in [0, 1]$ *and* $0 \leq s < t \leq 1$.

(b) $|\langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}}| \leq n^{-2H}$ *for any* $t \in [0, 1]$.

(c) $\sup_{t \in [0,1]} \sum_{k=0}^{n-1} |\langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}}| = O(1)$ *as* n *tends to infinity.*

(d) *For any integer* $q \geq 2$,

$$\sum_{k=0}^{n-1} \left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}} \right| = O\left(n^{-2H(q-1)} \right) \quad \text{as } n \text{ tends to infinity.} \quad (5.10)$$

(e) *Recall the definition* (5.2) *of* ρ_H. *We have*

$$\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = n^{-2H} \rho_H(k-j).$$

Consequently, for any integer $q \geq 1$, *we can write*

$$\sum_{k,j=0}^{n-1} |\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}}|^q = O(n^{1-2qH}) \quad \text{as } n \text{ tends to infinity.} \tag{5.11}$$

Proof We have

$$E(B_r(B_t - B_s)) = \frac{1}{2}(r^{2H} + t^{2H} - |t-r|^{2H}) - \frac{1}{2}(r^{2H} + s^{2H} - |s-r|^{2H})$$

$$= \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}(|s-r|^{2H} - |t-r|^{2H}).$$

Using the inequality $|b^{2H} - a^{2H}| \leq |b-a|^{2H}$ for any $a, b \in [0,1]$, we deduce (a).
Property (b) is an immediate consequence of (a). To show property (c), we use

$$\langle \varepsilon_t, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}}[(k+1)^{2H} - k^{2H} - |k+1-nt|^{2H} + |k-nt|^{2H}].$$

Property (d) follows from

$$\langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}} = \frac{1}{2n^{2H}}[(k+1)^{2H} - k^{2H} - 1]$$

and

$$\left| \langle \varepsilon_{k/n}, \partial_{k/n} \rangle_{\mathfrak{H}}^q - \frac{(-1)^q}{2^q n^{2qH}} \right| = \frac{1}{2^q n^{2qH}} |[(k+1)^{2H} - k^{2H} - 1]^q - (-1)^q|$$

$$= \frac{1}{2^q n^{2qH}} \sum_{i=1}^{q} \binom{q}{i} [(k+1)^{2H} - k^{2H}]^i$$

$$\leq \frac{1}{2^q n^{2qH}} [(k+1)^{2H} - k^{2H}] \sum_{i=1}^{q} \binom{q}{i}.$$

Finally, property (e) follows from

$$\sum_{k,j=0}^{n-1} |\langle \partial_{j/n}, \partial_{k/n} \rangle_{\mathfrak{H}}|^q \leq n^{-2qH} \sum_{k,j=0}^{n-1} |\rho_H(j-k)|^q$$

$$\leq n^{1-2qH} \sum_{r \in \mathbb{Z}} |\rho_H(r)|^q. \qquad \square$$

5.3 An Auxiliary Convergence Result

From now on, we fix $q \geq 2$ and make use of the following hypothesis on $f : \mathbb{R} \to \mathbb{R}$:

(H) f belongs to $\mathcal{C}^{2q}$ and, for any $p \geq 2$ and $i = 0, \ldots, 2q$,

$$E\left(\sup_{t \in [0,1]} \left| f^{(i)}(B_t) \right|^p \right) < \infty. \tag{5.12}$$

Notice that a sufficient condition for (5.12) to hold is that f satisfies an exponential growth condition of the form $|f^{(2q)}(x)| \leq k e^{c|x|^p}$ for some constants c, $k > 0$, and $0 < p < 2$.

The aim of this section is to prove the following auxiliary convergence result.

Theorem 5.2 *Suppose $H \in (\frac{1}{4q}, \frac{1}{2})$, and let f be a function satisfying Hypothesis* **(H)**. *Consider the sequence of q-parameter step processes defined by*

$$u_n = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \partial_{k/n}^{\otimes q}. \tag{5.13}$$

Then $u_n \in \mathrm{Dom}\,\delta^q$, and $\delta^q(u_n)$ converges stably to $\sigma_{H,q} \int_0^1 f(B_s)\, dW_s$, where W is a Brownian motion independent of B, and $\sigma_{H,q} > 0$ is defined in (5.8).

Proof The fact that u_n belongs to $\mathrm{Dom}\,\delta^q$ is a consequence of the inclusion $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q}) \subset \mathrm{Dom}\,\delta^q$ and hypothesis **(H)**. We are now going to show that the sequence $F_n = \delta^q(u_n)$ satisfies the conditions of Theorem 3.1. We make use of the notation

$$\alpha_{k,j} = \langle \varepsilon_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}}, \quad \beta_{k,j} = \langle \partial_{k/n}, \partial_{j/n} \rangle_{\mathfrak{H}}, \tag{5.14}$$

for $k, j = 0, \ldots, n-1$ and $n \geq 1$. Also, C will denote a generic constant.

Step 1. Let us show first that F_n is bounded in $L^2(\Omega)$. Taking into account the continuity of the Skorokhod integral from the space $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$ into $L^2(\Omega)$ (see (2.9)), it suffices to show that u_n is bounded in $\mathbb{D}^{q,2}(\mathfrak{H}^{\otimes q})$. Actually we are going to show that u_n is bounded in $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ for any integer $k \leq 2q$ and any real number $p \geq 2$. Using estimate (5.11), we obtain

$$\|u_n\|_{\mathfrak{H}^{\otimes q}}^2 = n^{2qH - 1} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \beta_{k,j}^q \leq C \sup_{0 \leq t \leq 1} \left| f(B_t) \right|^2.$$

Moreover, for any integer $k \geq 1$,

$$D^k u_n = n^{qH - \frac{1}{2}} \sum_{j=0}^{n-1} f^{(k)}(B_{j/n}) \varepsilon_{j/n}^{\otimes k} \otimes \partial_{j/n}^{\otimes q},$$

and we obtain in the same way

$$\left\| D^k u_n \right\|_{\mathfrak{H}^{\otimes(q+k)}}^2 = n^{2qH-1} \sum_{l,j=0}^{n-1} f^{(k)}(B_{l/n}) f^{(k)}(B_{j/n}) \langle \varepsilon_{l/n}, \varepsilon_{j/n} \rangle^k \beta_{l,j}^q$$

$$\leq C \sup_{0 \leq t \leq 1} \left| f^{(k)}(B_t) \right|^2.$$

Then the result follows from hypothesis (**H**).

Step 2. Let us show condition (i) of Theorem 3.1. Fix some integers $r, k_1, \ldots, k_{q-1}$ ≥ 0 such that $k_1 + 2k_2 + \cdots + (q-1)k_{q-1} + r = q$. Let $h \in \mathfrak{H}^{\otimes r}$. We claim that $\langle u_n, (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes h \rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$. Suppose first that $r \geq 1$. Without loss of generality, we can assume that h has the form $g \otimes \varepsilon_t$ with $g \in \mathfrak{H}^{\otimes(r-1)}$. Set $\Phi_n = (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1}F_n)^{\otimes k_{q-1}} \otimes g$. Then we can write

$$\langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \rangle_{\mathfrak{H}^{\otimes(q-1)}} \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}}.$$

As a consequence,

$$E\left(\left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right|\right) \leq n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} E\left(\left| f(B_{k/n}) \langle \partial_{k/n}^{\otimes(q-1)}, \Phi_n \rangle_{\mathfrak{H}^{\otimes(q-1)}} \right|\right) \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}} \right|.$$

Condition (c) of Lemma 5.1 implies

$$\sum_{k=0}^{n-1} \left| \langle \partial_{k/n}, \varepsilon_t \rangle_{\mathfrak{H}} \right| \leq C.$$

Hence,

$$E\left(\left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right|\right) \leq C n^{H-\frac{1}{2}} \left(E\left(\left\| \Phi_n \right\|_{\mathfrak{H}^{\otimes(q-1)}}^2\right)\right)^{\frac{1}{2}}.$$

On the other hand,

$$\left\| \Phi_n \right\|_{\mathfrak{H}^{\otimes(q-1)}}^2 = \left\| g \right\|_{\mathfrak{H}^{\otimes(r-1)}}^2 \prod_{m=1}^{q-1} \left\| D^m F_n \right\|_{\mathfrak{H}^{\otimes m}}^{2k_m},$$

and applying the generalized Hölder inequality, we have

$$E\left(\left\| \Phi_n \right\|_{\mathfrak{H}^{\otimes(q-1)}}^2\right) \leq C \prod_{m=1}^{q-1} \left(E\left(\left\| D^m F_n \right\|_{\mathfrak{H}^{\otimes m}}^{2k_m(q-1)}\right)\right)^{\frac{1}{q-1}}$$

$$= C \prod_{m=1}^{q-1} \left\| D^m F_n \right\|_{L^{2k_m(q-1)}(\Omega; \mathfrak{H}^{\otimes m})}^{2k_m}.$$

By Meyer's inequalities (2.9), for any $1 \leq m \leq q - 1$ and any $p \geq 2$, we obtain, using Step 1, that

$$
\left\| D^m F_n \right\|_{L^p(\Omega; \mathfrak{H}^{\otimes m})} = \left\| D^m \delta^q(u_n) \right\|_{L^p(\Omega; \mathfrak{H}^{\otimes m})}
$$
$$
\leq \left\| \delta^q(u_n) \right\|_{\mathbb{D}^{m,p}} \leq C \|u_n\|_{\mathbb{D}^{m+q,p}(\mathfrak{H}^{\otimes q})} \leq C.
$$

Therefore,

$$
E\left(\left| \langle u_n, \Phi_n \otimes \varepsilon_t \rangle_{\mathfrak{H}^{\otimes q}} \right| \right) \leq C n^{H - \frac{1}{2}},
$$

which converges to zero as n tends to infinity because $H < \frac{1}{2}$.

Suppose now that $r = 0$. In this case, we have $\Phi_n = (DF_n)^{\otimes k_1} \otimes \cdots \otimes (D^{q-1} F_n)^{\otimes k_{q-1}}$. Then

$$
\langle \partial_{j/n}^{\otimes q}, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = \langle \partial_{j/n}, DF_n \rangle_{\mathfrak{H}}^{k_1} \cdots \langle \partial_{j/n}^{\otimes (q-1)}, D^{q-1} F_n \rangle_{\mathfrak{H}^{\otimes(q-1)}}^{k_{q-1}}. \tag{5.15}
$$

From (5.15) and (5.13) we obtain

$$
\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}} = n^{qH - \frac{1}{2}} \sum_{k=0}^{n-1} f(B_{k/n}) \prod_{m=1}^{q-1} \langle \partial_{j/n}^{\otimes m}, D^m F_n \rangle_{\mathfrak{H}^{\otimes m}}^{k_m}. \tag{5.16}
$$

Notice that for any $m = 1, \dots, q - 1$, the term $\langle \partial_{j/n}^{\otimes m}, D^m F_n \rangle_{\mathfrak{H}^{\otimes m}}$ can be estimated by $n^{-mH} \| D^m F_n \|_{\mathfrak{H}^{\otimes m}}$. Then, taking into account that

$$
\sup_n E\left(\| D^m F_n \|_{\mathfrak{H}^{\otimes m}}^p \right) < \infty
$$

for any $p \geq 2$ and that $\sum_{m=1}^{q-1} m k_m = q$, we obtain for $E(|\langle u_n, \Phi_n \rangle_{\mathfrak{H}^{\otimes q}}|)$ an estimate of the form $C\sqrt{n}$, which is unfortunately not satisfactory. For this reason, a finer analysis of the terms $\langle \partial_{j/n}^{\otimes m}, D^m F_n \rangle_{\mathfrak{H}^{\otimes m}}$ is required.

First, we are going to apply formula (2.11) to compute the derivative $D^m F_n$, $m = 1, \dots, q - 1$:

$$
D^m F_n = \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \, \delta^{q-i}\left(D^{m-i} u_n \right)
$$
$$
= n^{qH - \frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \left(\varepsilon_{l/n}^{\otimes(m-i)} \otimes \partial_{l/n}^{\otimes i} \right) \delta^{q-i}\left(f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right). \tag{5.17}
$$

Set $\Psi_n^{m,j} = \langle \partial_{j/n}^{\otimes m}, D^m F_n \rangle_{\mathfrak{H}^{\otimes m}}$, and recall the definition of $\alpha_{k,j}$ and $\beta_{k,j}$ from (5.14). From (5.17) we obtain

$$
\Psi_n^{m,j} = n^{qH - \frac{1}{2}} \sum_{i=0}^{m} \binom{m}{i} \binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^i \, \delta^{q-i}\left(f^{(m-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right)
$$
$$
= \sum_{i=0}^{m} \Phi_n^{i,m,j} \tag{5.18}
$$

with

$$\Phi_n^{i,m,j} = n^{qH-\frac{1}{2}} \binom{m}{i}\binom{q}{i} i! \sum_{l=0}^{n-1} \alpha_{l,j}^{m-i} \beta_{l,j}^i \delta^{q-i}\left(f^{(m-i)}(B_{l/n})\partial_{l/n}^{\otimes(q-i)}\right).$$

By Meyer inequalities (2.9) we obtain, using also assumption **(H)**, that, for any $p \geq 2$,

$$\left\|\delta^{q-i}\left(f^{(m-i)}(B_{l/n})\partial_{l/n}^{\otimes(q-i)}\right)\right\|_{L^p} \leq C\left\|f^{(m-i)}(B_{l/n})\partial_{l/n}^{\otimes(q-i)}\right\|_{\mathbb{D}^{q-i,p}(\mathfrak{H}^{\otimes q-i})}$$

$$\leq Cn^{-(q-i)H}. \tag{5.19}$$

Using Lemma 5.1 (b) and (e), we have $|\alpha_{l,j}^{m-i}| \leq Cn^{-(m-i)2H}$ and $\sum_{l=0}^{n-1}|\beta_{l,j}^i| \leq Cn^{-2iH}$. Therefore, for any $i \geq 1$, we have

$$\left\|\Phi_n^{i,m,j}\right\|_{L^p} \leq Cn^{iH-\frac{1}{2}}\sum_{l=0}^{n-1}\left|\alpha_{l,j}^{m-i}\beta_{l,j}^i\right| \leq Cn^{-\frac{1}{2}-2mH+iH}. \tag{5.20}$$

On the other hand, if $i = 0$, Lemma 5.1(c) and (5.19) yield

$$\left\|\Phi_n^{0,m,j}\right\|_{L^p} \leq Cn^{-\frac{1}{2}-2mH+2H}. \tag{5.21}$$

Notice that the estimate for the $L^p(\Omega)$-norm of $\Phi_n^{0,m,j}$ in the case $i = 0$ is worse than for $i \geq 1$. We will see later that, for $p = 2$, we can get a better estimate for $\Phi_n^{0,m,j}$.

Because $\sum_{m=1}^{q-1} k_m \geq 2$, the number of factors in $\prod_{m=1}^{q-1}\langle\partial_{j/n}, D^m F_n\rangle_{\mathfrak{H}^{\otimes m}}^{k_m}$ is at least two. As a consequence, we can write

$$\langle u_n, \Phi_n\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}}\sum_{j=0}^{n-1} f(B_{j/n})\Psi_n^{\mu,j}\Psi_n^{\nu,j}\Theta_n^j$$

for some μ, ν (not necessarily distinct), where

$$\Theta_n^j = \left(\Psi_n^{\mu,j}\right)^{k_\mu-1}\left(\Psi_n^{\nu,j}\right)^{k_\nu-1}\prod_{\substack{m=1\\m\neq\mu,\nu}}^{q-1}\left(\Psi_n^{m,j}\right)^{k_m}. \tag{5.22}$$

Consider the decomposition

$$\langle u_n, \Phi_n\rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n,$$

where

$$A_n = n^{qH-\frac{1}{2}}\sum_{j=0}^{n-1} f(B_{j/n})\left(\sum_{i=0}^{\mu}\sum_{k=0}^{\nu}\mathbf{1}_{i+k\geq 1}\Phi_n^{i,\mu,j}\Phi_n^{k,\nu,j}\right)\Theta_n^j,$$

$$B_n = n^{qH-\frac{1}{2}}\sum_{j=0}^{n-1} f(B_{j/n})\Phi_n^{0,\mu,j}\Phi_n^{0,\nu,j}\Theta_n^j.$$

From (5.22) and the estimate $\|\Psi_n^{m,j}\|_{L^p} \leq Cn^{-mH}$, for all $p \geq 2$ and $1 \leq m \leq q$, we obtain

$$\|\Theta_n^j\|_{L^p} \leq Cn^{-H(q-\mu-\nu)}. \tag{5.23}$$

Then, from (5.20), (5.21), and (5.23) we obtain

$$E(|A_n|) \leq Cn^{qH+\frac{1}{2}}n^{-H(q-\mu-\nu)}\left(\sum_{i=1}^{\mu}\sum_{k=1}^{\nu}n^{-1-2(\mu+\nu)H+(i+k)H}\right.$$

$$\left.+\sum_{i=1}^{\mu}n^{-1-2(\mu+\nu)H+iH+2H}+\sum_{k=1}^{\nu}n^{-1-2(\mu+\nu)H+kH+2H}\right)$$

$$= Cn^{-\frac{1}{2}}+n^{-\frac{1}{2}+2H-\mu H}+n^{-\frac{1}{2}+2H-\nu H},$$

which converges to zero as n tends to infinity, because $\mu, \nu \geq 1$ and $H < \frac{1}{2}$.

For the term B_n, using again estimates (5.21) and (5.23), we get

$$E(|B_n|) \leq Cn^{qH+\frac{1}{2}-H(q-\mu-\nu)-1-2H(\mu+\nu)+4H} = Cn^{-\frac{1}{2}-H(\mu+\nu)+4H}$$

$$\leq Cn^{-\frac{1}{2}+2H},$$

which converges to zero as n tends to infinity if $H < \frac{1}{4}$. To handle the case $H \in [\frac{1}{4}, \frac{1}{2})$, we need more precise estimates for the $L^2(\Omega)$-norm of $\Phi_n^{0,\nu,j}$. We have, using formula (2.12),

$$E[(\Phi_n^{0,\nu,j})^2] = \binom{q}{i}^2\binom{m}{i}^2 i!^2 E\left(\left|n^{qH-\frac{1}{2}}\sum_{l=0}^{n-1}\alpha_{l,j}^{\nu}\delta^q\left(f^{(\nu)}(B_{l/n})\partial_{l/n}^{\otimes q}\right)\right|^2\right)$$

$$= n^{2qH-1}\binom{q}{i}^2\binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1}\alpha_{l,j}^{\nu}\alpha_{l',j}^{\nu}$$

$$\times E\left(\delta^q\left(f^{(\nu)}(B_{l/n})\partial_{l/n}^{\otimes q}\right)\delta^q\left(f^{(\nu)}(B_{l'/n})\partial_{l'/n}^{\otimes q}\right)\right)$$

$$= n^{2qH-1}\binom{q}{i}^2\binom{m}{i}^2 i!^2 \sum_{l,l'=0}^{n-1}\alpha_{l,j}^{\nu}\alpha_{l',j}^{\nu}\sum_{i=0}^{q}\binom{q}{i}^2 i!\alpha_{l,l'}^{q-i}\alpha_{l',l}^{q-i}\beta_{l,l'}^{2i}$$

$$\times E\left(f^{(\nu+q-i)}(B_{l/n})\,f^{(\nu+q-i)}(B_{l'/n})\right)$$

$$= \sum_{i=0}^{q}R_n^i.$$

If $i \geq 1$, then $\sum_{l,l'=0}^{n-1}\beta_{l,l'}^{2i} \leq Cn^{1-4iH}$, and we obtain an estimate of the form $\|R_n^i\|_{L^2} \leq Cn^\gamma$, where

$$\gamma = \frac{1}{2}(2qH-1-4\nu H-4(q-i)H+1-4iH) = -qH-2\nu H.$$

For $i = 0$, then $\sup_n \sum_{l,l'=0}^{n-1} |\alpha_{l,l'}\alpha_{l',l}| < \infty$, and we get

$$\gamma = \frac{1}{2}\big(2qH - 1 - 2H(2\nu + 2q - 2)\big) = -qH - 2\nu H - \frac{1}{2} + 2H.$$

We have obtained the estimate

$$\big\|\Phi_n^{0,\nu,j}\big\|_{L^2} \leq Cn^{-qH-2\nu H+2H-\frac{1}{2}}. \tag{5.24}$$

Fix $\frac{1}{4qH} < \alpha < 1$. This choice is possible because $\frac{1}{4qH} < 1$. We have, by Hölder's inequality,

$$E\big(|B_n|\big) \leq Cn^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} \big\|\Phi_n^{0,\mu,j}\big\|_{L^2}^\alpha \big\|\Phi_n^{0,\nu,j}\big\|_{L^2}^\alpha \big\|\big|\Phi_n^{0,\mu,j}\Phi_n^{0,\nu,j}\big|^{1-\alpha}\Theta_n^j\big\|_{L^{\frac{1}{1-\alpha}}}.$$

Using (5.24), (5.21), and (5.23), we obtain

$$E\big(|B_n|\big) \leq Cn^\gamma, \tag{5.25}$$

where

$$\begin{aligned}
\gamma &= qH + \frac{1}{2} + \big[-2qH - 2(\mu + \nu)H + 4H - 1\big]\alpha \\
&\quad - H(q - \mu - \nu) + (1-\alpha)\big(-1 - 2H(\mu + \nu) + 4H\big) \\
&= -\frac{1}{2} + 4H - H(\mu + \nu) - 2\alpha qH \\
&\leq -\frac{1}{2} + 2H - 2\alpha qH \leq \frac{1}{2} - 2\alpha qH < 0,
\end{aligned}$$

because $H < \frac{1}{2}$. Therefore, $E(|B_n|)$ converges to zero as n tends to infinity.

Step 3. Let us show condition (ii). We have

$$\big\langle u_n, D^q F_n \big\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{j=0}^{n-1} f(B_{j/n})\big\langle \partial_{j/n}^{\otimes q}, D^q F_n \big\rangle_{\mathfrak{H}^{\otimes q}}.$$

From (5.18) we get

$$\big\langle \partial_{j/n}^{\otimes q}, D^q F_n \big\rangle_{\mathfrak{H}^{\otimes q}} = n^{qH-\frac{1}{2}} \sum_{i=0}^{q} \binom{q}{i}^2 i! \sum_{l=0}^{n-1} \alpha_{l,j}^{q-i}\beta_{l,j}^i \delta^{q-i}\big(f^{(q-i)}(B_{l/n})\partial_{l/n}^{\otimes(q-i)}\big).$$

Therefore, we can make the decomposition

$$\big\langle u_n, D^q F_n \big\rangle_{\mathfrak{H}^{\otimes q}} = A_n + B_n + C_n,$$

where

$$A_n = n^{2qH-1}q! \sum_{l,j=0}^{n-1} \beta_{l,j}^q f(B_{l/n})f(B_{j/n}),$$

$$B_n = n^{2qH-1} \sum_{i=1}^{q-1} \binom{q}{i}^2 i! \sum_{l,j=0}^{n-1} \alpha_{l,j}^{q-i} \beta_{l,j}^i f(B_{j/n}) \delta^{q-i} \left(f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right),$$

$$C_n = n^{2qH-1} \sum_{l,j=0}^{n-1} \alpha_{l,j}^q f(B_{j/n}) \delta^q \left(f^{(q)}(B_{l/n}) \partial_{l/n}^{\otimes(q)} \right).$$

The term A_n converges to a nonnegative square-integrable random variable. Indeed,

$$A_n = \frac{q!}{2^q n} \sum_{k,j=0}^{n-1} f(B_{k/n}) f(B_{j/n}) \left(|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H} \right)^q$$

$$= \frac{q!}{2^q n} \sum_{p=-\infty}^{\infty} \sum_{j=0\vee -p}^{(n-1)\wedge(n-1-p)} f(B_{j/n}) f(B_{(j+p)/n}) \left(|p+1|^{2H} \right.$$

$$\left. + |p-1|^{2H} - 2|p|^{2H} \right)^q,$$

which converges in $L^1(\Omega)$ to

$$q! \left(\sum_{k \in \mathbb{Z}} \rho_H(k)^q \right) \int_0^1 f(B_s)^2 \, ds.$$

Then, it suffices to show that the terms B_n and C_n converge to zero in $L^2(\Omega)$. For the term B_n, we can write, using the fact that $\sum_{l,j=0}^{n-1} |\alpha_{l,j}^{q-i} \beta_{l,j}^i| \leq C n^{-2qH+1}$,

$$E(|B_n|) \leq C n^{2qH-1} \sum_{i=1}^{q-1} \sum_{l,j=0}^{n-1} |\alpha_{l,j}^{q-i} \beta_{l,j}^i| \left\| \delta^{q-i} \left(f^{(q-i)}(B_{l/n}) \partial_{l/n}^{\otimes(q-i)} \right) \right\|_{L^2}$$

$$\leq C \sum_{i=1}^{q-1} n^{-H(q-i)},$$

which converges to zero as n tends to infinity. Finally, for the term C_n, we can write

$$E(|C_n|) \leq C n^{qH+\frac{1}{2}} \sup_j \left\| \Phi_n^{0,q,j} \right\|_{L^2} \leq C n^{\frac{1}{2}-2qH+(2H-\frac{1}{2})\vee 0}$$

and $\frac{1}{2} - 2qH + (2H - \frac{1}{2}) \vee 0 < 0$, because if $2H - \frac{1}{2} \leq 0$, this is true due to $\frac{1}{2} - 2qH < 0$, and if $2H - \frac{1}{2} \geq 0$, then we get $2H(1-q) < 0$. This completes the proof of Theorem 5.2. □

5.4 Proof of the Stable Convergence (5.9)

As a consequence of Theorem 5.2, we can derive the following result, which is nothing but (5.9):

Theorem 5.3 *Suppose that f is a function satisfying Hypothesis* (**H**). *Let G_n be the sequence of random variables defined in* (5.5). *Then, provided $H \in (\frac{1}{4q}, \frac{1}{2})$, we have*

$$G_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \overset{\text{stably}}{\longrightarrow} \sigma_{H,q} \int_0^1 f(B_s)\, dW_s,$$

where W is a Brownian motion independent of B, and $\sigma_{H,q} > 0$ is defined by (5.8).

Proof We recall first that $H_q(n^H(\Delta B_{k/n})) = \frac{1}{q!} n^{qH} \delta^q(\partial_{k/n}^{\otimes q})$. Then, using (2.6) yields

$$f(B_{k/n}) \delta^q(\partial_{k/n}^{\otimes q}) = \sum_{r=0}^{q} \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}\big(f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)}\big),$$

where $\alpha_{k,k}$ is defined in (5.14). As a consequence,

$$G_n = \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{r=0}^{q} \sum_{k=0}^{n-1} \binom{q}{r} \alpha_{k,k}^r \delta^{q-r}\big(f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)}\big)$$

$$= \frac{1}{q!} \delta^q(u_n) + \sum_{r=1}^{q-1} \delta^{q-r}(v_n^{(r)}) + R_n,$$

where u_n is defined in (5.13),

$$v_n^{(r)} = \frac{1}{q!} \binom{q}{r} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^r\, f^{(r)}(B_{k/n}) \partial_{k/n}^{\otimes(q-r)},$$

and

$$R_n = \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^q f^{(q)}(B_{k/n}).$$

The proof will be done in two steps.

Step 1. We first show that if $H \in (0, \frac{1}{2})$, then, for $r = 1, \ldots, q-1$, $\delta^{q-r}(v_n^{(r)})$ converges to zero in $L^2(\Omega)$ as n tends to infinity. It suffices to show that $v_n^{(r)}$ converges to zero in the norm of the space $\mathbb{D}^{q-r,2}(\mathfrak{H}^{\otimes(q-r)})$. For $0 \le m \le q-r$, we can write, using the notation $\beta_{k,l}$ defined by (5.14),

$$E\big(\|D^m v_n^{(r)}\|_{\mathfrak{H}^{\otimes(q-r+m)}}^2\big)$$

$$= \left(\frac{1}{q!}\binom{q}{r}\right)^2 n^{2qH-1} \sum_{k,l=0}^{n-1} E\big(f^{(r+m)}(B_{k/n}) f^{(r+m)}(B_{l/n})\big) \alpha_{k,k}^r \alpha_{l,l}^r \alpha_{k,l}^m \beta_{k,l}^{q-r}$$

$$\le C n^{2qH-1} n^{-2H(2r-2+m+q-r)}$$

$$= C n^{2H-1-2Hm},$$

which converges to zero as n tends to infinity.

Step 2. To complete the proof, it suffices to check that

$$R_n - n^{-\frac{1}{2}-qH} \frac{(-1)^q}{2^q q!} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n})$$

converges to zero in $L^2(\Omega)$ as n tends to infinity. This follows from (5.10) and the estimates

$$\left\| \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_{k,k}^q f^{(q)}(B_{k/n}) - \frac{(-1)^q}{2^q q!} n^{-\frac{1}{2}-qH} \sum_{k=0}^{n-1} f^{(q)}(B_{k/n}) \right\|_{L^2}$$

$$\leq C n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \left| \alpha_{k,k}^q - \frac{1}{2^q n^{2qH}} \right| \leq C \, n^{-qH+2H-\frac{1}{2}}.$$

Notice that $-qH + 2H - \frac{1}{2} < 0$. The proof is now complete. $\qquad\square$

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