Graph Fuzzy Homomorphism Interpreted as Fuzzy Association Graphs

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Abstract

A new generic definition of graph fuzzy morphism is introduced that includes classical graph related problem definitions as subcases. Two practical interpretations as well as some properties are discussed. This definition is a first attempt towards a unified theoretic framework for graph morphism.

1. Introduction

Graph structures have been widely used in pattern recognition to model a group of objects sharing some relations. Examples of involved applications are numerous: character recognition (e.g. [1]), scene interpretation (e.g. [4]), model-based reasoning (e.g. [11]), robot vision (e.g. [12])... The most often addressed problem when using graph structure for pattern recognition is graph or subgraph isomorphism [7]. In many cases, this formulation has been applied successfully in spite of its NP-completeness [6]. But several similar problems call for less restrictive constraints when uncertainty and inaccuracy must be considered [13], or when the two patterns to be matched are known to be distinct [9, 3].

For such problems the literature does not propose a general definition of graph homomorphism (or graph morphism). In most papers, the definition and a matching criteria are gathered to produce an efficient algorithmic framework (e.g. [13, 3, 4, 2]...) that is unfavorable to genericity and comparisons. Moreover, imprecision on nodes or on relationships between nodes (in particular for describing spatial relations [8]), and on the morphism itself lead to propose a definition in the fuzzy set framework.

The original work presented in this paper is the proposal of a unified definition of graph morphism in the fuzzy set framework. The definition can handle classical definitions of graph or subgraph isomorphism, as well as related problems such as bipartite graph matching, graph edition or many structural pattern recognition problems such as labeling, or clustering. Our approach aims at giving a reference framework to compare models and algorithms, and to have a better understanding of the graph morphism problem thanks to a generic and concrete formalization.

Some background definitions about fuzzy relations are given in Section 2. Section 3 presents our generic definition of graph morphism, and its interpretation as a pair of fuzzy relations. This Section also discusses a new valuable emphasized view of the approach brought out by our formalization. The symmetry properties are important characteristics of the graph fuzzy morphism and are explained in Section 4. Section 5 offers a conclusion to this paper.

2. Fuzzy relations and fuzzy graphs

We briefly give in this Section some basic definitions of fuzzy relations.

Let S_1 and S_2 be two sets, and consider two subsets σ_1 and σ_2 of these sets, given by their membership functions: $\sigma_1 : S_1 \rightarrow [0,1]$ and $\sigma_2 : S_2 \rightarrow [0,1]$. The function $\mu : S_1 \times S_2 \rightarrow [0,1]$ is a fuzzy relation on $\sigma_1 \times \sigma_2$ if and only if we have :

$$\forall (x,y) \in S_1 \times S_2, \quad \mu(x,y) \leq \sigma_1(x) \wedge \sigma_2(y), \tag{1}$$

where $\wedge$ represents the minimum operator. This definition and some properties can be found in [10] and [5]. On fuzzy relations we can define an associative composition law called the sup-inf composition. Let μ_i, $i \in \{1,2\}$ be two fuzzy relations on $\sigma_i \times \sigma_{i+1}$, $i \in \{1,2\}$; for $i \in \{1,2\}$, $\mu_i : S_i \times S_{i+1} \rightarrow [0,1]$. The sup-inf composition denoted by $\mu_1 \circ \mu_2$ is defined by: $\forall (u_1, u_3) \in S_1 \times S_3$,

$$(\mu_1 \circ \mu_2)(u_1,u_3) = \sup_{u_2 \in S_2} \{ \mu_1(u_1,u_2) \wedge \mu_2(u_2,u_3) \}. \tag{2}$$

From the definition of a fuzzy relation on fuzzy sets, we can infer two definitions of fuzzy graphs. The first one was introduced by Kaufmann in [5]: a fuzzy graph μ is a function $\mu : S_1 \times S_2 \rightarrow [0, 1]$. This function corresponds to a fuzzy relation where the membership functions σ_1 and σ_2 are constant and equal to 1.

The second definition was introduced by Rosenfeld in [10] and assumes that $S_1 = S_2 = S$: a fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \rightarrow [0, 1]$, $\mu : S \times S \rightarrow [0, 1]$ that fulfills Equation 1. With this definition the notions of symmetry, reflexivity and transitivity can be defined [10] in order to generalize classical graph concepts to such fuzzy graphs.

3. Graph fuzzy morphism

This section introduces a generic definition of graph fuzzy morphism. We show that this definition includes many particular definitions about graph isomorphism, subgraph isomorphism, imperfect graph matching.

3.1 Definition

Let $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ be two graphs where N_1 and N_2 are the node sets, and $E_i \subseteq N_i \times N_i$, $i \in \{1, 2\}$ are the arc sets. To clarify notations all the elements of graph i will have the same subscript i. For instance u_1 belongs to N_1, u_2 to N_2, and (v_1, x_1) to E_1.

A fuzzy morphism (ρ_σ, ρ_μ) between two graphs G_1 and G_2 is a pair of functions $\rho_\sigma : N_1 \times N_2 \rightarrow [0, 1]$ and $\rho_\mu : N_1 \times N_2 \times N_1 \times N_2 \rightarrow [0, 1]$ such that $\forall (u_1, v_1) \in N_1 \times N_1$, $\forall (u_2, v_2) \in N_2 \times N_2$:

$$\rho_\mu(u_1, u_2, v_1, v_2) \leq \rho_\sigma(u_1, u_2) \wedge \rho_\sigma(v_1, v_2). \quad (3)$$

The application ρ_σ is called the *node morphism*, and ρ_μ the *arc morphism*.

The word morphism (or homomorphism) comes from algebra where it refers to an application between two spaces that does not change the internal composition laws of these spaces. Here, the same name has been chosen because the meaning is close. A graph is a set with a binary relation (arcs): the morphism must keep this relation, with a certain degree of correspondence in the fuzzy case.

It is easy to show that the definition of the node morphism corresponds to Kaufmann's definition of a fuzzy graph ($S_1 = N_1$ and $S_2 = N_2$). The arc morphism corresponds to Rosenfeld's definition of a fuzzy graph. Let us define $S = N_1 \times N_2$, ρ_σ is a fuzzy subset on S, and the arc morphism ρ_μ defined on $S \times S$ is a fuzzy graph in the sense of Rosenfeld.

Equation 3 means that two arcs cannot be matched with a degree higher than the minimum of the morphism degree of each pair of corresponding nodes. This relation is natural because an arc stands for a specific relation between two nodes. Even if this relation can have a proper semantic meaning, it is defined using its two extremal nodes.

3.2 Internal and external interpretations

One of the main features of the proposed definition is that it involves the Cartesian product $N_1 \times N_2 \times N_1 \times N_2$. In the interpretation of the arc morphism as a fuzzy graph, we have defined S as the Cartesian product $N_1 \times N_2$. The Cartesian products $N_i \times N_i \supseteq E_i$, $i \in \{1, 2\}$ are implicitly present in $S \times S$. We propose two complementary interpretations of the arc morphism:

- $S \times S = (N_1 \times N_2) \times (N_1 \times N_2)$ corresponds to the classical notion of association compatibility: $\rho_\mu(u_1, u_2, v_1, v_2)$ represents the compatibility between the association between u_1 and u_2, and the association between v_1 and v_2.
- $S \times S \rightsquigarrow (N_1 \times N_1) \times (N_2 \times N_2) \supseteq E_1 \times E_2$ is the new notion introduced in this paper thanks to the new formalism (Section 3.1). The arrow $\rightsquigarrow$ means that we have just implicitly emphasized a supset of $E_1 \times E_2$.

This distinction is of great importance and allows the deduction of many properties. The association compatibility is a pattern recognition notion used to measure the influence of the association between two objects on the association between two other objects. The measure is used to confirm or to invalidate the association. The arcs in a graph are always used as a compatibility measure between the association of two nodes and the association of two other adjacent nodes.

The arc morphism allows a different view by considering the arcs as a true semantic component of the graph. The vision of the arc morphism can be split into the internal and external vision. The internal vision refers to:

$$\Big(\overbrace{\overline{u_1},\ \overline{u_2}}^{N_1 \times N_2},\ \underbrace{\underline{v_1},\ \underline{v_2}}_{N_1 \times N_2}\Big).$$

The name "internal vision" is used because we can define an internal composition law using this view of the morphism. This composition is based upon the composition of fuzzy relations and will be the subject of another paper. The other view of the arc morphism will be called the external vision:

$$\underset{N_2 \times N_2}{\overset{N_1 \times N_1}{\Big(\overline{u_1},\ \underline{u_2},\ \overline{v_1},\ \underline{v_2}\Big)}}.$$

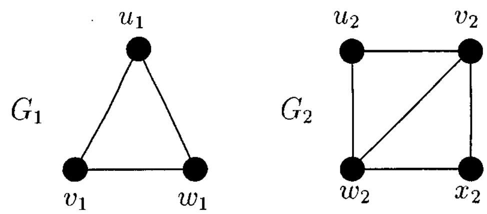

Figure 1. G_1 and G_2 are two examples of graphs.

3.3 Compatibility with crisp definitions

Graph morphism is defined in the crisp case as a pair of membership functions that have their values in $\{0, 1\}$ instead of $[0, 1]$. Graph isomorphism can therefore be defined as a graph morphism with the additional constraints: $|N_1| = |N_2|$, $|E_1| = |E_2|$, ρ_σ is a bijective function, and $\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\sigma(u_1, u_2) \wedge \rho_\sigma(v_1, v_2)$ if and only if $(u_1, v_1) \in E_1$ and $(u_2, v_2) \in E_2$. In this case, the definitions of the arc morphism and of the node morphism are redundant. The definition emphasizes the fact that we can separate the notions of node (object) matching, and arc (relation) matching. Here, we use the external view because the condition implies the presence of two arcs (u_1, v_1) and (u_2, v_2) in the constraint. Other definitions can be inferred from our formalism which we cannot develop in this paper for succinctness reasons.

3.4 Fuzzy association graphs

Our definition extends the classical notion of association graph using the fuzzy set framework and the linked notions of node and arc morphisms.

We can write the morphism using classical fuzzy adjacency matrices [5]. The node morphism adjacency matrix is straightforward. The arc morphism being defined on a 4-dimensional space, we can write it as a matrix by using $N_1 \times N_1$ and $N_2 \times N_2$ as the row and the column index sets. With the additional constraint that only existing arcs (of E_1 and E_2) can be matched, the morphism has no zero values on E_1 and E_2. Thus, the matrix can be written using the external view on $E_1 \times E_2$. Table 1 is a simple illustration of a random morphism between the two graphs of Figure 1. One must notice on this example that although the arcs of the graphs G_1 and G_2 of Figure 1 are not directed, they were split into two opposite directed arcs in the arc morphism matrix. This is because the definition of the graph morphism is not a priori symmetrical. The symmetry properties are developed in the next Section.

$N_1 \times N_2$	u_2	v_2	w_2	x_2
u_1	0.93	0.02	0.24	0.47
v_1	0.34	0.79	0.46	0.76
w_1	0.52	0.46	0.61	0.80

$E_1 \times E_2^1$	(u_2, v_2)	(u_2, w_2)	(v_2, u_2)	(v_2, w_2)	(v_2, x_2)	(w_2, u_2)	(w_2, v_2)	(w_2, x_2)	(x_2, v_2)	(x_2, w_2)
(u_1, v_1)	0.68	0.38	0.02	0.02	0.02	0.12	0.22	0.19	0.01	0.06
(u_1, w_1)	0.38	0.02	0.01	0.01	0.00	0.19	0.21	0.09	0.24	0.03
(v_1, u_1)	0.01	0.19	0.65	0.05	0.29	0.03	0.02	0.17	0.02	0.02
(v_1, w_1)	0.11	0.06	0.11	0.60	0.69	0.33	0.23	0.30	0.34	0.50
(w_1, u_1)	0.01	0.16	0.10	0.22	0.39	0.06	0.01	0.01	0.02	0.14
(w_1, v_1)	0.13	0.12	0.31	0.06	0.27	0.18	0.40	0.28	0.71	0.43

Table 1. Example of node and arc morphisms between G_1 and G_2 as two fuzzy adjacency matrices.

4. Symmetries

Symmetry is sometimes an immediate notion when dealing with graph, or simple correspondence. Here, this notion has several interpretations because of the different views (internal and external) of the morphism, and because of the complexity of graph correspondences. In fact, only the arc morphism is concerned by symmetry. The node morphism is defined on $N_1 \times N_2$, and no symmetry can be formulated unless $N_1 = N_2$. This case has not been studied yet. The notion of symmetry for the arc morphism is linked to the notion of undirected arc. If an arc (u_1, v_1) is not directed and is linked to the undirected arc (u_2, v_2), we would first want to have: $\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\mu(v_1, v_2, u_1, u_2)$.

We will call this symmetry the *weak symmetry*. This is related to the external view of symmetry if we consider that we want to reverse the direction of both arcs. The other interpretation of this property is the internal view. A morphism is weakly symmetrical if the information that brings the association of (u_2, v_2) on the association of (u_1, u_2) is the same as the information that brings the association of (u_2, v_2) on the association of (u_2, v_2). This situation is illustrated in Figure 2. As the arcs are not directed, the associations are not ordered among the arcs.
If the arcs are directed, we may act differently by following the order of the arcs. We will rather consider first the node morphism on (u_1, u_2) and then infer the morphism on (u_2, v_2) using the direction of the arrows. This situation is illustrated in Figure 3. We define strong symmetry by strengthening the condition of weak symmetry as follows:

$$\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\mu(u_1, v_2, v_1, u_2) = \rho_\mu(v_1, v_2, u_1, u_2). \quad (4)$$

The weak symmetry does not change Equation 3, but the strong symmetry implies a stricter constraint: the 4 implied nodes must be similar among each other to allow the arc morphism to be "significant". This type of situation is possible but is often considered only locally:

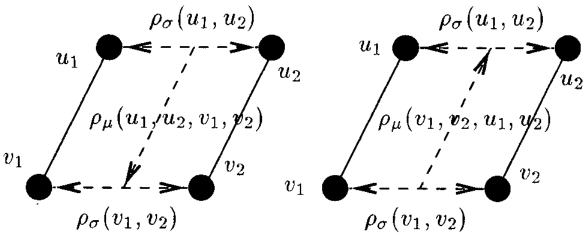

Figure 2. Internal view of the weak symmetrical morphism for an undirected arc: $\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\mu(v_1, v_2, u_1, u_2)$.

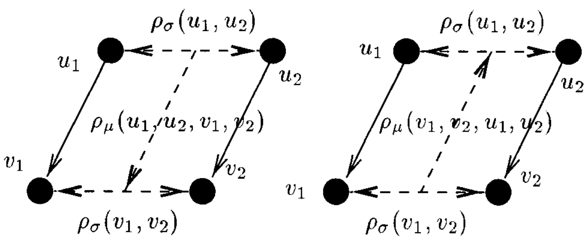

Figure 3. Internal view of the not weak symmetrical morphism for a directed arc: $\rho_\mu(u_1, u_2, v_1, v_2) \neq \rho_\mu(v_1, v_2, u_1, u_2)$.

$\exists(u_1, u_2, v_1, v_2) \in N_1 \times N_2 \times N_1 \times N_2$ such that Equation 4 is satisfied. This criterion may be used to handle one-to-many or many-to-one matching as in [13, 3, 8]: the idea is to characterize locally the morphism between two groups of similar objects.

5. Conclusion

In this paper we have proposed a generic and unified definition of graph fuzzy morphism that can handle a lot of classical structural pattern recognition problems based upon graphs. Problems related to inexact graph matching and one-to-many (or many-to-one) correspondences are handle with fuzzy sets by assigning a degree of correspondence between graph components, and with several properties (such as the local strong symmetry) based upon graph theory, association graphs or fuzzy graphs.

The fuzzy set framework was chosen to handle possible uncertainties and inaccuracies. One major advantage of our definition (apart from its compatilibity with former classical definition) is to formalize new properties and emphasize complementary aspects of the problem. More generally, the aim of combining fuzzy sets with morphisms (not restricted to isomorphism) was to build a theoretical background for imperfect graph matching.

Further works concern the analysis of the many properties that can be given about this definition, and to develop concrete applications such as [8, 9] concerning atlas-based brain structures recognition in medical imaging.

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