Donsker theorem in Wassertein 1 distance

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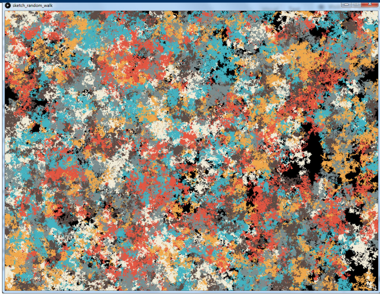
2019

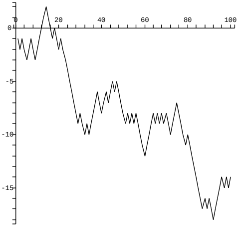


Random walk

Random walk

$$X^m(t) = \frac{1}{\sqrt{m}} \left(\sum_{j=1}^{[mt]} X_j + (mt - [mt])\, X_{[mt+1]} \right)$$

where $(X_j, j \geq 1)$ are i.i.d. such that $\mathbf{E}[X_1] = 0,\ \mathbf{E}\left[|X_1|^2\right] = 1$



Prohorov vs Kolmogorov-Rubinstein

Definition (Prohorov)

$$\mathbf{P}_m \xrightarrow{\text{Pro}(E)} \mathbf{P} \Longleftrightarrow \left(\int F \, \mathrm{d}\mathbf{P}_m \to \int F \, \mathrm{d}\mathbf{P} \text{ for all } F \in \mathfrak{C}^0(E) \right)$$

Definition (Kolmogorov-Rubinstein or Wasserstein 1)

$$\mathbf{P}_m \xrightarrow{\text{KR}(E)} \mathbf{P} \Longleftrightarrow \left(\sup_{F \in \mathfrak{C}^1(E)} \left(\int F \, \mathrm{d}\mathbf{P}_m - \int F \, \mathrm{d}\mathbf{P} \right) \to 0 \right)$$

where

- $\mathfrak{C}^0(E)$: bounded and continuous $E \to \mathbf{R}$
- $\mathfrak{C}^1(E)$: bounded and 1-Lipschitz continuous $E \to \mathbf{R}$

Donsker Theorem (1951)

Theorem

If $\mathbf{E}\left[|X_1|^2\right] < \infty$,

$$\mathbf{E}\left[F(X^m)\right] \xrightarrow{m\to\infty} \mathbf{E}\left[F(B)\right]$$

for all $F \in \mathfrak{C}^0(\mathcal{C})$.

Lamperti improvement (1961)

Theorem

$$\mathbf{E}\left[|X_1|^{2p}\right] < \infty \Longrightarrow \mathbf{E}\left[F(X^m)\right] \xrightarrow{m\to\infty} \mathbf{E}\left[F(B)\right]$$

for $F \in \mathfrak{C}^0(H_\alpha)$ *where*

$$\alpha < \frac{p-1}{2p}$$

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$$\alpha < \frac{p-1}{2p}$$

$$\mathbf{E}\left[|X_1|^{3}\right] < \infty \Longrightarrow \alpha < 1/6$$

$$p \to \infty \Longrightarrow \alpha \to 1/2$$

Choice of the functional space $\mathcal{F}$

Generalized problem

$$\sup_{F \in \mathcal{F}} \mathbf{E}\left[F(X^m)\right] - \mathbf{E}\left[F(B)\right]$$

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- H_α : α-Hölder continuous functions

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- Fractional Sobolev spaces

$$W_{\alpha,p} = \left\{ f,\ \iint_{[0,1]^2} \frac{|f(t)-f(s)|^p}{|t-s|^{1+\alpha p}}\,\mathrm{d}s\,\mathrm{d}t < \infty \right\}$$

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- Embeddings

$$H_{\alpha'} \subset W_{\alpha,p} \subset H_{\alpha-1/p}$$

for $\alpha' > \alpha > 1/p$

Stein's method and Donsker theorem

Barbour '90 Functional space : 3-times Fréchet differentiable functions on the Skorohod space, rate $n^{-1/2} \log n$

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Coutin and Laurent Decreusefond'18 following Shih 3-times weakly differentiable functions on $W_{\alpha,p}$ for $0 < \alpha - 1/p < 1/2$, rate $n^{\alpha-1/2}\log n$

Why “Lipschitz” is that important ?

Lipschitz functions of the sample-paths

- Local time, reflected process

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- CLT for Lipschitz test functions in dimension 1 immediate
- CLT in dimension d Gallouët, Mijoule, and Swan; Raič; Fang, Shao, and Xu (2017-2018)
- Rate $n^{-1/2} \log n$

Rate of convergence

Theorem (Coutin-D)

For $p \geq 3$ *and* $0 < \alpha - 1/p < 1/2$

$$\left|\mathbf{E}\left[F(X^m)\right] - \mathbf{E}\left[F(B)\right]\right| \leq c \, \|F\|_{\mathrm{Lip}_1(W_{\alpha,p})} \, \|X_1\|_{L^p} \; m^{-1/6+\alpha/3} \log m$$

end

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For $\alpha = 0, p = \infty$, *we set* $W_{\alpha,p} = \mathcal{C}$

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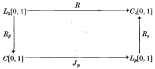
Abstract Wiener space

Theorem (Pietsch)

The triple $(I_{1,2},\ W_{\alpha,p}, \mathbf{P}_B)$ *is an abstract Wiener space for any admissible* (α, p) *where*

$$I_{1,2} = \left\{ f,\, \exists \dot{f} \in L^2, f(t) = \int_0^t \dot{f}(s)\, ds \right\} \subset W_{\alpha,p}$$



$$R: f(t) \to g(s) = \int_0^s f(t)\, dt.$$

25.6.3. According to 22.7.4 and 22.7.6 the following result is a special case of 22.4.13.

Proposition. *If* $2 < p < \infty$ *and* $0 < \lambda < 1/2 - 1/p$, *then*

$$R \in \mathfrak{P}_p(L_2[0,1], C_\lambda[0,1]).$$

Proof. Put $\alpha := 1/p + \lambda$ and $\beta := 1 - 1/p - \lambda$. Because of $R_{\alpha+\beta} = R_\alpha R_\beta$ and $\beta > 1/2$ we get the diagram

The assertion now follows from 17.3.5.

25.6.4. The *Brownian motion* can be described by the so-called *cylindrical Wiener probability* $\varrho := R(\gamma_{L_2[0,1]})$ defined on $C_0[0,1] := \{f \in C[0,1] : f(0) = 0\}$.

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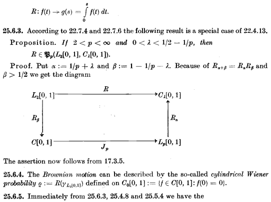
Theorem (Itô-Nisio)

Let $(g_n, n \geq 1)$ *be a CONB of* $I_{1,2}$*,* X_n *i.i.d.* $\sim \mathcal{N}(0,1)$

$$\sum_{n=1}^{N} X_n\, g_n \xrightarrow[\text{with prob. } 1]{\text{in } W_{\alpha,p}} B := \sum_{n=1}^{\infty} X_n\, g_n$$



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$$\begin{array}{ccc} L_2[0,1] & \xrightarrow{R} & C_\lambda[0,1] \\ \downarrow R_\beta & & \uparrow R_\alpha \\ C[0,1] & \xrightarrow[J_p]{} & L_p[0,1] \end{array}$$

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Dirichlet structure on $W_{\alpha,p}$

Ornstein-Uhlenbeck semi-group For $F : W_{\alpha,p} \to \mathbf{R}$, $x \in W_{\alpha,p}$

$$P_t F(x) = \mathbf{E}\left[F(e^{-t}x + \underbrace{\sqrt{1-e^{-2t}}}_{\beta_t} B) \right]$$

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Properties

- Stationarity

$$x \sim B \iff X_t(x) \sim B$$

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Generator, see Shih For $F \in \mathrm{Lip}_1(W_{\alpha,p})$

$$LF(x) = -\langle x, \nabla F(x)\rangle_{W_{\alpha,p},W^*_{\alpha,p}} + \sum_{j=1}^{\infty} \langle \nabla^{(2}F(x), g_j \otimes g_j\rangle_{I_{1,2}}$$

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An intermediate process

$$X^m(t) = \sum_{j=0}^{m-1} X_j \underbrace{\sqrt{m} \int_0^t \mathbf{1}_{[j/m,\ (j+1)/m[}(s)\, \mathrm{d}s}_{h_j^m(t)}$$

Affine interpolation of B

$$B^m(t) = \sum_{j=0}^{m-1} \underbrace{\sqrt{m}\Big(B(\frac{j+1}{m}) - B(\frac{j}{m})\Big)} \underbrace{\sqrt{m} \int_0^t \mathbf{1}_{[j/m,\ (j+1)/m)}(t)\, \mathrm{d}t}$$

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Remarks

- X^m and B^m share the same *functional* space :

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- Compare X and X^m Sample-paths
- Compare X^m and B^m Stein's method
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Sample-paths bounds

Theorem (Friz-Victoir)

For any $q \geq 1$*, for any* $\alpha < 1/2$*,*

$$\mathbf{E}\left[\|B^m - B\|_{H_\alpha}^q\right]^{1/q} \leq \frac{c}{m^{1/2-\alpha-\epsilon}}$$

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Theorem (Coutin, D.)

For any $q \geq 1$*, for any* $\alpha < 1/2$*,*

$$\mathbf{E}\left[\|X^m - X\|_{H_\alpha}^q\right]^{1/q} \leq \frac{c}{m^{1/2-\alpha}}$$

Stein method in dimension 1

Stein-Dirichlet representation formula

$$\mathbf{E}\left[F(\frac{S_n}{\sqrt{n}})\right] - \int F \, \mathrm{d}\mu = \mathbf{E}\Big[\int_0^\infty L \underbrace{\Big(\int_{\mathbf{R}} F(e^{-t}\frac{S_n}{\sqrt{n}} + \beta_t y) \, \mathrm{d}\mu(y)\Big)}_{P_tF(S_n/\sqrt{n})} \, \mathrm{d}t\Big]$$

where $\beta_t = \sqrt{1-e^{-2t}}$, $S_n = X_1 + \ldots + X_n$ and

$$Lf(x) = -xf'(x) + f''(x)$$

Into the deep

$$\mathbf{E}\left[\frac{S_n}{\sqrt{n}}\,(P_tF)'(\frac{S_n}{\sqrt{n}})\right] = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\left[X_j\,(P_tF)'(\frac{S_n}{\sqrt{n}})\right]$$

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$$=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\Big[X_j\,\underbrace{\Big((P_tF)'(\frac{S_n}{\sqrt{n}})-(P_tF)'(\frac{S_n}{\sqrt{n}}-\frac{X_j}{\sqrt{n}})\Big)}_{A}\Big]$$

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$$\sum_{j=1}^{n}\mathbf{E}\Big[X_j\ \underbrace{\Big((P_tF)'(\frac{S_n}{\sqrt{n}}) - (P_tF)'(\frac{S_n}{\sqrt{n}} - \frac{X_j}{\sqrt{n}})\Big)}_{A}\Big]$$

$$\mathbf{E}\left[X_j\Big((P_tf)'(\frac{S_n}{\sqrt{n}}) - (P_tf)'(\frac{S_n}{\sqrt{n}} - X_j/\sqrt{n})\Big)\right] = \frac{1}{\sqrt{n}}\int_0^1 \mathbf{E}\left[X_j^2(P_tf)''(\frac{S_n}{\sqrt{n}} + rX_j/\sqrt{n})\Big)\right]\,\mathrm{d}r$$

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$$= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{E}\Bigg[X_j\,\underbrace{\Big((P_tF)'(\frac{S_n}{\sqrt{n}}) - (P_tF)'(\frac{S_n}{\sqrt{n}} - \frac{X_j}{\sqrt{n}})\Big)}_{A}\Bigg]$$

$$\mathbf{E}\left[X_j\Big((P_tf)'(\frac{S_n}{\sqrt{n}}) - (P_tf)'(\frac{S_n}{\sqrt{n}} - X_j/\sqrt{n})\Big)\right]$$

$$= \frac{1}{\sqrt{n}}\int_0^1 \mathbf{E}\left[X_j^2(P_tf)''(\frac{S_n}{\sqrt{n}} + rX_j/\sqrt{n})\Big)\right]\,\mathrm{d}r$$

Consequence

Since

$$\int_0^1 \mathbf{E}\left[X_j^2 (P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}})\right] \, \mathrm{d}r = \mathbf{E}\left[(P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}})\right],$$

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we get

$$\begin{aligned} LP_t f(\frac{S_n}{\sqrt{n}}) \\ &= -\frac{1}{n}\sum_{j=1}^n \int_0^1 \mathbf{E}\left[X_j^2\Big((P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}} + rX_j/\sqrt{n}) - (P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}})\Big)\right] \,\mathrm{d}r \\ &\qquad + \frac{1}{n}\sum_{j=1}^n \mathbf{E}\left[(P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}}) - (P_t f)''(\frac{S_n^{\neg j}}{\sqrt{n}})\right] \end{aligned}$$

Key elements

- The bound depends on the Lipschitz continuity of $(P_t f)''$

$$\sup_{x\neq y} \frac{|(P_t f)''(x) - (P_t f)''(y)|}{|x-y|}$$

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- The $1/\sqrt{n}$ factor

Higher dimension

Finite dimensional Gaussian measure

$$Lf(x) = -x.\nabla f(x) + \frac{1}{2}\Delta f(x) = -x.\nabla f(x) + \frac{1}{2}\operatorname{trace}\nabla^{(2)} f(x)$$

$$P_t f(x) = \int_{\mathbf{R}^m} f(e^{-t}x + \beta_t y)\, \mathrm{d}\mu_m(y)$$

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Remind

$$X^m(t) = \sum_{j=0}^{m-1} X_j \; h_j^m(t)$$

$$B^m(t) = \sum_{j=0}^{m-1} \underbrace{\delta h_j^m}_{\mathcal{N}(0,1)} \; h_j^m(t)$$

Pushforward

Definition (Gaussian measure on $\mathcal{V}_m$)

$$\begin{aligned} S\,:\,\mathbf{R}^m &\longrightarrow \mathcal{V}_m = \operatorname{span}\{h_j^m, 1 \le j \le m\} \subset I_{1,2} \subset W_{\alpha,p} \\ x &\longmapsto \sum_j x_j\, h_j^m \end{aligned}$$

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$$P_t F(\underbrace{\sum x_j h_j^m}_{x}) = \int_{\mathcal{V}_m} F(e^{-t}x + \beta_t y)\, \mathrm{d}\mu_{\mathcal{V}_m}(y)$$

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Definition (Gaussian measure on $\mathcal{V}_m$)

$$\begin{aligned} S\,:\,\mathbf{R}^m &\longrightarrow \mathcal{V}_m = \operatorname{span}\{h_j^m, 1 \le j \le m\} \subset I_{1,2} \subset W_{\alpha,p} \\ x &\longmapsto \sum_j x_j\, h_j^m \\ \mu_m &\longmapsto \mu_{\mathcal{V}_m} = S^{\#}\mu_m \end{aligned}$$

$$P_t F(\underbrace{\sum x_j h_j^m}_{x}) = \int_{\mathcal{V}_m} F(e^{-t}x + \beta_t y)\, \mathrm{d}\mu_{\mathcal{V}_m}(y)$$

$$L_{\mathcal{V}_m} F(x) = -\langle x, \nabla F(x)\rangle_{I_{1,2}} + \frac{1}{2}\operatorname{trace} \nabla^{(2)} F(x)$$

Back to the deep

- We must compute

$$-\sum_{j=1}^{m} \mathbf{E}\left[X\left\langle h_j^m,\ \nabla P_t F(\sum_j X_j h_j^m)\right\rangle_{I_{1,2}}\right]$$

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$$-\sum_{j=1}^{m}\mathbf{E}\left[X_j\left\langle h_j^m,\ \left(\nabla P_tF(\sum_k X_kh_k^m)-\nabla P_tF(\sum_{k\neq j} X_kh_k^m)\right)\right\rangle_{I_{1,2}}\right]$$

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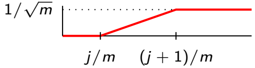
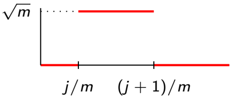
- The first term of the Taylor expansion is

$$\sum_{j=1}^{m} \mathbf{E}\left[X_j^2\left\langle h_j^m \otimes h_j^m,\ \nabla^{(2)} P_t F(\sum_{k\neq j} X_k h_k^m)\right\rangle_{I_{1,2}}\right]$$

Properties of h_j^m



h_j^m



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$$\|h_j^m\|_{\mathcal{C}} = \frac{1}{\sqrt{m}}$$

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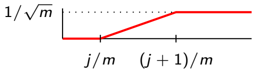

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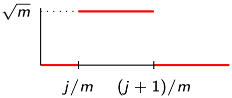

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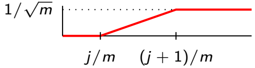

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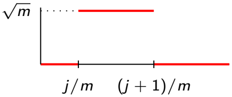

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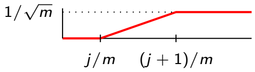

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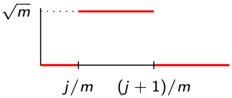

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Problem

No $m^{-1/2}$ factor since we work in $I_{1,2}$!

Representation formula

Theorem (Shih)

$$\left\langle \nabla^{(2)} P_\tau^m f(v),\ h\otimes h \right\rangle_{I_{1,2}{}^{\otimes 2}} = \frac{e^{-3\tau/2}}{\beta_{\tau/2}^2}\, \mathbf{E}\left[f\Big(w_\tau(v, B^m, \hat{B}^m)\Big) \delta h(B^m) \delta h(\hat{B}^m)\right]$$

where

$$w_\tau(v,y,z) = e^{-\tau/2}(e^{-\tau/2}v + \beta_{\tau/2} y) + \beta_{\tau/2} z$$

Projecting, conditionning, averaging

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$$\mathcal{D}_N = \{j/N, j = 0, \cdots, N-1\}$$

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  - B^N with B

The key lemma

$$\mathbf{E}\left[f\circ\pi_N\Big(w_\tau(v,B^m,\hat{B}^m)\Big)\delta h(B^m)\delta h(\hat{B}^m)\right]$$
$$=\mathbf{E}\left[f\Big(w_\tau(v,\pi_N B^m,\pi_N\hat{B}^m)\Big)\mathbf{E}\left[\delta h(B^m)\,|\,\pi_N B^m\right]\mathbf{E}\left[\delta h(\hat{B}^m)\,|\,\pi_N\hat{B}^m\right]\right]$$

Lemma

$$\mathit{var}\left(\mathbf{E}\left[\delta h(B^m)\,|\,\pi_N B^m\right]\right)\le c\,\frac{N}{m}$$

Questions ?

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