

FUZZY ADJACENCY BETWEEN IMAGE OBJECTS

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The notion of adjacency has a strong interest for image processing and pattern recognition, since it denotes an important relationship between objects or regions in an image, widely used as a feature in model-based pattern recognition. A crisp definition of adjacency often leads to low robustness in the presence of noise, imprecision, or segmentation errors. We propose two approaches to cope with spatial imprecision in image processing applications, both based on the framework of fuzzy sets. These approaches lead to two completely different classes of definitions of a degree of adjacency. In the first approach, we introduce imprecision as a property of the adjacency relation, and consider adjacency between two (crisp) objects to be a matter of degree. We represent adjacency by a fuzzy relation whose value depends on the distance between the objects. In the second approach, we introduce imprecision (in particular spatial imprecision) as a property of the objects, and consider objects to be fuzzy subsets of the image space. We then represent adjacency by a relation between fuzzy sets. This approach is, in our opinion, more powerful and general. We propose several ways for extending adjacency to fuzzy sets, either by using α -cuts, or by using a formal translation of binary equations into fuzzy ones. Since set equations are more easily translated into fuzzy terms, we shall privilege set representations of adjacency, particularly in the framework of fuzzy mathematical morphology. Finally, we give some hints on how to compare degrees of adjacency, typically for applications in model-based pattern recognition.

Keywords: Fuzzy image processing, fuzzy adjacency, fuzzy mathematical morphology.

1. Introduction

Adjacency has a large interest in image processing and pattern recognition, since it denotes an important relationship between image objects or regions²⁵, widely used as a feature in model-based pattern recognition.

A crisp definition of adjacency between crisp objects often leads to a low robustness in case of noise or segmentation errors. Let us consider for instance a problem of model-based pattern recognition, where spatial relationships are an important part of the recognition process. If two model objects are adjacent, we expect the corresponding image objects to be adjacent too, otherwise they will be difficult to recognize. However, if classical crisp adjacency is used, the fact that two objects are adjacent or not may depend on one point only. Figure 1 illustrates the sensitivity of

the binary adjacency concept to errors in segmentation or definition of the objects.

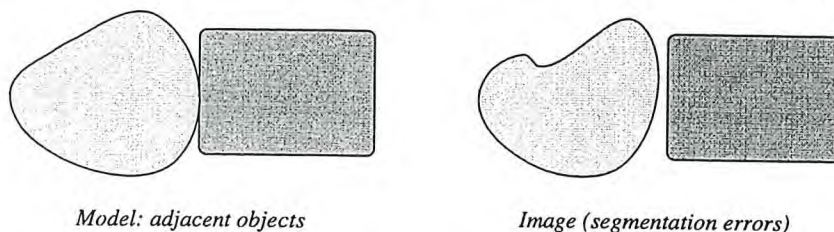


Figure 1: Sensitivity of crisp adjacency: small modifications in the shapes may completely change the adjacency relation, and thus prevent a correct recognition based on this relationship.

In order to include possible errors or imprecision in the processing and in the recognition, we use the framework of fuzzy sets that already proved to be useful for image processing under imprecision (see e.g. ¹⁶). Here we consider two completely different ways for representing imprecision. In the first one, the satisfaction of the adjacency property between two objects is considered to be a matter of degree; this can be more appropriate than a binary index ^{23, 24}. The second one consists in introducing imprecision in the objects themselves, and to deal with fuzzy objects, i.e. with objects considered as fuzzy sets on the space, with their attached membership function defined for every point. For instance, spatial imprecision due to the limited quality of image information can be represented in an adequate way by considering fuzzy objects. Then obviously adjacency is also a matter of degree.

In both cases, a need arises to find proper definitions for fuzzy adjacency between image regions.

Methods for dealing with uncertainty (in its most general sense) in image processing rely mainly on probabilistic approaches, and more recently on fuzzy sets and possibility theory and Dempster-Shafer evidence theory. They all deal with different types of uncertainty (like randomness, imprecision, or degrees of belief), and have been compared mainly from the point of view of image fusion (see e.g. ⁵). The problem of introducing imprecision and uncertainty at the level of adjacency parameters has not been addressed in the other frameworks to our knowledge. We argue that the fuzzy set framework is very appropriate to the management of spatial imprecision. Unfortunately, we could find only a few attempts in the literature to address the problem of fuzzy adjacency. Fuzzy topology was introduced in ²³. In that paper, Rosenfeld defines a fuzzy connectivity between points but without reference to the notion of fuzzy neighborhood, or to fuzzy adjacency. Similar approaches can also be found in ^{24, 30, 31}, where degrees of connectivity in a fuzzy set are also introduced, but neither the connectivity nor the adjacency between two fuzzy sets are defined. Rosenfeld and Klette ²⁶ define a degree of adjacency between two crisp sets, using a geometrical approach based on the notion of "visibility" of a set from another one. This definition is then extended to degree of adjacency between two fuzzy sets. However, this definition is not symmetrical, and probably

not easy to transpose to higher dimensions. We propose in this paper a completely different approach. The work the closest to ours is probably the one described in ¹⁰, where a degree of adjacency between two fuzzy sets is defined by extending binary definitions of contours, frontiers, and neighborhood. Again, the proposed definition of ¹⁰ is not symmetrical, and presents other drawbacks that will be described later.

In this paper we propose several definitions for degree of adjacency coping with spatial imprecision in image processing. This paper is organized as follows. Basic definitions for classical notions of adjacency are given in Section 2, both in the continuous and discrete domains. In Section 3, we propose a definition of degree of adjacency between crisp objects, depending on the distance between them, in order to deal with possible errors or imprecision in the segmentation or representation of these objects. In our opinion, a more powerful approach is to explicitly account for such imprecision in the representation of objects, consequently defined as spatial fuzzy sets, and to define a degree of adjacency between two fuzzy sets. In Section 4, we shortly describe some possible ways for extending adjacency to fuzzy sets. The first one is based on the α -cuts, and is developed in Section 5. The second one is based on a formal translation of binary equations into fuzzy ones and is detailed in Section 6. It leads to the best definitions with respect to properties, intuitive requirements, behavior and interpretation. Since set theoretical expressions are easy to translate into fuzzy terms, we shall privilege set representations of adjacency (instead of geometrical ones as in ²⁶). Fuzzy mathematical morphology provides a consistent framework for dealing with imprecision in set expressions, and we show how it can be used for defining fuzzy adjacency. Finally, Section 7 provides some hints on how to compare degrees of adjacency, typically for applications in model-based pattern recognition.

2. Crisp adjacency

2.1. Continuous domain

We assume here that objects are defined in a finite n -dimensional space, typically \mathbb{R}^n for sake of simplicity. A natural way to define adjacency relies on topology. We consider here the classical topology defined on \mathbb{R}^n . We denote by \mathcal{G} the set of open sets, \mathcal{F} the set of closed sets. For any subset X of \mathbb{R}^n , we denote by X^C its complementary set ($\mathbb{R}^n - X$), by \bar{X} its topological closure and by \dot{X} its interior (see e.g. ⁸). The boundary of X is then defined as:

$$\partial X = \bar{X} - \dot{X} = \bar{X} \cap \bar{X}^C. \quad (1)$$

Using these notations, adjacency between two subsets X and Y can be defined in the following simple way:

Definition 1 For any two subsets X and Y in \mathbb{R}^n , X and Y are adjacent if

$$\partial X \cap \partial Y \neq \emptyset \text{ and } X \cap \dot{Y} = \emptyset \text{ and } Y \cap \dot{X} = \emptyset. \quad (2)$$

Property 1 Definition 1 is equivalent to:

$$\bar{X} \cap \bar{Y} \neq \emptyset \text{ and } X \cap \dot{Y} = \emptyset \text{ and } Y \cap \dot{X} = \emptyset. \tag{3}$$

Property 2 Definition 1 implies that $\dot{X} \cap \dot{Y} = \emptyset$.

However, it is not sufficient to verify that $\partial X \cap \partial Y \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$ (or, equivalently, $\bar{X} \cap \bar{Y} \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$) in order to have a satisfactory definition of adjacency. Indeed, this would lead to consider the two sets in Figure 2 as adjacent, which is intuitively not convenient.

Definition 1 corresponds to the intuitive notion that X and Y are adjacent if they have no overlapping parts (overlap could be expressed as $X \cap \dot{Y} \neq \emptyset$ or $\dot{X} \cap Y \neq \emptyset$) and are not completely separated (separation could be expressed as $\bar{X} \cap \bar{Y} = \emptyset$).

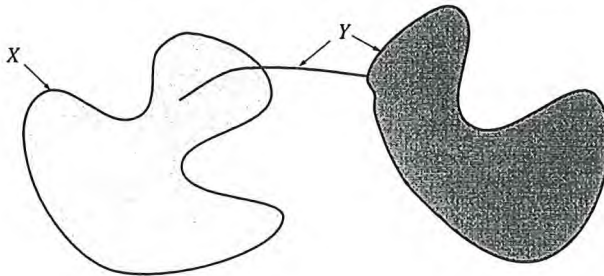


Figure 2: Two sets that are not adjacent but that satisfy $\partial X \cap \partial Y \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$.

Since neighborhood is an essential concept in image processing, a definition of adjacency can also be given with respect to it. It can take the following forms, for closed and open sets respectively.

Definition 2 For any two closed sets F and F' in \mathcal{F} , F and F' are adjacent in \mathcal{F} if

$$F \cap F' \neq \emptyset \text{ and } \forall x \in F \cap F', \forall V(x), \exists (y, y') \in V(x)^2, y \in F, y' \in F', y \notin \dot{F}', y' \notin \dot{F}, \tag{4}$$

where $V(x)$ denotes a neighborhood of x .

This means that any neighborhood of an intersection point contains points both in F and F' , as illustrated by Figure 3. The constraint stating that the point y in $V(x)$ that belongs to F should not belong to the interior of F' guarantees that situations like the one in Figure 2 are avoided.

Property 3 A consequence of definition 2 is that if F and F' are adjacent in \mathcal{F} , then $\dot{F} \cap \dot{F}' = \emptyset$ and $F \cap F' = \partial F \cap \partial F'$.

Property 4 Definition 2 is equivalent to definition 1 when applied to closed sets.

Definition 3 For any two open sets G and G' in \mathcal{G} , G and G' are adjacent in \mathcal{G} if

$$G \cap G' = \emptyset \text{ and } \exists x \in \mathbb{R}^n, \forall V(x), V(x) \cap G \neq \emptyset, V(x) \cap G' \neq \emptyset. \tag{5}$$

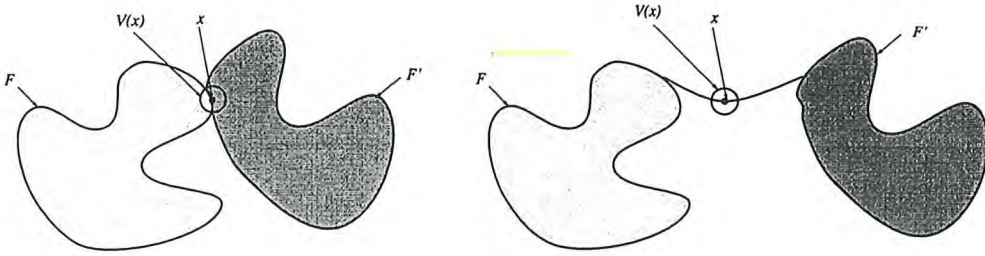


Figure 3: Two examples of adjacency between closed sets. On the right, the closed sets F and F' are adjacent in \mathcal{F} but their interiors \bar{F} and \bar{F}' are not adjacent in \mathcal{G} .

This means that there exists at least one point in a neighborhood of both G and G' .

Property 5 *If G and G' are adjacent in \mathcal{G} , then any x satisfying the equation of definition 3 belongs to $(G \cup G')^C$ which is a closed set, and moreover $x \in \partial(G \cup G')^C = \partial(G \cup G')$.*

Property 6 *Definition 3 is equivalent to definition 1 when applied to open sets.*

Property 7 *If two open sets G and G' are adjacent in \mathcal{G} according to definition 3, then \bar{G} and \bar{G}' are adjacent in \mathcal{F} according to definition 2.*

However, if two closed sets F and F' are adjacent in \mathcal{F} according to definition 2, their interior \bar{F} and \bar{F}' are not necessarily adjacent in \mathcal{G} according to definition 3. A counter-example is shown on the right example of Figure 3.

In practice, and in particular for morphological applications, mainly closed sets are considered ²⁷. They are extended to upper semi-continuous membership functions in the fuzzy case.

Moreover, the particular problematic situations illustrated by Figure 2 and Figure 3 (right) are avoided if we restrict ourselves to objects X of \mathbb{R}^n such that:

$$X = \bar{X}. \tag{6}$$

Such objects are closed sets and have everywhere a local dimension equal to n ^{14, 15}. For the 2D example illustrated by Figure 2, X is such an object, whereas Y is not since it contains parts of local dimension 1.

Property 8 *For objects X such that $X = \bar{X}$ (purely n -dimensional), the definition of adjacency between X and Y reduces to one of the following forms:*

$$\partial X \cap \partial Y \neq \emptyset \text{ and } \dot{X} \cap \dot{Y} = \emptyset, \tag{7}$$

$$X \cap Y \neq \emptyset \text{ and } \dot{X} \cap \dot{Y} = \emptyset, \tag{8}$$

$$X \cap Y \neq \emptyset \text{ and } \forall x \in X \cap Y, \forall V(x), \exists (y, y') \in V(x)^2, y \in (X - Y), y' \in (Y - X). \tag{9}$$

This property expresses a simpler way to assess the adjacency between objects, and holds for a large class of objects, that may be sufficient in practice.

2.2. Discrete images

In discrete images, the classical discrete topology has little meaning⁹. Since this topology considers each point of the discrete space as an open set, the only connected components according to this topology contain only one point. This is obviously not convenient for image analysis and pattern recognition, since we expect connected objects being made of several points. Therefore, classical discrete topology is not used in image processing. The most common approach consists in directly defining a discrete connectivity and a neighborhood between points, from which connected components can be deduced in a graph-like manner. More details on discrete topology in 2D images can be found e.g. in²⁵, and these notions can be extended to any number of dimensions. In this Section, we adopt this approach based on discrete connectivity for defining adjacency between two image regions X and Y , subsets of the discrete space. Consider an n -dimensional discrete space (typically \mathbb{Z}^n), and any discrete connectivity defined on this space, denoted c -connectivity (for instance, for $n = 3$, we may consider 6-, 18- or 26-connectivity on a cubic grid).

Existing definitions of adjacency in the discrete case can be found e.g. in²², where Rosenfeld defines adjacency in 4- and 8-connectivity in 2D binary images as follows:

Definition 4²² X and Y are 4-adjacent (resp. 8-adjacent) if $\exists x \in X, \exists y \in Y$ such that x and y are neighbors according to the 4-connectivity (resp. 8-connectivity).

This definition does not take into account possible intersections between X and Y . In particular, according to this definition, intersecting objects are adjacent, which rather corresponds to a notion of connectedness.

Since we would like to distinguish between connectedness and adjacency relationships, we prefer another definition, where we consider that the adjacency concept makes sense for regions that do not intersect*. We denote by $n_c(x, y)$ the Boolean variable stating that x and y are neighbors in the sense of the discrete c -connectivity.

Definition 5 For any subsets X and Y in \mathbb{Z}^n , X and Y are adjacent according to the c -connectivity if:

$$X \cap Y = \emptyset \text{ and } \exists x \in X, \exists y \in Y : n_c(x, y). \quad (10)$$

This definition is in agreement with the notion of adjacency used in¹⁸, stating that adjacent objects are those that are edge to edge without overlapping parts.

We consider for the discrete boundary of a set X its interior boundary defined as:

$$\partial X = X - E(X, B_c) \quad (11)$$

*Actually, the proposed definitions could be slightly modified for regions intersecting only at their boundaries, if needed, as follows: $\forall X \subset \mathbb{Z}^n, \forall Y \subset \mathbb{Z}^n$, X and Y are adjacent if $X \cap Y \neq \emptyset$ and $\forall z \in X \cap Y, \exists x \in (X - Y), \exists y \in (Y - X)$ such that $n_c(x, z)$ and $n_c(y, z)$. This is equivalent to $X \cap Y = \partial X \cap \partial Y$.

where $E(X, B_c)$ denotes the morphological erosion of X by the structuring element B_c of size 1 defined according to the chosen discrete connectivity²⁷. Using this definition, the discrete adjacency can be related to the boundary in the following way:

Property 9 *A consequence of definition 5 is that, if X and Y are adjacent, then any $x \in X$ and $y \in Y$ that satisfy $n_c(x, y)$ belong to the boundary of X and Y respectively.*

Therefore the fuzzy extension of definition 5 can be obtained either by considering only the constraint on the neighborhood, or by considering also the constraint on the boundary, as will be seen in Section 6.

Property 10 *Definition 5 can also be expressed equivalently in terms of morphological dilation[†] as:*

$$X \cap Y = \emptyset \text{ and } D(X, B_c) \cap Y \neq \emptyset, D(Y, B_c) \cap X \neq \emptyset, \quad (12)$$

where $D(X, B_c)$ denotes the dilation of X by the structuring element B_c , since the following equivalence holds (for $x \neq y$):

$$n_c(x, y) \Leftrightarrow x \in B_{cy} \Leftrightarrow y \in B_{cx}, \quad (13)$$

where B_{cx} denotes the structuring element B_c translated at point x .

This will provide a third way to extend the definition to fuzzy sets, either directly from fuzzy dilation, or by means of distance computation, which is closely related to dilation (equation 12 means that the minimum (nearest point) distance between X and Y is equal to 1).

However, these definitions may have a drawback depending on the connectivity. If all objects are considered using the same connectivity, this may lead to well known ambiguous and paradoxal situations. This problem does not arise for 6-connectivity on an hexagonal grid, but it is typically the case for 4- or 8-connectivity on a square grid²⁵. If such paradoxes have to be avoided, one classical solution consists in separating the objects and their boundaries. In this way, the space (or the image) is partitioned in objects (using one type of connectivity, e.g. 4-connectivity) and a set of points constituting the boundaries of all these objects, defined using the dual connectivity (e.g. 8-connectivity). Let us denote by \mathcal{B} this boundary set (the image is then partitioned in $\mathcal{I} = \mathcal{B} \cup X \cup Y \cup \dots$). Then adjacency can be defined as follows:

Definition 6 *Two regions X and Y of a partition are adjacent if:*

$$X \cap Y = \emptyset \text{ (i.e. } X \neq Y) \text{ and } \exists z \in \mathcal{B}, \exists x \in X, \exists y \in Y : n_c(x, z), n_c(y, z), \quad (14)$$

where the objects are considered in c -connectivity, while the boundary is defined according to the dual connectivity.

The extension of this definition to fuzzy sets is the scope of Section 6.6.

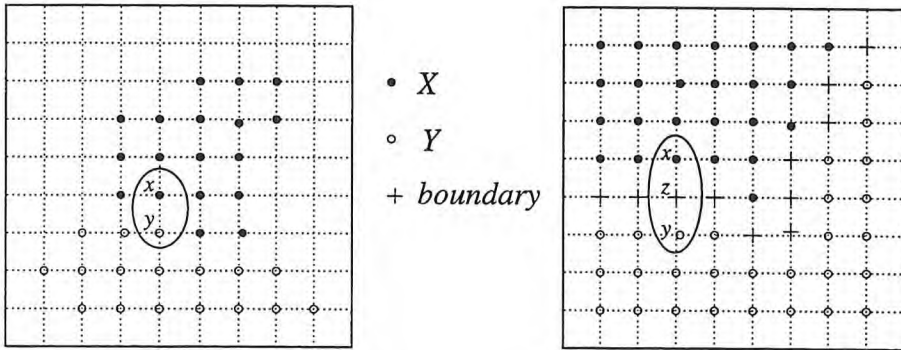


Figure 4: Illustration of definitions 5 (left) and 6 (right) for crisp discrete adjacency between two image regions X and Y . On the left, $x \in X$, $y \in Y$ and x and y are 4-neighbors. On the right, the objects are considered in 4-connectivity, while the boundary is 8-connected. On this example, z belongs to the boundary and is a 4-neighbor of x and of y .

Definitions 5 and 6 are illustrated by the simple example in Figure 4.

Note that for several applications spatial relationships are used for recognizing objects in images from a previous segmentation. In such cases the regions in the image form a partition. Then definition 4 can be used (i.e. we can ignore the condition on empty intersection of definition 5).

3. Degree of adjacency between crisp objects

In this Section, we consider adjacency between crisp objects to be a matter of degree, in order to account for possible errors in the detection of the objects, as mentioned in the introduction (see Figure 1). The simplest idea consists in defining a degree of adjacency between two crisp sets that depends on the distance between them. Intuitively, we require that this degree be equal to 1 if the sets are strictly adjacent in the sense of Section 2, and that it decreases if the objects are far from being strictly adjacent (either intersecting or separated). The speed of the decreasingness has to be chosen according to the desired degree of fuzziness and to the expected imprecision or errors in the definition of the objects. For instance, if we assume that the detected objects are likely to have at most one pixel width errors around them, then the degree of adjacency should decrease very rapidly with respect to the distance between the objects.

Degree of adjacency can then be defined depending on how far the objects are from strict adjacency. This can be expressed in terms of transformations that would bring two non-adjacent objects in a position where they are strictly adjacent (i.e. where they satisfy definition 5):

[†]The reader may refer to ²⁷ for more details on mathematical morphology, a theory which is widely used in image processing.

Definition 7 The degree of adjacency between two sets X and Y is defined as:

$$\mu_{adj}(X, Y) = f[d(X, Y)] \quad (15)$$

where $d(X, Y)$ denotes the size of the minimal admissible transformation between X and Y (or from one set with respect to the other) achieving strict adjacency, and f is a decreasing function of d , taking values in $[0, 1]$.

This definition is rather intuitive and calls for a more precise statement of several terms: admissible transformation, size and minimal size, and function f .

For the transformation of one set with respect to the other, we may think of very simple transformations like translations in \mathbb{Z}^n or \mathbb{R}^n . But we may also define a more general set \mathcal{T} of admissible transformations, related to the pattern recognition problem at hand. Indeed, several recognition problems are defined up to some transformations. Typically, the recognition of an object will not differ if the object is defined up to a translation, a rotation and a scaling. Moreover, a study of the imprecision in the images and in the detected objects may lead to a knowledge of the possible transformations that may affect the objects.

The size of a transformation is strongly related to the type of transformation we consider. For the example of translations, it is simply the norm of the translation vector. More examples are given in Appendix 1. Note that in case of complex transformations, size may not be straightforward to define. Having defined the set of transformations and the size of a transformation (denoted by s), the size of the minimal transformation d is such that:

$$d = \min\{s(t), t \in \mathcal{T}\}. \quad (16)$$

Finally, the function f has to be defined as a decreasing function of d . For instance, f can be chosen as $\exp(-\lambda d)$, where λ ($\lambda > 0$) controls the decreasingness of μ_{adj} . A generic description of such functions is given in Section 4.4.

Two advantages of this definition can be highlighted: first it has a generic component, that may find different instantiations depending on the problem; second, it applies when the objects intersect as well as when they are far from each other, as opposed to definitions relying on a classical notion of distance (see Appendix 1).

Although this method would obviously increase the robustness to segmentation errors compared to crisp definitions, it suffers from several drawbacks:

- The choice of the function f is somewhat arbitrary and probably difficult to tune for specific applications. It can be expected that a compromise has to be found between:

robustness i.e. adjacency should not be too sensitive to small changes in the objects; this calls for a function f with a smooth shape; and

specificity i.e. the degree of adjacency should be informative enough, close to binary definition; this calls for a function f with a steep shape.

- In order to avoid such arbitrariness, another way to proceed consists in introducing explicitly the possible imprecision in the detected objects (for instance using a fuzzy dilation ⁴), and then using a definition of fuzzy adjacency between two fuzzy sets, as proposed in Sections 5 and 6. Indeed, it appears in many applications that the imprecision, for instance due to segmentation or to some geometric processing, is known or can be estimated, and therefore the objects can be modeled as fuzzy objects in a non-arbitrary way (see e.g. ⁶). In such cases, it is better to deal directly with fuzzy objects than to introduce imprecision in each relationship of interest. This also guarantees a more consistent processing, since the imprecision attached to the objects are considered in the same way by all relationships.
- Defining separately a degree of adjacency may be questionable if other spatial relationships have to be taken into account in the recognition process (which is very likely). Typically, it could be more meaningful to have a coherent framework for defining degrees of adjacency, and degrees of inclusion and intersection (see e.g. ^{28, 4} for the definitions of degrees of intersection and inclusion, related to fuzzy mathematical morphology). Indeed, we can expect a continuous variation between degree of inclusion, degree of intersection and degree of adjacency[†] (for instance if an object is progressively moved with respect to the other one, as shown in Figure 5). This is left for future work.

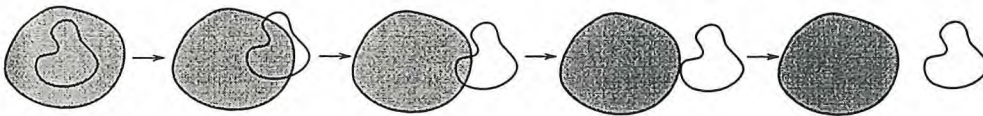


Figure 5: Continuous variation from one relationship to another (inclusion, two different degrees of intersection, adjacency, etc.).

4. Extending adjacency to fuzzy objects

In the rest of this paper we consider fuzzy objects (i.e. fuzzy sets defined on the considered space by means of their membership function) and define fuzzy adjacency between such objects. This Section presents shortly the possible principles that can be used for extending adjacency to fuzzy sets and the requirements posed to this extension.

In the continuous case (in \mathbb{R}^n), we consider upper semi-continuous (u.s.c.) membership functions[‡] since they generalize closed sets. Indeed, the α -cuts of a u.s.c. membership function are closed sets, and so is its subgraph (see e.g. ²⁷). In the

[†]This idea has been suggested by Michel Roux, ENST, 1996.

[‡]A function f is upper semi-continuous (u.s.c.) at x if:

$$\forall t > f(x), \exists V(x), \forall y \in V(x), t > f(y),$$

where $V(x)$ is a neighborhood of x in \mathbb{R}^n . If f is u.s.c. at every point, it is simply said to be u.s.c.

discrete case, a fuzzy object is simply defined by its membership function, defined on \mathbb{Z}^n and taking values in $[0, 1]$. For practical applications, membership functions having a bounded support are often considered. No specific property is necessary in terms of convexity, compactness or connectivity. However, since problems related to adjacency are often considered on connected objects, we may restrict to membership functions having a compact connected support. All what follows is valid without any additional assumption on the membership functions.

In this Section, we consider the general problem of extending a relationship R_B between two binary objects to its fuzzy equivalent R (fuzzy relationship between two fuzzy objects). Instantiations of the described methods to the case of adjacency are provided in the next two Sections.

4.1. Using the α -cuts

One way to define crisp sets from a fuzzy set consists in taking the α -cuts of this set. Therefore a first class of methods relies on the application of the relationship R_B to each α -cut. This gives rise to two different "fuzzification" methods in the literature.

The first one consists in "stacking" the results obtained with binary operations on the α -cuts: let us denote by μ and ν the membership functions of two fuzzy objects defined on the considered space and taking values in $[0,1]$, the fuzzy equivalent R of R_B is then defined as (see e.g. ^{11, 4, 17}):

$$R(\mu, \nu) = \int_0^1 R_B(\mu_\alpha, \nu_\alpha) d\alpha, \tag{17}$$

or by a double integration as:

$$R(\mu, \nu) = \int_0^1 \int_0^1 R_B(\mu_\alpha, \nu_\beta) d\alpha d\beta. \tag{18}$$

Other fuzzification equations are possible, like:

$$R(\mu, \nu) = \sup_{\alpha \in [0,1]} \min(\alpha, R_B(\mu_\alpha, \nu_\alpha)) \text{ or } R(\mu, \nu) = \sup_{\alpha \in [0,1]} (\alpha R_B(\mu_\alpha, \nu_\alpha)). \tag{19}$$

Examples of this approach concern for instance connectivity ²⁴, fuzzy mathematical morphology ⁴, distances ^{11, 2, 3}, etc.

The second method is the extension principle ³³, which leads in the general case to a fuzzy number (rather than a crisp number):

$$\forall n \in \mathcal{V}(R_B), R(\mu, \nu)(n) = \sup_{R_B(\mu_\alpha, \nu_\alpha)=n} \alpha, \tag{20}$$

where $\mathcal{V}(R_B)$ denotes the image of R_B , i.e. the set of values taken by R_B .

If the relationship to be extended only takes binary values (0/1, or true/false), then the extension principle reduces to:

$$R(\mu, \nu) = \sup_{R_B(\mu_\alpha, \nu_\alpha)=1} \alpha, \tag{21}$$

and is equivalent to the two fuzzification procedures given by equations 19. This is typically the case for binary adjacency between binary sets as defined in Section 2.

4.2. *By formal translation of equations*

A second class of methods consists in translating binary equations into their fuzzy equivalent: intersection is replaced by a t-norm, union by a t-conorm, sets by membership functions, etc. Examples can be found for defining fuzzy morphology⁴, fuzzy inclusion²⁸, etc.

This translation is particularly straightforward if the binary relationship can be expressed in set theoretical and logical terms. This can be obtained in a natural way from the definitions given in Section 2. Moreover, this remark endows methods based on mathematical morphology with a particular interest, since mathematical morphology is mainly based on set theory.

These two classes of methods (Sections 4.1 and 4.2) may lead to the same definitions for particular choices of the fuzzy operators.

4.3. *Minimal properties for a fuzzy adjacency*

We conclude this Section by examining which properties are required for fuzzy adjacency. In³¹ the concept of fuzzy connectedness is developed based on the following properties: symmetry, reflexivity, decreasingness with respect to the distance between points. Here, our purpose is slightly different since we are interested in adjacency as noted in Section 2.2. Although both concepts are closely related, adjacency is more strict than connectedness. Typically in the binary case, adjacency implies connectedness, but the reverse is not true (intersecting sets are considered as connected to each other but they are not adjacent in a strict sense). Adjacency can also be interpreted as connectedness restricted to the cases where the local dimension of the intersection is at most $n - 1$ in a n -dimensional space. Therefore reflexivity is not necessary here.

Finally, the required properties for fuzzy adjacency are the following:

- symmetry (i.e. the degree of adjacency between μ and ν has to be the same than between ν and μ);
- consistency with binary definitions (i.e. if μ and ν are crisp sets, their degree of adjacency should be equal to 1 or 0, depending on whether or not they are adjacent, according to the definitions of Section 2): this consistency will be automatically satisfied if we use one of the above methods; note that this corresponds to the case where no imprecision has to be taken into account (in contrary to the problem addressed in Section 3);
- decreasingness with respect to the distance between both sets: this has already led to the definitions proposed in Section 3, and in the fuzzy case, this will be taken into account in the notion of fuzzy neighborhood;

- a last property, often desirable although not mandatory, is invariance with respect to geometrical transformations. Invariance with respect to translation or rotation of the image (i.e. simultaneously for all objects) is directly verified by extending adjacency using one of the above methods (up to discretization problems in case of rotations, that are however not specific to the fuzzy aspect). More difficult is invariance with respect to a scaling (zoom) of the image: this requirement will induce the use of normalized measures (this can be done using a scaling factor adapted to the scale of the image, or using the power $\frac{1}{n}$ of spatial measures like the volume)[¶] Other geometrical invariances are not considered here, since translation, rotation and scaling are the most used in pattern recognition.

4.4. Transforming a spatial measure into a degree in [0,1]

Since we want measures of adjacency that depend on spatial information, we construct in the next sections definitions that involve some spatial measurements in \mathbb{R}^n or \mathbb{Z}^n , and that need to be transformed into degrees in [0,1]. The aim of this Section is to provide some generic definition of a function f that transforms a space measure m defined on fuzzy sets in \mathbb{R}^n or \mathbb{Z}^n into a degree in [0,1]. In the following sections, m will be typically a distance or a fuzzy volume, and therefore takes values in $[0, +\infty]$. Then f has to be a function from $[0, +\infty]$ to $[0, 1]$. Such functions have been extensively used for defining generic t-norms and t-conorms satisfying specific properties ¹².

Here, the required properties for f will generally be:

- continuity,
- decreasingness,
- limit conditions: $f(0) = 1$ and $f(+\infty) = 0$.

This implies bijectivity of f , and allows therefore to recover the spatial measure from the degree in [0,1].

Examples that are useful in the following are:

$$f(m) = \exp(-\lambda m) \quad (\lambda > 0), \tag{22}$$

$$f_{b,S}^c(m) = \frac{1 + \exp(-b)}{1 + \exp b(\frac{m}{S} - 1)} \quad (b > 0, S > 0), \tag{23}$$

or in the discrete case with $m \in [1, +\infty]$:

$$f_{b,S}^d(m) = \frac{1 + \exp(-b)}{1 + \exp b(\frac{m-1}{S} - 1)}. \tag{24}$$

[¶]A fuzzy set μ defined on an image leads to a fuzzy set μ' on the scaled image by a factor λ defined as: $\forall x, \mu'(\lambda x) = \mu(x)$. This may cause problems only in the discrete case for non integer scale factor, but this is not considered in this paper.

Examples of functions that apply on measures of a combination of two fuzzy sets (e.g. using a t-norm) are:

$$f_t[m(t(\mu, \nu))] = 1 - \frac{m(t(\mu, \nu))}{\min[m(\mu), m(\nu)]}, \tag{25}$$

$$f_t[m(t(\mu, \nu))] = \frac{1}{1 - \frac{m(t(\mu, \nu))}{\min[m(\mu), m(\nu)]}} - 1. \tag{26}$$

Similar examples can be provided if f has to be increasing instead of decreasing.

5. Fuzzy adjacency from the α -cuts

This Section is dedicated to the application of the first class of extension methods to the case of fuzzy adjacency. We denote by $Adj(X, Y)$ the adjacency between two crisp sets X and Y (taking values 0 and 1), and by $\mu_{adj}(\mu, \nu)$ the degree of adjacency between two fuzzy sets μ and ν (not necessarily normalized). All proposed definitions lead to degrees of adjacency that take values in $[0, 1]$.

5.1. Extension principle

5.1.1. Continuous case

Definition 8 According to the extension principle, adjacency between two fuzzy sets μ and ν is defined as:

$$\mu_{adj}(\mu, \nu) = \sup\{\alpha, \alpha \in [0, 1], Adj(\mu_\alpha, \nu_\alpha) = 1\}, \tag{27}$$

with $\mu_{adj}(\mu, \nu) = 0$ if the supremum is taken over the empty set.

This equation expresses that the degree of adjacency corresponds to the maximum height where the α -cuts (which are closed sets if we consider u.s.c. membership functions) just intersect at their boundaries (see Figure 6).

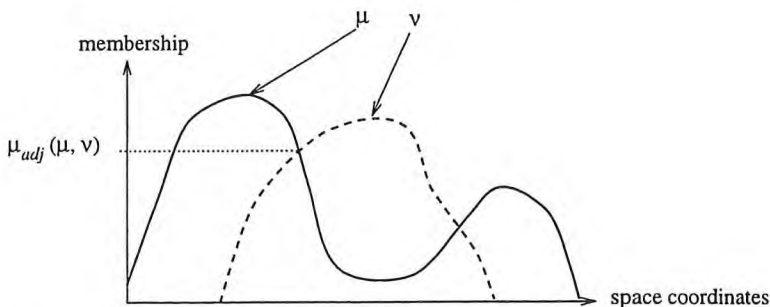


Figure 6: Degree of adjacency between two fuzzy sets (defined in a one-dimensional space) according to the extension principle in the continuous case.

Property 11 Using this definition, fuzzy adjacency is a symmetrical relationship, and is consistent with the binary definition (this is guaranteed by the extension principle). The decreasingness with respect to the distance is also satisfied, if we use a distance between fuzzy sets that include both spatial distance and comparison of membership values (see e.g. ³, ²).

We may have a more operational definition in some particular cases. One of them corresponds to fuzzy sets forming a fuzzy partition, e.g. in the sense of Bezdek ¹:

Definition 9 A fuzzy partition of \mathcal{I}' is defined in ¹ as a set of fuzzy sets μ_i ($i = 1 \dots n$) such that:

$$\forall x \in \mathcal{I}', \sum_{i=1}^n \mu_i(x) = 1,$$

$$\forall i \in \{1, \dots, n\}, 0 < \sum_{x \in \mathcal{I}'} \mu_i(x) < |\mathcal{I}'|,$$

where \mathcal{I}' is a finite subset of the underlying space \mathcal{I} (\mathbb{R}^n or \mathbb{Z}^n), and $|\mathcal{I}'|$ denotes the cardinality of \mathcal{I}' .

Property 12 In the continuous case, if we consider u.s.c. membership functions such that μ and ν belong to a fuzzy partition, definition 8 is equivalent to:

$$\mu_{adj}(\mu, \nu) = \sup\{\min[\mu(x), \nu(x)], x \in \mathbb{R}^n\} = \mu_{adj}^{\min}(\mu, \nu). \tag{28}$$

This equation suggests a more general definition of fuzzy adjacency, where the minimum is replaced by any t-norm t . It is called t-adjacency, and is defined as follows:

Definition 10 A degree of t-adjacency between μ and ν belonging to a fuzzy partition is defined from a t-norm t as:

$$\mu_{adj}^t(\mu, \nu) = \sup\{t[\mu(x), \nu(x)], x \in \mathbb{R}^n\}. \tag{29}$$

Property 13 This definition is still symmetrical, consistent with the binary definition, and decreasing with the distance between μ and ν . It is more strict than the definition obtained by property 12, since the following property holds for any t-norm t :

$$\forall(\mu, \nu), \mu_{adj}^t(\mu, \nu) \leq \mu_{adj}^{\min}(\mu, \nu). \tag{30}$$

This property can be exploited in practical applications by choosing a t-norm appropriate to the degree of "strictness" we want to attach to the adjacency relationship.

Note that property 12 holds not only for fuzzy sets belonging to a fuzzy partition, but for a larger class of fuzzy sets (like the example of Figure 6).

A strong drawback of definition 8 derived from the extension principle is that it is not able to discriminate situations that have intuitively different adjacency properties. For instance, in Figure 7, the degree of adjacency between μ and ν would be the same than the degree of adjacency between μ and ν' . Intuitively, we would rather say that $\mu_{adj}(\mu, \nu') < \mu_{adj}(\mu, \nu)$ since μ and ν' strongly overlap. Note that this problem does not arise if we work on a fuzzy partition.

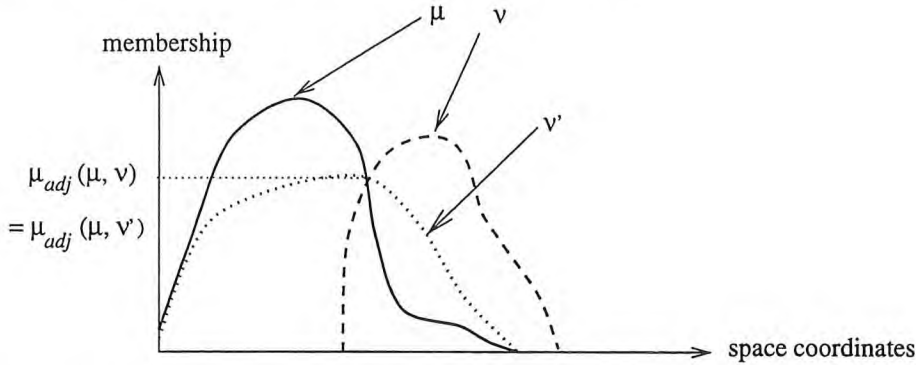


Figure 7: Low discrimination power of the definition of degree of adjacency between two fuzzy sets according to the extension principle: $\mu_{adj}(\mu, \nu') = \mu_{adj}(\mu, \nu)$, although μ and ν' strongly overlap and should be considered as less adjacent than μ and ν .

5.1.2. Discrete case

In the discrete case, we can still apply definition 8. However, it is no more equivalent to the maximum height of the intersection (defined as the minimum of μ and ν), as shown in Figure 8. In this example (in a one-dimensional space), $\mu_{adj}(\mu, \nu) = a_2$, since a_2 is the highest value for which μ_{a_2} and ν_{a_2} are adjacent (the neighbor points are x and y), according to definition 5. For instance, this property does not hold anymore for a_1 . The maximum height of the intersection is a_3 , for which μ_{a_3} and ν_{a_3} are adjacent (also with neighbor points x and y), but a_3 is not the highest value sharing this property!

It should be noted that in the fuzzy discrete case, the hypothesis of non intersecting sets used in the crisp case (related to the fact that we often deal with a partition of the image) does not appear in the fuzzy definition derived from definition 8. The empty intersection is checked on the α -cuts, but there is no hypothesis on the fuzzy intersection between both fuzzy sets, and no hypothesis about a fuzzy partition of the considered space. This is obviously no more the case when using the height of the intersection (property 12).

||This effect is mainly due to digitization.

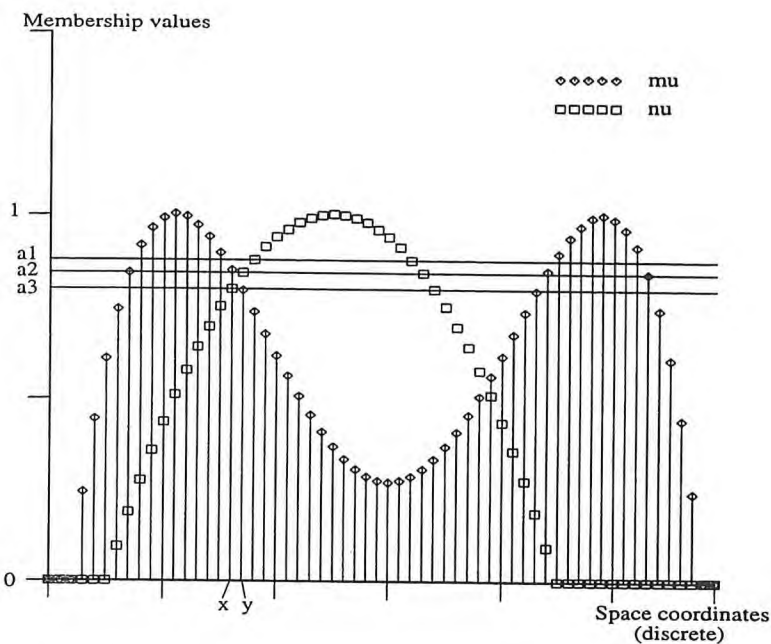


Figure 8: Applying the extension principle as in definition 8 in the discrete case: $\mu_{adj}(\mu, \nu) = a_2$ (in the cuts of height a_2 , x and y are neighbors and belong respectively to μ_{a_2} and ν_{a_2}). The maximum of the intersection (defined as the minimum of μ and ν) would provide in this case a_3 .

5.2. Fuzzification by integration

As mentioned above, equations 19 provide exactly the same results as the one obtained by the extension principle in the case of fuzzification of binary adjacency. Therefore we consider here only the fuzzification method based on integration over the α -cuts.

Using equation 17, we obtain:

$$\mu_{adj}(\mu, \nu) = \int_0^1 Adj(\mu_\alpha, \nu_\alpha) d\alpha. \tag{31}$$

If we consider that μ_α and ν_α are not adjacent if the intersection of their interior is not empty, as stated in definitions 1 and 5, this equation is of limited interest. For instance, in the example in Figure 6, $Adj(\mu_\alpha, \nu_\alpha) = 1$ only for one value of α . If we consider a weaker definition of adjacency, where we ignore the condition on intersection (i.e. we enlarge the notion of adjacency to that of connectedness), then integrating this relationship on the α -cuts is again equivalent to the equation provided by the extension principle. This is due to the fact that, for $\alpha < \alpha'$, if $\mu_{\alpha'}$ and $\nu_{\alpha'}$ are connected, then so are μ_α and ν_α (since $\mu_{\alpha'} \subset \mu_\alpha, \nu_{\alpha'} \subset \nu_\alpha$).

Using a double integration, according to equation 18, again we obtain a formula of interest only by considering weak adjacency. However, there is no more equivalence with the extension principle, and computation is much more expensive.

6. Formal translation of binary adjacency equations into fuzzy adjacency

In this Section, we make use of the principle shortly described in Section 4.2 in order to define a degree of adjacency between two fuzzy sets μ and ν . As above, in the continuous case, fuzzy sets are supposed to be defined on \mathbb{R}^n as u.s.c. functions, while in the discrete case, they are defined on \mathbb{Z}^n endowed with a discrete connectivity.

Since binary definitions always involve constraints on the intersection of the two sets and a notion of neighborhood, the first two Subsections are dedicated to defining fuzzy equivalents of these concepts. Then, we extend definition 5, using only neighborhood relationships, and in the next Subsection, we add boundary constraints, as introduced in property 9. We also consider fuzzy adjacency derived from fuzzy dilation and from fuzzy distance. In Subsection 6.6, we consider the case where the representation of fuzzy sets is provided under the form of a fuzzy partition of the space into objects and object boundaries, obtained for instance from a preliminary segmentation or pre-processing step.

6.1. Degree of intersection between two fuzzy sets

The degree of intersection between two fuzzy sets is obtained by translating the set equation $X \cap Y \neq \emptyset$ into fuzzy terms. This equation is equivalent to $\exists x \in \mathbb{R}^n$ (resp. \mathbb{Z}^n), $x \in X \cap Y$). The simplest fuzzy translation provides:

$$\mu_{int}(\mu, \nu) = \sup_x t[\mu(x), \nu(x)], \quad (32)$$

where t is a t-norm. The supremum is taken over the whole space (\mathbb{R}^n or \mathbb{Z}^n).

A degree of empty intersection (or of disjointness) is then derived as:

$$\mu_{\neg int}(\mu, \nu) = c[\mu_{int}(\mu, \nu)], \quad (33)$$

where c is a fuzzy complementation (for instance defined as $\forall a \in [0, 1], c(a) = 1 - a$).

This form has already been widely used in the fuzzy set literature. In particular, it is often interpreted as a degree of conflict between two fuzzy sets or two possibility distributions¹³.

However this form is not always adequate for image processing purposes since it does not include any spatial information.

The degree of intersection and of non-intersection can therefore be reformulated in order to better represent the notion of spatial overlapping. This also avoids counter-intuitive results as mentioned in Section 5 (see Figure 7). The expression $\sup_x t[\mu(x), \nu(x)]$ only represents the maximum height of the intersection. Although it is generally low for fuzzy sets that have almost disjoint supports, its value does not account for different overlapping situations, as illustrated by Figure 7. A better

solution consists in defining a degree of intersection by considering the fuzzy hypervolume of the intersection. This also corresponds to a translation process, in the sense that we have:

$$X \cap Y = \emptyset \Leftrightarrow V_n(X \cap Y) = 0. \tag{34}$$

For defining the hypervolume of a fuzzy set, we simply use the classical fuzzy cardinality. This provides for a fuzzy set μ (having bounded support) in the discrete case:

$$V_n(\mu) = \sum_{x \in \mathbb{Z}^n} \mu(x), \tag{35}$$

and in the continuous case:

$$V_n(\mu) = \int_{x \in \mathbb{R}^n} \mu(x). \tag{36}$$

From the hypervolume of $t(\mu, \nu)$, we can derive a degree of intersection in $[0, 1]$. It should be equal to 0 if μ and ν have completely disjoint supports, high if one set is included in the other, and increasing with respect to the hypervolume of the intersection. Therefore, we can use the functions proposed in Section 4.4 for transforming the volume into a degree of intersection. The following definition (using e.g. equation 25) satisfies these requirements:**

Definition 11 *The degree of intersection between two fuzzy sets μ and ν , depending on the hypervolume of their intersection, is defined by:*

$$\mu_{int}(\mu, \nu) = \frac{V_n[t(\mu, \nu)]}{\min[V_n(\mu), V_n(\nu)]}. \tag{37}$$

Again a degree of non-intersection can be derived from this expression using equation 33.

Property 14 *The intersection degrees defined by equation 32 and definition 11 are both consistent with the binary definition and invariant with respect to geometrical transformations.*

In the example shown in Figure 9, μ and ν have the same degree of intersection according to equation 32 (maximum of the intersection, equal to 0.63 in this case) than μ and ν' . Using the fuzzy volume as in definition 11, we obtain $\mu_{int}(\mu, \nu) = 0.31$ and $\mu_{int}(\mu, \nu') = 0.66$. This corresponds well to the fact that μ and ν' have a larger overlap than μ and ν .

In the following definitions of fuzzy adjacency, we may use either expressions derived from the height of the intersection or expressions involving the fuzzy hypervolume of the intersection. We will see that this leads to different adjacency degrees in situations like the one of Figure 9.

**Other definitions leading to similar properties are possible.

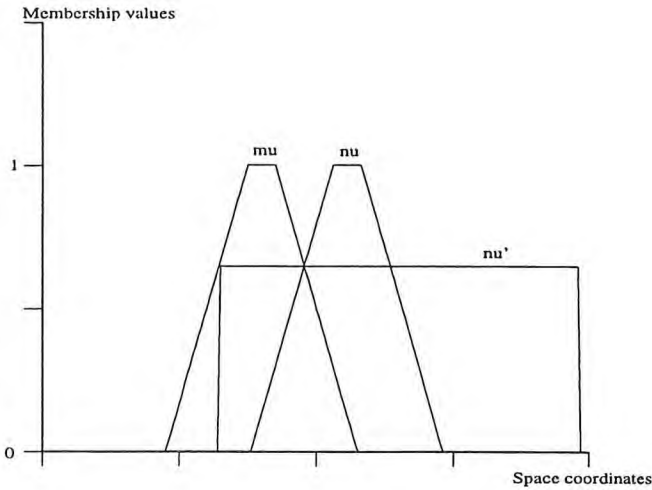


Figure 9: μ and ν have the same degree of intersection than μ and ν' using the maximum of the intersection, while they have a lower one using the fuzzy hyper-volume.

6.2. Fuzzy neighborhood

In this Section, we examine several possibilities for defining a degree of neighborhood n_{xy} between two points x and y in \mathbb{Z}^n endowed with a discrete connectivity or in \mathbb{R}^n . They will be used in the definitions of fuzzy adjacency.

Let us first consider binary definitions of n_{xy} . In the continuous case, we set $n_{xy} = 1$ if $x \in V(y)$, $n_{xy} = 0$ otherwise. In the discrete case, we set $n_{xy} = 1$ if x and y are neighbors in the sense of the considered discrete connectivity, and $n_{xy} = 0$ otherwise (i.e. $n_{xy} = n_c(x, y)$). With these definitions, the consistency with the binary case is guaranteed.

We now concentrate on fuzzy versions of n_{xy} .

A first definition has been proposed in ¹⁰, which depends on the distance between x and y . Let us denote by $d(x, y)$ the Euclidean distance on \mathbb{Z}^n (resp. \mathbb{R}^n).

Definition 12 *The degree of neighborhood between any two points x and y is defined in ¹⁰ as:*

$$n_{xy} = \frac{1}{1 + d(x, y)}. \tag{38}$$

This definition is illustrated on the left of Figure 10. It has the drawback to provide a degree of 0.5 for two points being at distance 1 (i.e. two neighbors in the discrete case), leading therefore to a definition which is not consistent with the binary one. This could be avoided in the discrete case by considering this formula only for $x \neq y$ (then $d(x, y) \geq 1$) and by multiplying all values by a factor 2.

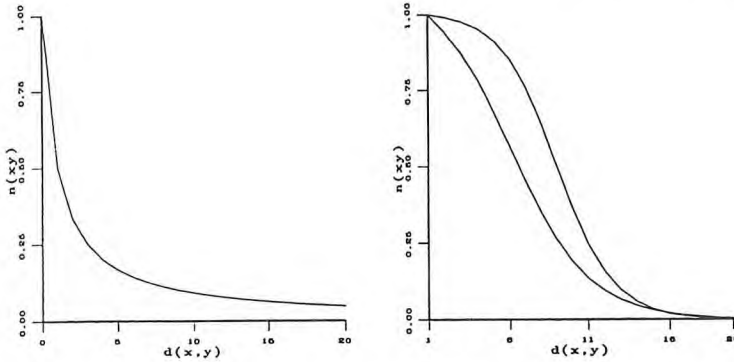


Figure 10: Illustration of two possible definitions for n_{xy} . Left: definition 12 ¹⁰. Right: definition 13 for two different sets of parameters b and S .

We propose here another definition, which avoids this problem. We have chosen a parameterized function, as suggested in Section 4.4 (equations 23 and 24), which allows for flexibility in the spatial extent of the fuzzy neighborhood. Of course, other functions sharing these properties could be used (see e.g. the S-function in ²⁰).

Definition 13 *The degree of neighborhood between any two points x and y is defined in the discrete case as:*

$$n_{xy} = f_{b,S}^d[d(x,y)] = \frac{1 + \exp(-b)}{1 + \exp b(\frac{d(x,y)-1}{S} - 1)}, \tag{39}$$

and in the continuous case as:

$$n_{xy} = f_{b,S}^c[d(x,y)] = \frac{1 + \exp(-b)}{1 + \exp b(\frac{d(x,y)}{S} - 1)}, \tag{40}$$

where b and S are two positive parameters controlling respectively the flattening (or slope) and the width of the curve representing n_{xy} as a function of $d(x,y)$.

This definition is illustrated on the right of Figure 10, for two different values of b and S .

These definitions of fuzzy neighborhood depend on local information and are therefore not invariant with respect to scaling. If this property is mandatory for the application at hand, it can be satisfied by simply normalizing the distance $d(x,y)$ by the scaling factor. Invariance with respect to translation and rotation are always satisfied.

In the discrete case, the Euclidean distance can be replaced by any discrete distance, for instance using a chamfer algorithm. This allows to adapt the distance to the type of space digitization, to the considered discrete connectivity, and to possible anisotropy of the data (see e.g. ¹⁹).

Both definitions 12 and 13 rely only on spatial information. Another point of view could be to combine this spatial information with the membership of x and y to the considered fuzzy sets. We suggest to do this in one of the two following ways:

- the first one consists in defining a fuzzy neighborhood as a fuzzy structuring element, and to use the result of the fuzzy dilation as degree of neighborhood (this has for instance been used for introducing spatial imprecision in fuzzy sets in ⁶);
- the second one consists in combining one of the previous definitions with the degree of connectivity defined in ²⁴, using a t-norm.

6.3. Using neighborhood constraints

We propose to fuzzify definition 5 by combining a degree of empty intersection ($X \cap Y = \emptyset$) with a degree of existence of neighbors ($\exists x \in X, \exists y \in Y, n_c(x, y)$) using a t-norm t (expressing the simultaneous satisfaction of both conditions).

For the first part, we can use a degree of non-intersection derived either from the height of the intersection or from its fuzzy hypervolume, as suggested in Section 6.1.

For the second part, existence is translated by means of a supremum (taken over the whole space), leading to:

$$\sup_x \sup_y t[\mu(x), \nu(y), n_{xy}], \tag{41}$$

where n_{xy} represents the degree to which x and y are neighbors!^{††}It can be either crisp or fuzzy (as defined in Section 6.2).

Finally, we obtain the following definition for fuzzy adjacency.

Definition 14 *The degree of adjacency between μ and ν involving only neighborhood constraints is defined as:*

$$\mu_{adj}(\mu, \nu) = t \left[\mu_{\neg int}(\mu, \nu), \sup_x \sup_y t[\mu(x), \nu(y), n_{xy}] \right]. \tag{42}$$

Property 15 *This definition is symmetrical, consistent with the discrete binary definition (i.e. in the case where μ and ν are crisp and $n_{xy} = n_c(x, y)$), and decreasing with respect to the distance between the two fuzzy sets. It is invariant with respect to geometrical transformations (for scaling, only if n_{xy} is itself invariant).*

For instance in Figure 11, this definition provides a degree of adjacency equal to 1 if we use 8-connectivity, and equal to 0 if we use 4-connectivity.

Property 16 *In the case of binary μ and ν , definition 14 is equivalent to definition 7 for non intersecting sets if n_{xy} is a decreasing function of $d(x, y)$, as is the case for the definitions of n_{xy} proposed in Section 6.2.*

^{††}In such expressions $t(a, b, c)$ stands for $t[t(a, b), c]$. This notation is adopted for sake of simplicity and justified since any t-norm is commutative and associative.



Figure 11: Definition 14 reduces to the binary definition in the crisp case. The degree of adjacency between the black boxes is equal to 1 if we use 8-connectivity, and equal to 0 if we use 4-connectivity.

Note that although this definition is a direct translation of definition 5, which had been proposed in the discrete case, it applies as well in a continuous space. However the equivalence with the continuous binary case is satisfied only if the intersection is checked on the interior of the sets (i.e. without boundary), since in the continuous case on closed sets, the definition of adjacency implies that the sets are intersecting only at their boundary. Therefore, the definition involving boundary constraints provided in the next Subsection is more convenient in this case.

Another solution consists in assuming that the considered fuzzy sets constitute a fuzzy partition, e.g. in the sense of Bezdek ¹ (see definition 9). If the two considered fuzzy sets belong to such a partition, it is no more needed to translate the first part of definition 5, and the degree of empty intersection can be ignored. This leads to a simpler expression of fuzzy adjacency, providing not necessarily the same absolute values but the same ranking.

However, obtaining such a partition is not always straightforward. It can be directly obtained by using for instance fuzzy C-means algorithm ¹ to perform the first segmentation, but the constraints of definition 9 can be more difficult to satisfy if the fuzzy segmentation is obtained by any other method.

Property 17 *In the continuous case, if we set $n_{xy} = 1$ if $x = y$, $n_{xy} = 0$ otherwise, the second term in definition 14 is equivalent to the definition of t -adjacency introduced in Section 5 (definition 10).*

Property 18 *In the discrete case, if we set $n_{xy} = 1$ if x and y are neighbors in the sense of the considered discrete connectivity, and $n_{xy} = 0$ otherwise, then the second term in definition 14 is equivalent to definition 8 obtained from the extension principle, for t taken as the minimum.*

In both cases, if we assume that we have a fuzzy partition, and thus ignore the first term in definition 14, we obtain exactly the same definition as the one derived from the extension principle.

Therefore, definition 14 appears as a more general definition than definition 8

in the following sense:

- it includes more information through the degree of empty intersection that weights the degree of adjacency, which decreases when the degree of intersection between μ and ν increases,
- the choice of a fuzzy n_{xy} allows to give more or less importance to the neighborhood relationships,
- the choice of the t-norm provides a flexibility in the degree of strictness we want to attach to the adjacency relationship,
- definition 8 is a particular case of definition 14 when:
 - degree of empty intersection is ignored,
 - n_{xy} is crisp and adequately chosen,
 - the t-norm is the minimum (otherwise we obtain the extended definition of t-adjacency).

Using the fuzzy hypervolume in the definition of μ_{-int} leads in the case illustrated by Figure 7 to a lower degree of adjacency between μ and ν' than that between μ and ν . This is in better agreement with the intuition than using the height of the intersection.

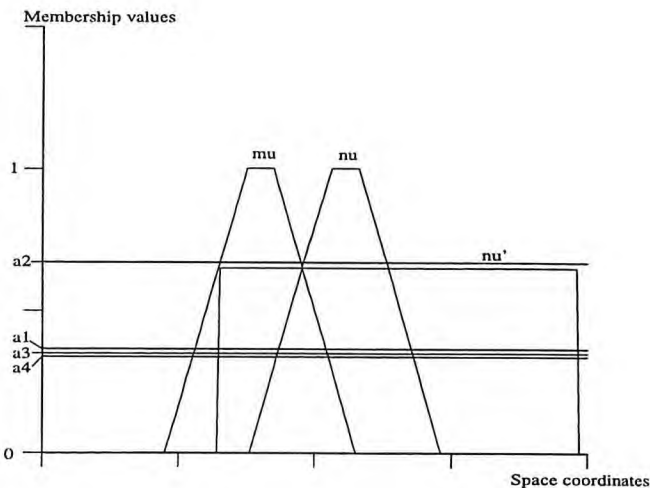


Figure 12: Illustration of definition 14 when using different definitions for the degree of intersection. Using the maximum of the intersection we obtain $\mu_{adj}(\mu, \nu) = \alpha_1 (= 0.36)$ and $\mu_{adj}(\mu, \nu') = \alpha_3 (= 0.35)$, and using the fuzzy hypervolume $\mu_{adj}(\mu, \nu) = \alpha_2 (= 0.67)$ and $\mu_{adj}(\mu, \nu') = \alpha_4 (= 0.34)$.

Let us consider the example of Figure 9. Figure 12 illustrates the results obtained with definition 14 with the t-norm minimum and both definitions of degree of intersection. Using the maximum of the intersection we obtain $\mu_{adj}(\mu, \nu) = 0.36$ and $\mu_{adj}(\mu, \nu') = 0.35$, which are very similar values. On the contrary, using the fuzzy hypervolume, definition 14 accounts for the differences in intersection and provides $\mu_{adj}(\mu, \nu) = 0.67$ and $\mu_{adj}(\mu, \nu') = 0.34$, which are this time very different.

Let us consider now a 2D example. Figure 13 shows on the left a slice of a magnetic resonance (MR) image of the human brain, where several structures have been segmented and serve as a model (or atlas), and on the right a slice (at approximately the same level) of another MR image where the same structures have to be recognized. A rough fuzzy segmentation of this image is shown in Figure 14. The adjacency degrees between some of the obtained fuzzy objects are given in Table 1. These are obtained using definition 14 using the maximum of intersection as intersection degree and the t-norm minimum. 4-connectivity was used. The results are in agreement with what can be expected from the model. In this case, crisp adjacency would provide completely different results in the model and in the image, such preventing its use for recognition. This suggests that fuzzy adjacency degree can indeed be used for pattern recognition purposes, of course combined with other spatial relationships.



Figure 13: MR image of a brain with a few segmented structures (left). MR image of another brain (right).

6.4. Adding boundary constraints

Another way to extend fuzzy adjacency from definition 5 consists in introducing a constraint on the boundary of the considered sets, as given by property 9, i.e. the neighbor points involved in definition 5 are on the boundary of the sets.

A similar work has already been done in ¹⁰. In that paper, the authors successively define:

- a fuzzy neighborhood depending on the distance to the considered point (the

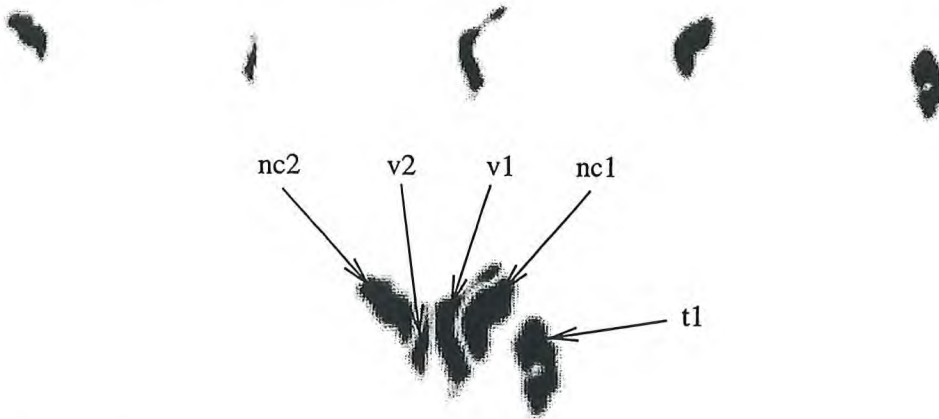


Figure 14: Top: 5 fuzzy objects resulting from a rough fuzzy segmentation of the right image of Figure 13 (membership values rank between 0 and 1, from white to black). Bottom: superposition of these fuzzy objects (the maximum membership value is displayed at each point) and labels (used in Table 1).

Fuzzy object 1	Fuzzy object 2	Degree of adjacency	Adjacency in the model (binary)
v1	v2	0.368	1
v1	nc1	0.463	1
v1	t1	0.000	0
v1	nc2	0.035	0
v2	nc2	0.427	1
nc1	t1	0.035	0

Table 1: Results obtained using definition 14 with the maximum of intersection as intersection degree, the t-norm minimum, and 4-connectivity. Labels of structures are given in Figure 14. High degrees are obtained between structures where adjacency is expected (i.e. between v1 and v2, v1 and nc1, v2 and nc2, according to the model), while very low degrees are obtained in the opposite case.

idea of defining a fuzzy neighborhood instead of just taking the crisp discrete definition is very interesting and inspired our work in Section 6.2);

- the membership $\mu_C(R_i)$ to the "contour" of a fuzzy set R_i , using a fuzzy translation (as in Section 4.2) of the binary definition of a contour (a point belongs to the contour of a set A if it belongs to A and its neighborhood contains at least one point in A^C);
- the membership to the "frontier" between two fuzzy regions $\mu_F(R_i, R_j)$, again by translating the equivalent binary notion (a point belongs to the frontier if it is a contour point of R_i and its neighborhood contains at least one contour

point of R_j).

From these definitions, they deduce a degree of adjacency between two fuzzy regions α_{ij} as the fuzzy cardinality of $\mu_{F(R_i, R_j)}$ normalized by the fuzzy cardinality of $\mu_{C(R_i)}$.

Although the approach is very attractive, these definitions suffer from several drawbacks, with respect to the requirements we imposed in this paper:

- $\mu_{F(R_i, R_j)}$ is not symmetrical;
- α_{ij} is not symmetrical;
- the definition is not consistent with the binary case (according to the definitions we used in the previous Sections): if R_i and R_j are binary sets, their degree of adjacency is equal to the number of points of $C(R_i)$ that are adjacent to R_j , divided by the number of points of $C(R_i)$ (therefore it is not equal to 1 if R_i and R_j are strictly adjacent according to definition 5, and not equal to 0 if the sets intersect);
- α_{ij} is equal to 1 if $R_i = R_j$ in the binary case;
- α_{ij} represents rather the relative length (or surface) of the adjacency;
- α_{ij} does not depend on any condition on the intersection between both regions (however, the application mentioned in ¹⁰ concerns fuzzy objects obtained using fuzzy C-means algorithm, and therefore building a fuzzy partition; thus, for this application, the condition on intersection is not necessary, as mentioned in the previous Section; but the definition would not apply to arbitrary fuzzy regions);
- α_{ij} does not satisfy our requirement that it ought to depend on the distance between the two fuzzy sets.

Based on these remarks, we propose a new definition that overcomes these drawbacks and better matches our requirements.

Instead of defining fuzzy contours, frontiers and neighborhood, our approach consists in defining only the fuzzy boundary of a fuzzy set, which is then combined with neighborhood relationship.

Definition 15 *The fuzzy boundary of a fuzzy set μ is defined by the membership function b_μ as:*

$$\forall x \in \mathbb{Z}^n, b_\mu(x) = t[\mu(x), \sup_{z \in \mathbb{Z}^n} t[c(\mu)(z), n_{xz}]], \tag{43}$$

where t is a t -norm, c a fuzzy complementation, and n_{xz} the degree to which x and z are neighbors. A similar definition can be given in \mathbb{R}^n .

This definition is illustrated by Figure 15.

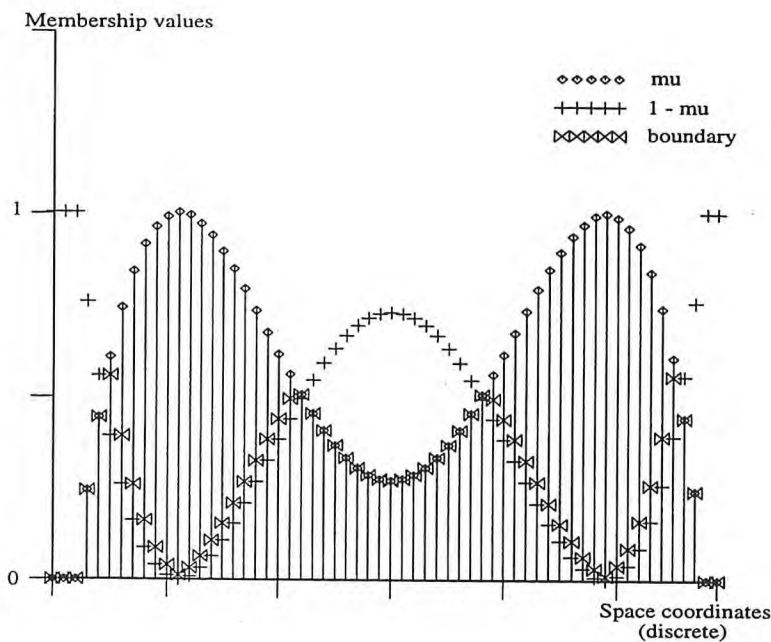


Figure 15: Illustration of the fuzzy boundary b_μ of a fuzzy set μ in the discrete one-dimensional case (with the complementation $1 - \mu$, $n_{xz} = n_c(x, z)$, and the t-norm minimum).

Property 19 *In the binary case (μ and n_{xz} binary), this definition is consistent with the classical definition of the boundary of a crisp set X (set of points of X that have a neighbor in X^C). It is also invariant with respect to geometrical transformations (for scaling, only if n_{xy} is itself invariant).*

The translation of definition 5 along with the property on boundary leads now to the following definition:

Definition 16 *The degree of adjacency between μ and ν involving neighborhood and boundary constraints is defined by:*

$$\mu_{adj}(\mu, \nu) = t \left[\mu_{\neg int}(\mu, \nu), \sup_x \sup_y t[b_\mu(x), b_\nu(y), n_{xy}] \right], \tag{44}$$

where the supremum is taken over \mathbb{Z}^n .

This definition is illustrated in Figure 16 by the same example as in Figure 8.

Again, as in the previous Subsection, if we assume that the considered fuzzy sets constitute a fuzzy partition, we can ignore the first term corresponding to the degree of empty intersection.

Property 20 *This definition is symmetrical, consistent with the binary definition if μ, ν and n_{xy} are binary, and decreases if the distance between μ and ν increases.*

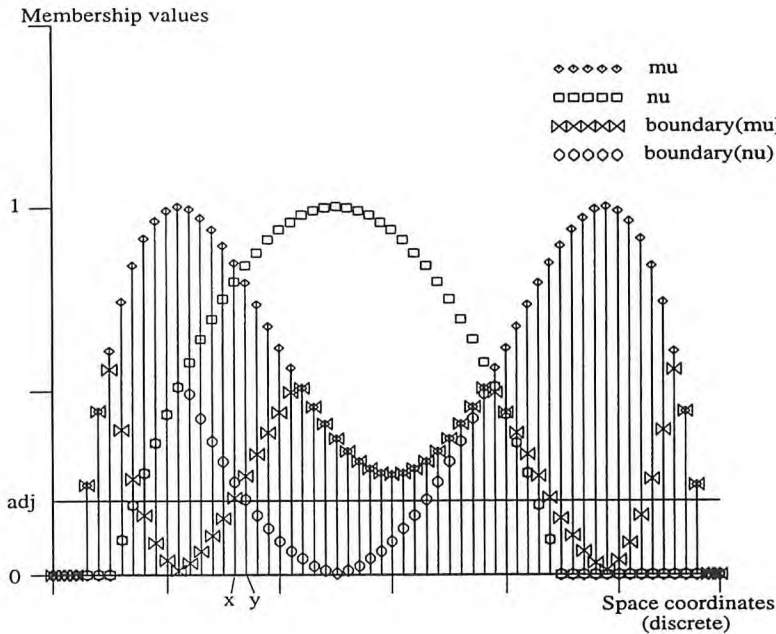


Figure 16: Fuzzy boundary b_μ and b_ν of the two fuzzy sets μ and ν of Figure 8 in the discrete one-dimensional case (with the complementation $1 - \mu$ and the t-norm minimum). Fuzzy adjacency is then equal to adj according to definition 16, which is less than α_2 provided by definition 14. The neighbor points for which the adjacency value is attained are again x and y .

It is invariant with respect to geometrical transformations (for scaling, only if n_{xy} is itself invariant).

Property 21 *This definition is equivalent to definition 14 in the binary case. In the fuzzy case, it is more severe, i.e. leads to a lower degree of adjacency.*

The differences between definition 14 and definition 16 are illustrated by Figure 17 and compared with the results derived from extension principle.

Property 22 *In the case of binary μ and ν , definition 16 is equivalent to definition 7 for non intersecting sets if n_{xy} is a decreasing function of $d(x, y)$, as is the case for the definitions of n_{xy} proposed in Section 6.2.*

In the continuous case, adjacency between closed sets means that they are intersecting only at their boundary. This can be expressed as:

$$\partial X \cap \partial Y \neq \emptyset \text{ and } (\partial X)^C \cap (\partial Y)^C = (X \cup Y)^C. \tag{45}$$

This can be translated as a conjunction of three terms, one representing the degree of intersection between the boundaries, and the two others the inclusion of the intersection of the complementary of boundaries in the complementary of the union

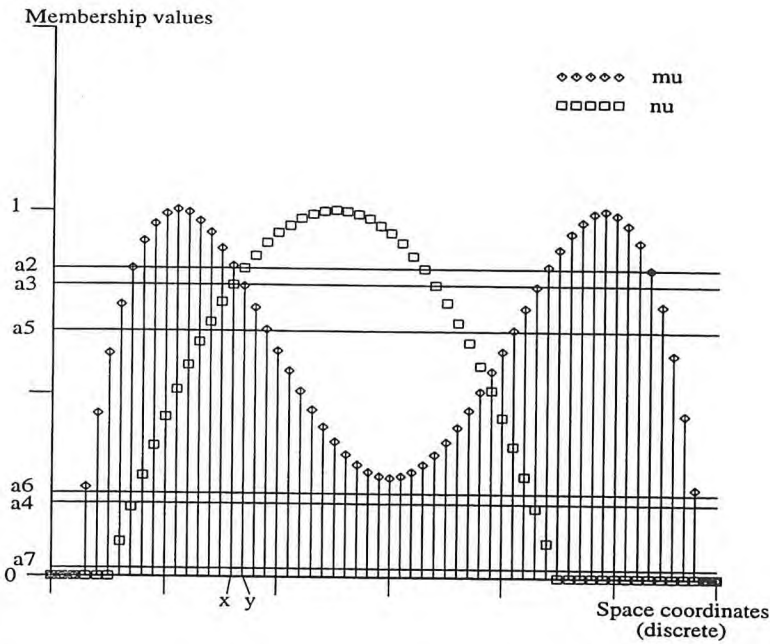


Figure 17: Comparison of the definitions of degree of adjacency between the two fuzzy sets μ and ν of Figure 8 in the discrete one-dimensional case (with the complementation $1 - \mu$). Degree of non-intersection is computed using the maximum height of the intersection. The extension principle provides α_2 . The maximum of the intersection is equal to α_3 . Definition 14 and definition 16 provide a degree of adjacency equal to α_4 with the t-norm minimum. Using the product as t-norm, we obtain α_5 with the extension principle, α_6 with definition 14 and α_7 (which is lower than α_6) with definition 16.

of both sets and the reverse, which can be expressed using the degree of non-intersection (since $A \subset B \Leftrightarrow A \cap B^C = \emptyset$). This leads to the following definition of adjacency in the continuous case:

Definition 17 *The adjacency between two fuzzy sets defined on \mathbb{R}^n involving boundary constraints is expressed as:*

$$\mu_{adj}(\mu, \nu) =$$

$$t[\mu_{int}(b_\mu, b_\nu), \mu_{-int}[c[T(b_\mu, b_\nu)], T(\mu, \nu)], \mu_{-int}[T(b_\mu, b_\nu), c[T(\mu, \nu)]]], \quad (46)$$

where t is a t-norm, c a complementation and T the t-conorm dual of t with respect to c .

Note that in this definition, the notion of neighborhood only appears in that of boundary.

Property 23 *This definition is symmetrical, and consistent with the binary definition if μ, ν are binary. It is moreover invariant with respect to geometrical transformations.*

6.5. Using fuzzy morphological operators

In a morphological context, it may also be interesting to define adjacency from fuzzy dilation, by translating property 10 into fuzzy terms. A direct translation of this property leads to the following definition.

Definition 18 *The degree of adjacency between μ and ν involving fuzzy dilation is defined as:*

$$\mu_{adj}(\mu, \nu) = t[\mu_{\neg int}(\mu, \nu), \mu_{int}[D(\mu, B_c), \nu], \mu_{int}[D(\nu, B_c), \mu]]. \quad (47)$$

This definition represents a conjunctive combination of a degree of non-intersection between μ and ν and a degree of intersection between one fuzzy set and the dilation of the other. B_c can be taken as the elementary structuring element related to the considered connectivity, or as a fuzzy structuring element, representing for instance spatial imprecision (i.e. the possibility distribution of the location of each point). This definition is illustrated by Figure 18 and shows that the same result is obtained as with definition 14.

Property 24 *This definition is symmetrical, consistent with the binary definition if μ, ν and B_c are binary, and decreases if the distance between μ and ν increases.*

Fuzzy dilation can also serve for defining the fuzzy boundary of a fuzzy set, as follows.

Definition 19 *The fuzzy boundary b_μ of a fuzzy set μ is defined from fuzzy dilation as:*

$$b_\mu(x) = t[\mu(x), D(c(\mu), B_c)(x)]. \quad (48)$$

Property 25 *This definition is equivalent to definition 15 if the structuring element is consistent with the choice of the fuzzy neighborhood (typically if we take the elementary structuring element defined from the discrete connectivity used in a binary definition of n_{xy}).*

Property 26 *The definition of adjacency obtained from this boundary definition (with any structuring element) is still symmetrical, consistent with the binary case and decreasing when the distance between both fuzzy sets increases.*

Another link between adjacency and mathematical morphology in the continuous case is given by the observation that adjacent sets are those whose boundaries have a non empty intersection while their eroded sets have an empty intersection. The translation in fuzzy terms leads to the following definition:

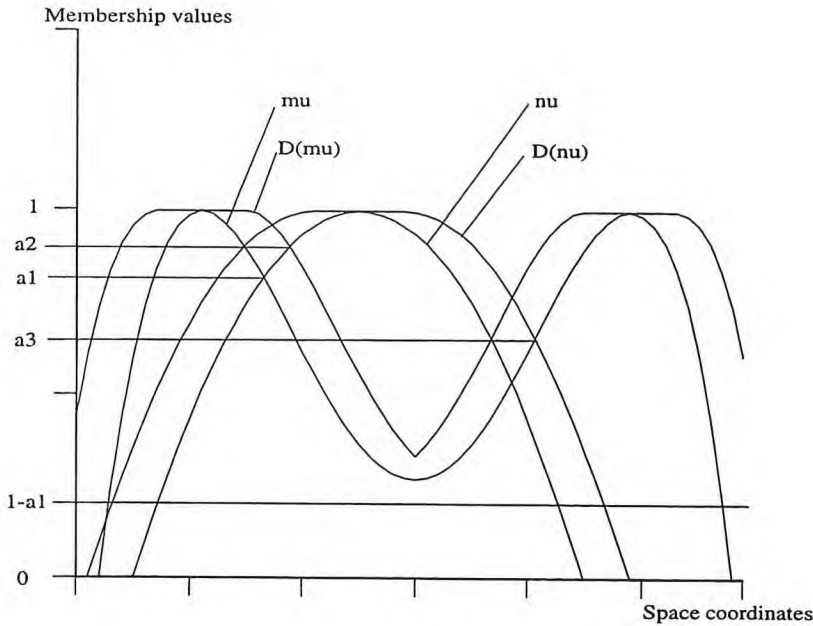


Figure 18: Illustration of the use of fuzzy mathematical morphology on the same example than in Figure 17. The conjunction of the degree of non-intersection between μ and ν ($1 - \alpha_1$) and a degree of intersection between one fuzzy set and the dilation of the other (α_2 and α_3) leads to the same result as with definition 14 (here $1 - \alpha_1$ when using the t-norm minimum). (On this figure, the dilation has been amplified for the sake of legibility.)

Definition 20 *The degree of adjacency between μ and ν defined on \mathbb{R}^n involving fuzzy erosion is defined as:*

$$\mu_{adj}(\mu, \nu) = t[\mu_{int}[b_\mu, b_\nu], \mu_{-int}[E(\mu, B_c), E(\nu, B_c)]], \tag{49}$$

where $E(\mu, B_c)$ denotes the fuzzy erosion of μ by B_c , where B_c is either the elementary structuring element or a fuzzy structuring element.

Property 27 *The definition of adjacency obtained from fuzzy erosion is still symmetrical, consistent with the binary case and decreasing when the distance between both fuzzy sets increases.*

This definition applies as well to the discrete case if we consider that adjacent sets are intersecting just at their boundary.

Finally, since in the discrete binary case equation 12 means that the minimum (nearest point) distance between X and Y is equal to 1, we can also exploit this fact in the fuzzy case, by using the fuzzy minimum distance, defined from fuzzy dilation as in ². We do not go into further details for this approach, since it leads

to similar definitions, sharing the same properties as the previous ones.

6.6. Image partition in objects and boundaries of objects

In this Section, we consider that the space is partitioned into objects and a set representing the boundary of these objects. In the fuzzy case, we consider a fuzzy partition, in the sense of ¹ (see definition 9), in fuzzy objects μ_i and a fuzzy set μ_B representing the boundary of the μ_i 's. Then definition 6 is simply translated as follows.

Definition 21 *The degree of adjacency between μ_i and μ_j involving a fuzzy partition into objects and boundary of objects is defined as:*

$$\mu_{adj}(\mu_i, \mu_j) = \sup_z t[\mu_B(z), \sup_x \sup_y t[\mu_i(x), \mu_j(y), n_{xz}, n_{yz}]]. \quad (50)$$

Property 28 *This definition is symmetrical and consistent with its binary equivalent (definition 6) (i.e. if the membership functions and the neighborhood function are binary).*

However, obtaining such a partition is not always simple, and therefore this definition is likely to give rise to heavy computation.

7. Similarity measure between adjacency values

The main application that is anticipated for this work concerns model-based pattern recognition. Recognition will be based both on properties of objects themselves and also on relationships between objects. Adjacency is such a relationship. Typically, we may think of a graph-based approach, where interpretation of an image graph is inferred from a model graph. Nodes in these graphs will be regions, objects or object parts, represented as spatial fuzzy sets, while arcs will represent several kinds of fuzzy relationships, including adjacency.

In order to perform a fuzzy relaxation, that would allow us to label the image graph according to the model, similarity measures between nodes and between arcs are needed. Objects and their relationships may be represented as a collection of features or properties (like shape, size, adjacency, relative position, etc.), and similarity between the model and the image has to take into account all these features and properties (see e.g. ²⁹). In this Section, we define measures of similarity for comparing adjacency relationships between model objects and image objects.

Let us denote by \mathcal{M} the model and \mathcal{I} the image (of any dimension) to be recognized, and by $\mu_{\mathcal{M}}, \nu_{\mathcal{M}}$ (resp. $\mu_{\mathcal{I}}, \nu_{\mathcal{I}}$) two (fuzzy) objects in the model (resp. in the image). The previous Sections provide measures of adjacency $\mu_{adj}(\mu_{\mathcal{M}}, \nu_{\mathcal{M}})$ and $\mu_{adj}(\mu_{\mathcal{I}}, \nu_{\mathcal{I}})$, whatever the chosen definition. A measure of similarity between these two values has to check if the arcs between $\mu_{\mathcal{M}}$ and $\nu_{\mathcal{M}}$ on the one hand, and $\mu_{\mathcal{I}}$ and $\nu_{\mathcal{I}}$ on the other hand are of the same type, i.e. there is a similar relation. This provides a measure of the quality of a possible matching between $\mu_{\mathcal{I}}$ and $\mu_{\mathcal{M}}$ on the one hand, and $\nu_{\mathcal{I}}$ and $\nu_{\mathcal{M}}$ on the other hand, according to the arcs. Indeed,

this matching is more plausible if the relationships between $\mu_{\mathcal{I}}$ and $\nu_{\mathcal{I}}$ are similar to the ones between $\mu_{\mathcal{M}}$ and $\nu_{\mathcal{M}}$. Let us denote by $s_{adj}[(\mu_{\mathcal{I}}, \nu_{\mathcal{I}}), (\mu_{\mathcal{M}}, \nu_{\mathcal{M}})]$ this measure.

Similarity measures (or t-indistinguishability measures) have been defined in ³² in the following way:

Definition 22 ³² A similarity measure s between fuzzy sets satisfies the three following properties (for any fuzzy sets μ, ν, ξ):

- reflexivity: $s(\mu, \mu) = 1$;
- symmetry: $s(\mu, \nu) = s(\nu, \mu)$;
- t-transitivity: $t[s(\mu, \nu), s(\nu, \xi)] \leq s(\mu, \xi)$ (where t is a t-norm).

What are the properties of s_{adj} we need for the anticipated applications?

- if we have exactly the same regions in the image and in the model (and therefore the same values for the degrees of adjacency), we want to recognize $\mu_{\mathcal{I}}$ and $\nu_{\mathcal{I}}$ as $\mu_{\mathcal{M}}$ and $\nu_{\mathcal{M}}$; therefore reflexivity is needed;
- symmetry is also desirable, since we may want to reverse the respective roles of \mathcal{I} and \mathcal{M} ;
- since we compare pairs of objects belonging to two different sets (\mathcal{I} and \mathcal{M}), transitivity has little meaning and we can relax this constraint.

Therefore, more precisely, since we require only reflexivity and symmetry, s_{adj} is a resemblance measure ^{21, 7}.

The simplest definition for s_{adj} is as follows:

Definition 23 A similarity between adjacency relationships is defined as:

$$s_{adj}[(\mu_{\mathcal{I}}, \nu_{\mathcal{I}}), (\mu_{\mathcal{M}}, \nu_{\mathcal{M}})] = 1 - |\mu_{adj}(\mu_{\mathcal{I}}, \nu_{\mathcal{I}}) - \mu_{adj}(\mu_{\mathcal{M}}, \nu_{\mathcal{M}})|. \quad (51)$$

Property 29 The measure proposed in definition 23 is reflexive and symmetrical.

This measure takes high values if the values of adjacency are similar, i.e. of the same order of magnitude. However, for pattern recognition purposes, it may be more interesting to have a measure that is high only if both values are high. Indeed, the fact that two objects are adjacent like in the model is more relevant to recognition than the fact that they are not adjacent like in the model. The following definition can then be more useful for such applications.

Definition 24 A similarity between adjacency relationships is defined as:

$$s_{adj}[(\mu_{\mathcal{I}}, \nu_{\mathcal{I}}), (\mu_{\mathcal{M}}, \nu_{\mathcal{M}})] = f[\mu_{adj}(\mu_{\mathcal{I}}, \nu_{\mathcal{I}}), \mu_{adj}(\mu_{\mathcal{M}}, \nu_{\mathcal{M}})], \quad (52)$$

where f is either a t-norm, or a symmetrical sum with the appropriate behavior.

Convenient symmetrical sums are for instance associative ones (but median) that behave in a conjunctive way if both values are low (less than 0.5), in a disjunctive way if both values are high (greater than 0.5) or else like a compromise¹². An example of such a symmetrical sum is:

$$\forall(a, b) \in [0, 1]^2, \sigma(a, b) = \frac{ab}{1 - a - b + 2ab}. \quad (53)$$

Property 30 *The measure proposed in definition 24 is symmetrical. It is reflexive only if f is an idempotent t -norm (i.e. the minimum).*

These measures are located at an intermediate level, in the sense that they do not apply directly to the considered objects but to some global feature extracted from these objects. In order to cope with the summarization aspect of such a feature, it may be interesting to incorporate in the similarity measure a weight representing the quality of the adjacency. Typically a low confidence should be attached to an adjacency between two objects that concerns only a few points, like on the example on the right of Figure 3.

A possible solution consists in weighting the similarity measure by a normalized length of adjacency. For instance, the definition of adjacency proposed in¹⁰ could be used for this purpose, since it represents rather the relative length of adjacency than the degree of adjacency, as already mentioned.

8. Conclusion

The aim of this research was to investigate notions of fuzzy adjacency that could serve for model-based pattern recognition in image processing under imprecision. We proposed several definitions, both for crisp objects and for fuzzy objects, that show good properties with respect to binary definitions and to the requirements we imposed about their behavior. In our opinion, the most interesting definition is obtained if imprecision is represented directly in the considered objects (i.e. definition on fuzzy objects), and if the hypervolume of the intersection is taken into account (see Section 6 and definitions 14 and 16). This definition have nice properties, and a good behavior with respect to intuitive requirements. Moreover, it allows for a consistent representation and management of imprecision. We have shown that these definitions can be expressed in terms of fuzzy morphological operators. This has two advantages, from an interpretation point of view in terms of spatial imprecision management, and from a theoretical one, since adjacency inherits some interesting properties of fuzzy mathematical morphology. Future work aims at incorporating adjacency to other relationships between fuzzy objects in a structural pattern recognition approach. This may also provide a second level of comparison between the proposed definitions, by looking at the results of a whole recognition process, which can lead to the choice of the most appropriate definition for a specific application.

Appendix 1: Adjacency from admissible transformations

In this Appendix, we give more examples for admissible transformations and

their size, that can be used in definition 7. We already mentioned the case where \mathcal{T} is the set of translations of \mathbb{R}^n or \mathbb{Z}^n . In this case, the minimal size corresponds to the nearest point distance between the two sets X and Y , if they do not intersect. In the other cases, the problem amounts to choosing the appropriate distance between X and Y . In order to fulfill the mentioned requirements on decreasingness, we propose to define $d(X, Y)$ as follows:

- If $X \cap Y = \emptyset$, $d(X, Y)$ is taken as the minimum (or nearest point) distance (see Figure 19 left):

$$d(X, Y) = \inf_{x \in X, y \in Y} d(x, y), \tag{54}$$

where $d(x, y)$ is the Euclidean distance defined on the considered space (this corresponds to the size of the minimal translation to be applied to one set in order to achieve strict adjacency). On compact sets, as well as in a finite discrete space, this distance is attained at the boundary of X and Y . Note that this is not really a distance since separability and triangular inequality are not satisfied. In particular, $d(X, Y) = 0$ does not imply $X = Y$, but only $X \cap Y \neq \emptyset$. However, this is exactly the property we need for defining adjacency from this distance (since we need a null distance as soon as the objects touch each others).

- If $X \cap Y \neq \emptyset$ and none set is included in the other, $d(X, Y)$ is taken as the largest distance between boundary points inside the intersection (see Figure 19 right):

$$d(X, Y) = \sup_{x \in \partial X \cap Y, y \in \partial Y \cap X} d(x, y). \tag{55}$$

This expresses in some way to what extent the objects intersect and how far they are from an intersection reduced to boundary points (which would represent exact adjacency). This form guarantees a continuity with the previous case when X and Y just intersect at their boundaries. However, this definition is not convenient if the two sets have a narrow elongated intersection. In this case, this definition of distance provides a much higher value than the size of the minimal translation that makes the two sets just adjacent.

- If $X \subset Y$ or $Y \subset X$, then the distance is set to infinity, or to a value such that $f(d) = 0$. Unfortunately, continuity with the previous case is then lost.
- In order to preserve continuity, another solution consists in taking for d a function of the hypervolume V_n of the intersection, for instance, for $X \cap Y \neq \emptyset$:

$$d(X, Y) = \frac{1}{1 - \frac{V_n(X \cap Y)}{\min[V_n(X), V_n(Y)]}} - 1. \tag{56}$$

This applies either if none of the sets is included in the other one, or if one is included in the other (in this case $d(X, Y) = +\infty$), and guarantees continuity with the case $X \cap Y = \emptyset$. Other possibilities for deriving a degree in $[0, 1]$

from a spatial measure like the volume are proposed in Section 4.4. The transformation corresponding to such expressions is the one that adds in case of empty intersection (resp. removes in case of intersecting sets) points from (resp. to) one set in order to achieve strict adjacency.

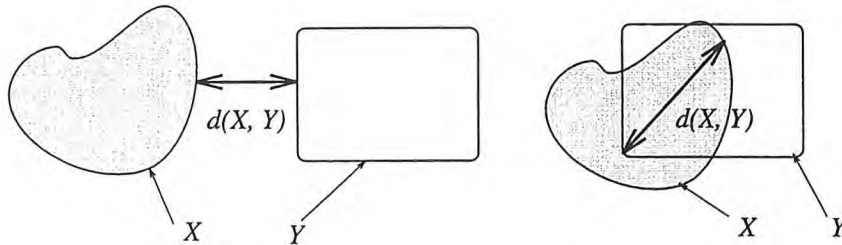


Figure 19: Appropriate distance between crisp sets for defining degrees of adjacency, in case of non intersecting sets (left) and intersecting sets (right).

In the discrete case, similar definitions of the distance are also appropriate. However, since we have chosen to define crisp adjacency between non intersecting sets (see definition 5), $f(d)$ should be chosen in the discrete case such that $f(1) = 1$.

Other transformations can be used, for instance rotations. Indeed, several pattern recognition methods are rotation invariant. If the rotation is defined as an axis and an angle (in 3D), the size of the transformation can be defined as the absolute value of the angle. Another better solution consists in using the isomorphism between the set of 3D rotations and the space of normalized quaternions, and in defining the size of the transformation by the norm of the corresponding quaternion.

Finally, in order to guarantee the invariance property with respect to a scale factor of the image (see Section 4.3), the proposed measures can be easily normalized with respect to this scale factor.

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