Explanatory Reasoning for Image Understanding Using Formal Concept Analysis and Description Logics

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***Abstract*—In this paper, we propose an original way of enriching description logics with abduction reasoning services. Under the aegis of set and lattice theories, we put together ingredients from mathematical morphology, description logics, and formal concept analysis. We propose computing the best explanations of an observation through algebraic erosion over the concept lattice of a background theory that is efficiently constructed using tools from formal concept analysis. We show that the defined operators are sound and complete and satisfy important rationality postulates of abductive reasoning. As a typical illustration, we consider a scene understanding problem. In fact, scene understanding can benefit from prior structural knowledge represented as an ontology and the reasoning tools of description logics. We formulate model based scene understanding as an abductive reasoning process. A scene is viewed as an observation and the interpretation is defined as the best explanation, considering the terminological knowledge part of a description logic about the scene context. This explanation is obtained from morphological operators applied on the corresponding concept lattice.**

***Index Terms*—Description logics, explanatory reasoning, formal concept analysis, image understanding, mathematical morphology.**

I. Introduction

Automatic image interpretation has been an active field of research for several years. In this large field, this paper focuses on extracting high level information from images or video sequences, when the detection and recognition of structures can benefit from prior structural knowledge (such as spatial interactions). This is, in particular, the case in video sequences related to a specific context (sport events for instance), in medical imaging (using anatomical knowledge), or in aerial and satellite imaging (man-made structures such as airports and towns for instance).

Description logics (DL) are an important paradigm of logic-based knowledge representation [1]. They are a decidable family of first-order logics that spans numerous applications in areas such as semantic web, cognitive robotics, spatial reasoning, computer vision, including logical-based scene understanding and semantic interpretation.

Scene interpretation can benefit from prior knowledge expressed as ontologies and from description logics endowed with spatial reasoning tools, as illustrated in our previous work [2], [3]. The challenge in this paper was to derive reasoning tools that are able to handle in a unified way quantitative information supplied by the image domain and qualitative pieces of knowledge supplied by the ontology level. The interpretation task is performed in a sequential way by maintaining the consistency between the information extracted from the image and the corresponding expert knowledge encoded at the terminological level. In other words, object recognition and interpretation are seen as the coherence of a current situation (spatial configuration) encoded in the assertional box (ABox) of the DL with the terminological box (TBox) part. However, when the expert knowledge is not strictly consistent with the observed situation, which is common in image interpretation, then this approach does not apply or leads to inconsistent results. Moreover, in the context of image interpretation, a given structural configuration can be consistent with different prior knowledge parts (or consistent to some degree). These facts call for adapting DL reasoning tools to such situations, and abduction seems to be an appealing framework toward this aim. In the context of nonmonotonic reasoning paradigms in AI, abduction refers to the reasoning process of forming a hypothesis that explains observed phenomena. More precisely, it allows computing the best explanation of the observed phenomena, which suits situations where the knowledge at hand is not strictly consistent with the observations. Formally, given a background theory $\mathcal{K}$ representing the expert knowledge and a formula C representing an observation on the problem domain, abductive reasoning searches for an explanation formula D such that D is satisfiable with respect to $\mathcal{K}$ and it holds that $\mathcal{K} \models (D \rightarrow C)$ ($\mathcal{K} \cup D \models C$).

For readability convenience and in order to illustrate the potential of our approach in the context of image interpretation as well as for other AI-based applications, we will consider two running examples. The first one was introduced by Elsenbroich *et al.* [4] to argue the need of developing computational tools of abduction in the context of ontologies.

Manuscript received February 17, 2012; revised August 23, 2012; accepted December 29, 2012. Date of publication September 23, 2013; date of current version April 11, 2014. This paper was recommended by Associate Editor A. Petrosino.

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Digital Object Identifier 10.1109/TSMC.2013.2280440

Example 1 (SHD): Elsenbroich *et al.* [4] considered the following medical ontology-based diagnosis: suppose a disease, called the shake-hands-disease (SHD), that always develops when one shakes hands with someone else who carries the shake-hands-disease-virus ($SHDV$). Suppose further a medical ontology containing:

1) roles: $has\_symptom$, $carries\_virus$, etc.;
2) concepts: SHD, $SHDV$, $Laziness$, $Pizza\_Appetite$, $Google\_Lover$, etc.;

and a set of axioms (Section III-A) specifying that:

1) if someone has the disease SHD then he or she suffers from laziness and pizza appetite;
2) a researcher is someone that has symptoms laziness, pizza appetite, and Google lover;
3) finally someone who shakes hands with someone who carries the SHDV virus has a disease SHD.

Suppose that one wants to explain why someone has symptoms laziness and pizza appetite. A tailored answer would be that this happens because he or she shakes the hands of someone who carries the shake-hands-disease virus. In Section IV, we will discuss how to computationally come up with such a result in a direct way.

The second example arises from brain image interpretation, and is within the scope of our application domain (Fig. 1). As explained before, the image interpretation task in the framework of ontologies consists of extending the knowledge base with new assertions about the regions of interest in the image and their relations, within their global context. We assume to have at disposal a background theory describing the brain knowledge enriched with spatial relations [2], [5], and a series of image processing algorithms allowing us to extract initial regions of interest from the image. The ABox and the concepts are detailed in Section V. The interpretation task within this context consists of explaining the presence of a nonenhanced tumor located in the peripheral cerebral hemisphere and that is far from the lateral ventricle, by taking into account the background theory on the brain domain and the first objects recognized in the image. A typical answer to this question is that the image represents a brain disease, and this disease is a peripheral small deforming tumor.

In this paper, we propose adding abductive reasoning tools to description logics. Under the aegis of set and lattice theories, we put together ingredients from mathematical morphology, description logics, and formal concept analysis. We propose computing the best explanations of an observation through algebraic erosion over the concept lattice of a background theory that is efficiently constructed using tools from formal concept analysis. This paper extends and develops preliminary ideas presented in [6]. We show that the defined operators satisfy important rationality postulates of abductive reasoning.

We first motivate our approach by the need of explicit human expert knowledge in the image interpretation process (Section II). We then introduce in Section III the necessary background for constructing the abductive engine of a known description logic, $\mathcal{EL}$. Section IV is dedicated to the introduction of mathematical morphology operators on concept lattices, and the definition of explanatory relations. This is the core section of this paper since we introduce two new original classes of operators that are proven to satisfy the rationality postulates of explanatory reasoning. The proposed approach is illustrated on the brain example in Section V and is discussed with respect to related work and regarding some of its features in Section VI. Finally, we draw some conclusions and point out some future research directions.

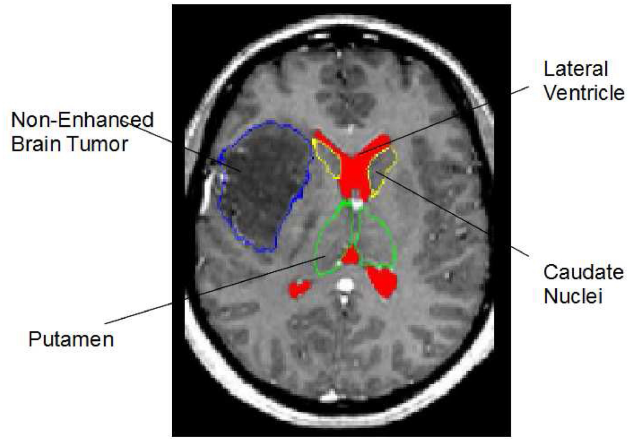


Fig. 1. Example of cerebral image interpretation problem. The interpretation problem consists of explaining the presence of objects such as a nonenhanced brain tumor, located in the cerebral hemisphere and far from the lateral ventricle. A typical solution is that the image represents a brain disease, and this disease is a peripheral small deforming tumor.

II. Human Expert Knowledge

In many domains, images are a very important source of information. Hence, automatic image understanding has been an active field of research for several years to extract meaningful content and provide higher level description and interpretation. Two major approaches coexist for image interpretation: the numerical and statistical methods, and the model-based methods. Nevertheless, major problems still remain open and the research on automatic image interpretation calls for intensive investigation and concerns. In particular, one challenging issue is to extract high level semantics from an image in a form that is close to and suitable for application domain decision making. This issue is often defined as the semantic gap [7]. Indeed, the importance of semantics in images has been highlighted in different domains, such as scene analysis, image interpretation but also image retrieval. In numerical approaches, *a priori* knowledge is often related to perceptual manifestations of semantics. Nevertheless, in many image interpretation domains, the image semantics cannot be considered being included explicitly in the image itself. It rather depends on prior knowledge on the domain and on the context of use of the image. Introducing explicit human expert knowledge in the image interpretation process is not a new idea, as evidenced by the numerous works on knowledge based systems for computer vision [8]–[12]. However, these types of approaches suffer from several shortcomings, in particular because of the lack of genericity (many systems are rather ad hoc), and the difficulty and the cost of acquiring and representing prior knowledge.

Recent developments in the field of knowledge engineering, including ontology engineering, allow answering some of these questions [13]. Ontologies are defined as a formal, explicit specification of a shared conceptualization [14]. An ontology encodes a partial view of the world with respect to a given domain. It is composed of a set of concepts, their definitions, and their relations that can be used to describe and reason about a domain. Ontological modeling of knowledge and information is crucial for conceptual modeling in many real world applications such as medicine for instance [15], [16], or geosciences [17].

Moreover, ontological reasoning can also be used to formulate image interpretation tasks. For instance, Dasiopoulou *et al.* [18] and Meghini *et al.* [20] proposed using uncertain ontological reasoning (through fuzzy description logics) to evaluate the consistency of the interpretation obtained with statistical learning techniques. Explicit semantics, represented by ontologies, have also been intensely used in the field of image and video indexing and retrieval [21], [22]. In most of these approaches, only the descriptive part of ontologies is used, as a common multilevel language to describe image content [23], or more recently as hierarchical semantic concept networks to refine image annotation [24] or to perform image classification [22], [25], [26].

The role of the human expert is of prime importance in all these domains, in particular to set the vocabulary, the context, and the useful knowledge so as to guarantee a shared conceptualization and to allow storage, reasoning, and communication. All useful concepts should be explicitly given, while leaving room for reasoning capabilities to derive higher level knowledge and interpretation. For instance, the interpretation of the image in Fig. 1 requires the ontology to contain concepts, such as brain, cerebral hemisphere, lateral ventricle, tumor, far from... Moreover, in medicine, noticeable efforts have led to the development of the neuronames brain hierarchy[1] and the foundational model of anatomy (FMA)[2] at the University of Washington, or Neuranat[3] in Paris at CHU La Pitié-Salpêtrière. All these developments required contributions from human experts. An important part of the modeling also concerns spatial relations, and again they are provided by expert knowledge. For instance in neuro-anatomy, descriptions such as "the left caudate nucleus is to the left of the lateral ventricles" are often found in textbooks. Such linguistic descriptions have to be formalized and also encoded for each specific application in order to fill the semantic gap. To this aim, fuzzy representations of spatial entities and spatial relations in concrete domains have been proposed in our previous work [2] and later in [27] .

In this paper, we propose linking the different ways in which human expert knowledge can be expressed by combining description logics, formal concept analysis, and mathematical morphology. The first framework provides formal tools to exploit ontological knowledge and to reason on it. Formal concept analysis allows encoding explicitly objects and their attributes, and provides a complete lattice, suitable for algebraic reasoning using mathematical morphology. Finally, mathematical morphology operators can be used for a number of reasoning tasks, such as fusion, revision, abduction, mediation, and will be developed here for the aim of image interpretation expressed as an abduction process.

[1]Available at http://braininfo.rprc.washington.edu/.

[2]Available at http://sig.biostr.washington.edu/projects/fm/AboutFM.html.

[3]Available at http://www.chups.jussieu.fr/ext/neuranat.

III. Background

A. Description Logics

In this section, we consider the description logics $\mathcal{EL}$ and $\mathcal{EL}_{gfp}$ which belong to the family of the description logics enjoying the finite model property. This property is useful in our framework since abduction operators will be performed via a concept lattice representation, which is built offline using tools from formal concept analysis. Let N_C and N_R be pairwise disjoint and finite sets of concept names and role names, respectively. We use the letter R for role names, and the letters C and D for concepts. The symbol $\top$ denotes the universal concept. The set of $\mathcal{EL}$ concepts is the smallest set such that: 1) every concept name is a concept and 2) if C and D are concepts and R a role name, then the following expressions are also concepts: $C \sqcap D$ (concept conjunction), $\exists R.C$ (existential restriction on role names). An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a set $\Delta^{\mathcal{I}}$, called the domain of $\mathcal{I}$, and a function $\cdot^{\mathcal{I}}$ which maps every concept C to a subset $C^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ and every role R to a subset $R^{\mathcal{I}}$ of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ such that, for all concepts C, D, and all roles R, the following properties are satisfied: 1) $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$; 2) $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$ and 3) $(\exists R.C)^{\mathcal{I}} = \{x \mid \exists y \ s.t. \ (x, y) \in R^{\mathcal{I}} \ \ and \ \ y \in C^{\mathcal{I}}\}$.

A DL knowledge base (KB) $\mathcal{K}$ consists of two components, the TBox and the ABox. The TBox $\mathcal{T}$ describes the terminology by listing concepts and roles and their relationships. In $\mathcal{EL}$, the TBox contains axioms of type $C \sqsubseteq D$ (a general concept inclusion, GCI, where C and D are $\mathcal{EL}$ concepts) and of type $A \equiv C$ (concept definition where A is an atomic concept and C an $\mathcal{EL}$ concept). The ABox $\mathcal{A}$ contains assertions about objects. Concept assertions are of the form $a : C$ which reads as a is a C, and role assertions write $(a, b) : R$ and read as a is R-related to b.

An interpretation $\mathcal{I}$ is a model of a DL (TBox or ABox) axiom if it satisfies this axiom, and it is a model of a DL knowledge base $\mathcal{K}$ if it satisfies every axiom in $\mathcal{K}$. A concept C is satisfiable if it admits a model, i.e., $C^{\mathcal{I}} \neq \emptyset$.

One of the most important reasoning services in DL is computing the subsumption relationships between concept descriptions. Given two concept descriptions C and D, one says that D subsumes C (denoted by $C \sqsubseteq D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all interpretations $\mathcal{I}$. A concept C is equivalent to D ($C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$. The subsumption relation $\sqsubseteq$ is a preorder (i.e., reflexive and transitive), but not an order (it does not need to be antisymmetric: it may hold that two equivalent concept descriptions are not syntactically equal). The preorder $\sqsubseteq$ induces a partial order $\sqsubseteq_{\equiv}$ on the equivalence classes of concept descriptions

$$[C_1] \sqsubseteq_{\equiv} [C_2] \ \text{ iff } C_1 \sqsubseteq C_2$$

where $[C_i] = \{D \mid C_i \equiv D\}$ is the equivalence class of C_i ($i = 1, 2$). The subsumption hierarchy should be understood with respect to this induced partial order. In the presence of a KB $\mathcal{K}$, the subsumption is constructed according to this KB: $C_1 \sqsubseteq_{\mathcal{K}} C_2$ iff $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{K}$.

In our approach, we assume the knowledge base to be complete in the following sense. Let $\mathcal{I}$ be any model of $\mathcal{K}$. We say that $\mathcal{K}$ is complete for $\mathcal{I}$ if for every two complex concept descriptions C and D the GCI $C \sqsubseteq D$ holds in $\mathcal{I}$ iff $\mathcal{K} \models (C \sqsubseteq D)$. If $\mathcal{K}$ is complete for $\mathcal{I}$, then $\mathcal{I}$ is called a free model of $\mathcal{K}$. In the following, we assume that $\mathcal{K}$ is complete for some finite model. This is admittedly a strong assumption. However, there are methods to obtain a complete knowledge base using an expert assisted formalism [28].

We now consider the set $\mathcal{L}$ of all $\mathcal{EL}$-concept descriptions over the signature of $\mathcal{K}$, partially ordered by subsumption with respect to $\mathcal{K}$ ($\sqsubseteq_{\mathcal{K}}$). We then consider the induced partial order $\sqsubseteq_{\equiv}$ on the quotient set $\mathcal{L}/_{\equiv_{\mathcal{K}}}$.

Proposition 1: $(\mathcal{L}/_{\equiv_{\mathcal{K}}}, \sqsubseteq_{\equiv})$ forms a finite lattice.

Proof: Let $\mathcal{I}$ be a free and finite model of $\mathcal{K}$. For any two concept descriptions C and D it holds that $C \sqsubseteq_{\mathcal{K}} D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and therefore $C \equiv_{\mathcal{K}} D$ iff $C^{\mathcal{I}} = D^{\mathcal{I}}$. Since $\mathcal{I}$ is finite there are only finitely many choices of $C^{\mathcal{I}}$ and $D^{\mathcal{I}}$, and by restricting concept definitions to noncyclic and nonredundant ones, thus $\mathcal{L}_{\equiv_{\mathcal{K}}}$ must also be finite. It remains to show that infima and suprema exist.

Let $[C], [D], [E] \in \mathcal{L}/_{\equiv_{\mathcal{K}}}$ be three equivalence classes such that $[E] \sqsubseteq_{\equiv} [C]$ and $[E] \sqsubseteq_{\equiv} [D]$. This is equivalent to $E^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ and $E^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, i.e., $E^{\mathcal{I}} \subseteq C^{\mathcal{I}} \cap D^{\mathcal{I}} = (C \sqcap D)^{\mathcal{I}}$. Moreover, $[C \sqcap D]$ is a lower bound of $[C]$ and $[D]$. Thus, $[C \sqcap D]$ is the infimum of C and D.

For the supremum, consider $[E_1], [E_2] \in \mathcal{L}/_{\equiv_{\mathcal{K}}}$ that are upper bounds for both $[C]$ and $[D]$. From $[C] \sqsubseteq_{\equiv} [E_1]$ and $[C] \sqsubseteq_{\equiv} [E_2]$ we get $C^{\mathcal{I}} \subseteq E_1^{\mathcal{I}}$ and $C^{\mathcal{I}} \subseteq E_2^{\mathcal{I}}$ and hence $C^{\mathcal{I}} \subseteq (E_1 \sqcap E_2)^{\mathcal{I}}$, which implies $[C] \sqsubseteq_{\equiv} [E_1 \sqcap E_2] = \inf\{[E_1], [E_2]\}$, where the inf on equivalence classes is related to the partial order $\sqsubseteq_{\equiv}$ induced by $\sqsubseteq$, and analogously $[D] \sqsubseteq_{\equiv} \inf\{[E_1], [E_2]\}$. This means that the infimum of two upper bounds for $[C]$ and $[D]$ is also an upper bound. Since the set $\mathcal{L}/_{\equiv_{\mathcal{K}}}$ is finite, the infimum

$$\inf\{[E] \in \mathcal{L}/_{\equiv_{\mathcal{K}}} \mid [C] \sqsubseteq_{\equiv} [E] \text{ and } [D] \sqsubseteq_{\equiv} [E]\}$$

exists and is the supremum of $[C]$ and $[D]$. ■

This proof can be directly extended to any family of equivalence classes. Note that for any free model $\mathcal{I}$ of $\mathcal{K}$ the lattice $(\mathcal{L}/_{\equiv_{\mathcal{K}}}, \sqsubseteq_{\equiv})$ is isomorphic to $(S, \subseteq)$ where $S = \{C^{\mathcal{I}} \mid C \in \mathcal{L}\}$. The corresponding isomorphism is

$$\varphi : \left| \begin{array}{l} \mathcal{L}/_{\equiv_{\mathcal{K}}} \longrightarrow S \\ \quad [C] \longmapsto C^{\mathcal{I}}. \end{array} \right.$$

Example 2 (SHD [4]): Using the $\mathcal{EL}$ syntax, the SHD ontology axioms are as follows:

1) $\exists has\_disease.SHD \sqsubseteq \exists has\_symptom.(Laziness \sqcap Pizza\_Appetite)$;
2) $Researcher \sqsubseteq \exists has\_symptom.(Laziness \sqcap Pizza\_Appetite \sqcap Google\_Lover)$;
3) $\exists shake\_hands.\exists carries\_virus.SHDV \sqsubseteq \exists has\_disease.SHD$.

A possible model of this TBox is as follows:

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{peter, paul, mary, shd, shdv, l, p, g, x, v\} \\
SHD^{\mathcal{I}} &= \{shd\} \\
SHDV^{\mathcal{I}} &= \{shdv\} \\
Laziness^{\mathcal{I}} &= \{l\} \\
PizzaAppetite^{\mathcal{I}} &= \{p\} \\
GoogleLover^{\mathcal{I}} &= \{g\} \\
Researcher^{\mathcal{I}} &= \{peter\} \\
HasSymptom^{\mathcal{I}} &= \{(peter, l), (peter, p), (peter, g), \\
&\quad (paul, p), (paul, g), (mary, l), (mary, p)\} \\
CarriesVirus^{\mathcal{I}} &= \{(mary, shdv), (x, v)\} \\
HasDisease^{\mathcal{I}} &= \{(peter, shd), (mary, shd)\} \\
ShakeHands^{\mathcal{I}} &= \{(peter, mary)\}.
\end{aligned}$$

When cyclic concept definitions are allowed (e.g., $A \equiv B \sqcap \exists r.A$), a greatest fixpoint semantics is used rather than a descriptive one as defined above. We distinguish then the set of primitive concepts $\mathcal{N}_{prim}$ and the set $\mathcal{N}_{def}$ of defined concepts. For more details on the greatest fixpoint semantics, please refer to [1].

Another nonclassical reasoning service, which will be helpful in the sequel for constructing the concept lattice (see Algorithm 4), is computing the most specific concept of a subset belonging to the domain. It is defined as the least concept description containing this subset. This formally states as follows.

Definition 1 (Most Specific Concept (msc) [29]): Let $\mathcal{T}$ be a TBox and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a model of $\mathcal{T}$. Let $X \subseteq \Delta^{\mathcal{I}}$ be some subset of the domain of $\mathcal{I}$ and E a defined concept in $\mathcal{T}$. The concept E is called the most specific concept of X with respect to $\mathcal{I}$ if the following conditions hold:

1) $X \subseteq E^{\mathcal{I}}$;
2) If $\mathcal{T}'$ is a conservative extension[4] of $\mathcal{T}$ that uses the same primitive concept names and role names then for every defined concept F in $\mathcal{T}'$ with $X \subseteq F^{\mathcal{I}}$, it holds that $E \sqsubseteq_{\mathcal{T}'} F$.

The most specific concept definition considered here (from [29]) differs from the one traditionally found in DL papers which consider the problem related to finding a most specific concept of an ABox instance. One should also note that the most specific concept does not exist for arbitrary DLs. In [29], it is shown that the *msc* always exists for the DL $\mathcal{EL}$ with cyclic concept definition endowed with greatest fixpoint semantics ($\mathcal{EL}_{gfp}$). In the sequel whenever we note $\mathcal{EL}$, it should be understood that a greatest fixpoint semantics is considered.

B. Abduction in Description Logics

Abduction, originally introduced by Charles Sanders Peirce in the late 19th century, refers to the ability to reason from observations to explanations, and is a fundamental source

[4]A conservative extension of a TBox $\mathcal{T}$ is a TBox $\mathcal{T}'$ such that $\mathcal{T} \subseteq \mathcal{T}'$, and if A and B are concept names used in $\mathcal{T}$ then $A \sqsubseteq_{\mathcal{T}'} B$ iff $A \sqsubseteq_{\mathcal{T}} B$.

of new knowledge, i.e., learning. It is a fundamental form of reasoning next to induction and deduction. It is often understood as a form of backward reasoning from a set of observations back to a cause. Hence, it represents an appealing framework for image interpretation: a scene is viewed as an observation and the task of interpretation consists in finding the best explanation considering the terminological knowledge part of a description logic about the scene context. The challenge then is to fill the gap between abductive reasoning and description logics. To the best of our knowledge, few work has addressed this subject [4], [30], [31]. The first work reported in the literature is the one in [31], where a tableaux-based algorithm is proposed to account for matchmaking tasks. The abduction problem is considered at the terminological level and is seen as the way of finding all sub-concepts of a given concept. However, the considered DL $\mathcal{ALN}$ does not allow existential restrictions, which are mandatory in our context for representing spatial relations between scene objects. Later, in a position paper, Elsenbroich *et al.* [4] discussed, without providing hints for computational tools, the usefulness of abductive reasoning in DL, provided several application scenarios, and introduced rigorous definitions and postulates of abductive reasoning in the context of ontologies. Other abduction-like nonmonotonic services are reported in the literature. In [32], debugging incoherent terminologies is considered, i.e., finding a minimally unsatisfiable subset of TBox axioms, and in [33], and later [34], the authors addressed the problem of finding justifications, i.e., minimal sets of axioms of an ontology that make a particular entailment of the ontology hold. These works are pointed out in [35] where some computational complexity results in the DL $\mathcal{EL}$ are reported. Recently, based on the correspondence between DL and modal logic, Klarman *et al.* [30] introduced reasoning calculi for solving ABox abduction problems in the DL $\mathcal{ALC}$. The algorithms are based on regular connection tableaux and resolution with set-of-support and are proven to be sound and complete. Finally, in a context similar to the one claimed in this paper, the task of multimedia interpretation as abductive reasoning over DL rules is considered in [36] and [37]. An inference service for ABox abduction restricted to rules is introduced. A more detailed discussion of the other approaches can be found in Section VI.

Based on the TBox $\mathcal{T}$ and ABox $\mathcal{A}$ parts of the knowledge base, abduction in the framework of DL can be viewed from different standpoints [4], [30]: concept abduction, TBox abduction, ABox abduction, and knowledge base abduction.

In this paper, we consider the case of concept abduction with respect to a background theory or knowledge base. The following definition formally states our purpose.

Definition 2 (Concept Abduction [4]): Let Γ be an arbitrary DL, $\mathcal{K}$ a knowledge base, and C a concept in Γ such that C is satisfiable with respect to $\mathcal{K}$. A concept abduction problem, denoted as $\langle \mathcal{K}, C \rangle$, consists of finding a set $Expla(C)$ of complex concepts γ in a possibly different DL Γ' (a sublogic of Γ) such that $\mathcal{K} \models \gamma \sqsubseteq C$. An explanatory relation is a binary relation $C \rhd \gamma$ where the intended meaning of $C \rhd \gamma$ is γ is a preferred explanation of C.

Explanatory reasoning is concerned with preferred explanations rather than just plain explanations. So, explaining an observation requires that some formulas must be selected as preferred explanations.

Rationality postulates for abduction have been widely studied in the context of propositional logic [38]. In this paper, we consider the rationality postulates introduced in [39] adapted to the DL context

LLE$_\mathcal{K}$: $\dfrac{C \equiv_\mathcal{K} D \ \ , \ \ C \rhd \gamma}{D \rhd \gamma}$

RLE$_\mathcal{K}$: $\dfrac{\gamma \equiv_\mathcal{K} \gamma' \ \ ; \ \ C \rhd \gamma}{C \rhd \gamma'}$

E-CM: $\dfrac{C \rhd \gamma \ \ ; \ \gamma \sqsubseteq_\mathcal{K} D}{(C \sqcap D) \rhd \gamma}$

E-C-Cut: $\dfrac{(C \sqcap D) \rhd \gamma \quad \forall \delta \, [C \rhd \delta \ \Rightarrow \ \delta \sqsubseteq_\mathcal{K} D \,]}{C \rhd \gamma}$

RS: $\dfrac{C \rhd \gamma \ \ \gamma' \sqsubseteq_\mathcal{K} \gamma \ \ ; \ \gamma' \not\sqsubseteq_\mathcal{K} \bot}{C \rhd \gamma'}$

ROR: $\dfrac{C \rhd \gamma \ \ ; \ C \rhd \delta}{C \rhd (\gamma \sqcup \delta)}$

LOR: $\dfrac{C \rhd \gamma \ \ ; \ D \rhd \gamma}{(C \sqcup D) \rhd \gamma}$

E-DR: $\dfrac{C \rhd \gamma \ \ ; \ D \rhd \delta}{(C \sqcup D) \rhd \gamma \ \text{ or } \ (C \sqcup D) \rhd \delta}$

E-R-Cut: $\dfrac{(C \sqcap D) \rhd \gamma \ \ ; \ \exists \delta \, [C \rhd \delta \ \& \ \delta \sqsubseteq_\mathcal{K} D]}{C \rhd \gamma}$

E-Reflexivity : $\dfrac{C \rhd \gamma}{\gamma \rhd \gamma}$

E-Con$_\mathcal{K}$: $\mathcal{K} \not\models \neg C(a)$ iff there is γ such that $C \rhd \gamma$.

The intended meaning and motivation for these postulates can be found in [39]. It is worth noting that in the context of the relatively inexpressive DL $\mathcal{EL}$ not allowing for disjunction and negation, ROR, LOR, E-DR and E-Con$_\mathcal{K}$ postulates are not considered.

The rationality postulates can be satisfied by operators (such as the subsumption: $\gamma \sqsubseteq_\mathcal{K} C$ for instance) which we do not consider restrictive enough. Therefore, to enhance the notion of preferred explanation, in addition to the rationality postulates detailed above we consider the following minimality constraints.

Definition 3 (Minimality Constraint): Let us consider the concept abduction problem $\langle \mathcal{K}, C \rangle$, with $Expla(C)$ the set of explanations and γ a preferred solution, i.e., $C \rhd \gamma$:

> γ is $\sqsubseteq$-minimal if there is no explanation ζ of $\langle \mathcal{K}, C \rangle$ such that $\zeta \sqsubset_\mathcal{K} \gamma$ and $\gamma \not\sqsubseteq_\mathcal{K} \bot$.

This should be read: γ is minimal if there is not a more specific explanation than γ. The trivial solution $\bot$ is excluded. Other minimality constraints for abduction in DL can be found in [35] in addition to complexity analysis in the particular case of the DLs $\mathcal{EL}$ and $\mathcal{EL}^{++}$.

When the abduction problem is restricted to concept names, the set of all explanatory solutions is obvious. It is exactly the set of concept names that are subsumed by the observation C. This amounts to going down in the subsumption hierarchy, starting from the concept to explain. In this paper, we are interested in complex concepts, i.e., concepts that are not defined in the TBox and are not explicitly represented in the subsumption hierarchy.

Example 3 (SHD Cont'd): Within the context of the SHD example,[5] given the concept $\exists has\_symptom.(Laziness \sqcap Pizza\_Appetite)$, if we are restricted to GCI in the TBox, a solution obtained by simple backward chaining on the classification tree would be $\exists shake\_hands.\exists carries\_virus.SHDV$. However, we are looking for complex $\mathcal{EL}$-concepts, and in this case our approach allows for abducing the following complex concept (see details in Section IV):

$$\begin{aligned}&\exists shake\_hands.(\exists carries\_virus.SHDV \sqcap\\ &\exists has\_disease.SHD \sqcap\\ &\exists has\_symptom.Pizza\_Appetite \sqcap\\ &has\_symptom.Laziness).\end{aligned}$$

One should remark that this concept is not a named one; hence, our approach goes beyond simple backward chaining in the classification tree. It involves the largest number of atomic concepts (dually the model size is small) and is in this sense most central and satisfies the minimality constraint (Definition 3).

C. Formal Concept Analysis

Formal concept analysis (FCA) is a theory of data analysis, knowledge representation, and information management that aims at identifying conceptual structures from data sets [40]. It relies on a lattice-theoretic formalization of the notions of concept and conceptual hierarchy. A formal context is defined as a triple $\mathbb{K} = (G, M, I)$, where G consists of the set of objects, M the set of attributes, and $I \subseteq G \times M$ a relation between the objects and attributes. A pair $(g, m) \in I$ stands for the object g has the attribute m. The formal concepts of the context $\mathbb{K}$ are all pairs (X, Y) with $X \subseteq G$ and $Y \subseteq M$ such that (X, Y) is maximal with the property $X \times Y \subseteq I$. The set X is called the extent and the set Y is called the intent of the formal concept (X, Y). The set of all formal concepts of a given context can be hierarchically ordered by inclusion of their extent: $(X_1, Y_1) \leq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2$ ($\Leftrightarrow Y_2 \subseteq Y_1$). This order, which reflects the subconcept-superconcept relation, always induces a complete lattice that is called the concept lattice of the context (G, M, I), denoted $\mathbb{C}(\mathbb{K})$. For $X \subseteq G$ and $Y \subseteq M$, the derivation operators α and β are defined as $\alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\}$, and $\beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}$. For $X_1 \subseteq X_2 \subseteq G$ (resp. $Y_1 \subseteq Y_2 \subseteq M$), the following holds: 1) $\alpha(X_2) \subseteq \alpha(X_1)$ (resp. $\beta(Y_2) \subseteq \beta(Y_1)$) and 2) $X_1 \subseteq \beta(\alpha(X_1))$ and $\alpha(X_1) = \alpha(\beta(\alpha(X_1)))$ (resp. $Y_1 \subseteq \alpha(\beta(Y_1))$ and $\beta(Y_1) = \beta(\alpha(\beta(Y_1)))$). Moreover, the pair (α, β) induces a Galois connection between the partially ordered powersets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$. Saying that (X, Y) with $X \subseteq G$ and $Y \subseteq M$ is a formal concept is equivalent to $\alpha(X) = Y$ and $\beta(Y) = X$. For $Y_1, Y_2 \subseteq M$, the implication $Y_1 \rightarrow Y_2$ holds in $\mathbb{K}$ ($\mathbb{K} \models Y_1 \rightarrow Y_2$) iff $\beta(Y_1) \subseteq \beta(Y_2)$ (or $Y_2 \subseteq \alpha\beta(Y_1)$). This means that the implication holds if every object having all attributes from Y_1 also has all attributes from Y_2.

In a concept lattice, infimum and supremum of a family of formal concepts $(X_t, Y_t)_{t \in T}$ are calculated as follows:

$$\bigwedge_{t \in T}(X_t, Y_t) = \left(\bigcap_{t \in T} X_t, \alpha(\beta(\bigcup_{t \in T} Y_t))\right), \tag{1}$$

$$\bigvee_{t \in T}(X_t, Y_t) = \left(\beta(\alpha(\bigcup_{t \in T} X_t)), \bigcap_{t \in T} Y_t\right). \tag{2}$$

Every complete lattice can be viewed as a concept lattice. A complete lattice $(L, \leq)$ is isomorphic to the concept lattice $\mathbb{C}(L, L, \leq)$.

A pair (formal concept) (X', Y') is said to be a descendant of a pair (X, Y) if $X \subset X'$. A pair (X', Y') is said to be a successor of a pair (X, Y) if $X \subset X'$ and there is no intermediate pair (X'', Y'') such that $X \subset X'' \subset X'$. The set of successors of a given pair is called the cover of this pair and will be denoted in the sequel as $\uparrow (X, Y)$. The successors of the bottom element are called atoms.

Dually, a pair (X', Y') is said to be an ancestor of a pair (X, Y) if $X' \subset X$. A pair (X', Y') is said to be a predecessor of a pair (X, Y) if $X' \subset X$ and there is no intermediate pair (X'', Y'') such that $X' \subset X'' \subset X$. The set of all ancestors of a given pair will be denoted in the sequel as $\downarrow (X, Y)$.

Given a formal context, the key problem is to efficiently compute the underlying formal concept lattice, i.e., the set of all implications holding in this context. Adopting a brute force approach by enumerating all the possible implications ($2^{2|M|}$) is very time consuming and generates a redundant implication set. A less naive strategy can exploit the facts that: 1) for any subset Y of M, the implication $Y \rightarrow \alpha\beta(Y)$ always holds in $\mathbb{K}$ and 2) if $Y_1 \rightarrow Y_2$ holds in $\mathbb{K}$ then $Y_2 \subseteq \alpha\beta(Y_1)$. One can then define the implication set by enumerating all ($2^{|M|}$) subsets Y of M and generate the implications $Y \rightarrow \alpha\beta(Y)$. However, this approach still generates redundant implications, which makes it ineffective in particular for large scale applications. A natural question then is to ask whether there exists an implication set that constitutes a basis, i.e., an implication set that is nonredundant and from which all the implications holding in a given context can be derived. The following definitions will be helpful for the definition of an efficient concept lattice construction algorithm.

Definition 4 (Implication Base): Given a formal context $\mathbb{K}$, a set of implications $\mathcal{B}$ defines a basis for the implication set in $\mathbb{K}$ ($imp(\mathbb{K})$), if it is:

1) sound, i.e., every implication $Y_1 \rightarrow Y_2$ from $\mathcal{B}$ holds in $\mathbb{K}$);
2) complete, i.e., every implication $Y_1 \rightarrow Y_2$ holding in $\mathbb{K}$ can be derived from $\mathcal{B}$;
3) minimal, i.e., no strict subset of $\mathcal{B}$ is complete.

Of particular interest is the stem base (also called the Guigues–Duquenne base) defined as $\mathcal{B} = \{Y \rightarrow \alpha\beta(Y) \mid Y \text{ is a pseudointent of } \mathbb{K}\}$, where a pseudointent of a formal context $\mathbb{K}$ is recursively defined as the set Y of attributes satisfying $Y \neq \alpha\beta(Y)$ and $\alpha\beta(\tilde{Y}) \subseteq Y$ for each pseudointent $\tilde{Y} \subset Y$ [41].

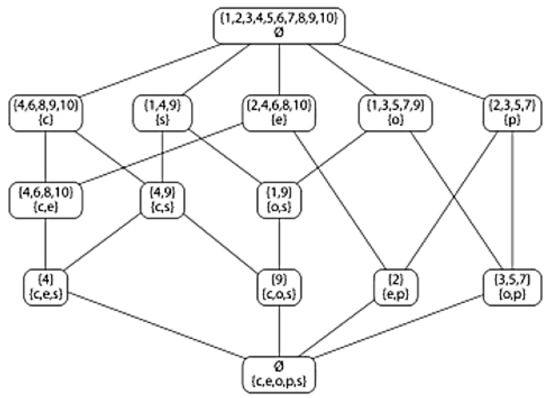
[5] Although the example has been originally introduced as an ABox Abudction problem, we found it simple and clear enough to adapt it to the context of concept abduction.

Require: formal context $\mathbb{K} = (G, M, I)$
Ensure: $\mathcal{B}$ - the stem base
  $\mathcal{B} := \varnothing$
  Define a strict total order on attributes, e.g. $m_1 < m_2 < \cdots < m_n$
  Encode attribute sets as bit-vectors of length $|M|$, e.g. $\{m_1, m_4, m_5\}$ as $[1, 0, 0, 1, 1, 0, 0]$ for an attribute set of cardinality 7.
  $Y := [0, \ldots, 0]$
  loop
    if $Y \neq \alpha\beta(Y)$ **then**
      $\mathcal{B} := \mathcal{B} \cup \{Y \rightarrow \alpha\beta(Y)\}$
    end if
    $k := |M| + 1$
    while $(k \neq 0$ or $(Y[k] = 0$ and $\mathcal{B}(Y + k)[i] = 1, \forall i > k))$ **do**
      $k := k - 1$
    end while
    if $k = 0$ **then return** $\mathcal{B}$, **exit**
    end if
    $Y := \mathcal{B}(Y + k)$
  end loop

Fig. 2. Algorithm for computing the stem base [42].

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			

(a)



(b)

Fig. 3. Simple example of a concept lattice from Wikipedia (objects are integers from 1 to 10, and attributes are composite, even, odd, prime, and square). Table (a) depicts the formal context and Figure (b) its induced concept lattice.

An efficient approach to construct the concept lattice is to enumerate the pseudointents of $\mathbb{K}$ in a lectic order which is defined as follows.

Definition 5 (Lectic order): The lectic order is a linear order on the powerset of M. It is defined as follows: fix an arbitrary strict total order $<$ on the set $M = \{m_1, \ldots, m_n\}$ of attributes, say $m_1 < \cdots < m_n$. Let $Y_1, Y_2 \subseteq M$ be two sets of attributes. Define

$$Y_1 <_i Y_2 \text{ iff } \exists m_i \in Y_2 \backslash Y_1 \text{ and } Y_1 \cap \{m_1, \ldots, m_{i-1}\} = Y_2 \cap \{m_1, \ldots, m_{i-1}\}.$$

The lectic order is the union of all $<_i$ for $i = 1, \ldots, n$.

The algorithm in Fig. 2 computes the stem base by lectically enumerating the pseudointents of $\mathbb{K}$. In this algorithm $Y + i$ amounts to setting the ith bit to 1 and all subsequent bits to 0, i.e., $Y[i] := 1$ and $\forall j > i,\ Y[j] := 0$. $\mathcal{B}(Y)$ means applying implications to attribute sets, e.g. for $\mathcal{B} = (\{m_1\} \rightarrow \{m_1, m_4, m_5\})$ and $Y = \{m_1, m_2, m_3\}$, $\mathcal{B}(Y) = \{m_1, m_2, m_3, m_4, m_5\}$.

Example 4: We consider a classical example to illustrate the definitions and the algorithm introduced above. Furthermore, this example will be used throughout this paper to illustrate and discuss the proposed operators. The considered formal context and the associated concept lattice are depicted in Fig. 3.

The bottom element is
$(\emptyset, \{composite, even, odd, prime, square\})$.
The atoms are: $(\{4\}, \{composite, even, square\})$, $(\{9\}, \{composite, odd, square\})$, $(\{2\}, \{even, prime\})$, and $(\{3, 5, 7\}, \{odd, prime\})$. $(\{1, 9\}, \{odd, square\})$ is a successor of $(\{9\}, \{composite, odd, square\})$. The cover of $(\{9\}, \{composite, odd, square\})$ is the set $\{(\{4, 9\}, \{composite, square\}), (\{1, 9\}, \{odd, square\})\}$.

The computed Guigues–Duquenne base using the algorithm introduced above is:

1) {composite, odd} → {composite, odd, square};
2) {even, square} → {composite, even, square};
3) {even, odd} → {composite, even, odd, prime, square};
4) {composite, prime} → {composite, even, odd, prime, square};
5) {odd, square} → {composite, even, odd, prime, square}.

D. Using FCA in Description Logics

Description logics and formal concept analysis have been first developed independently until the seminal work of [43].

Now, the gap between both theories has been significantly reduced. On the one hand, researchers from the FCA community tried to enrich formal contexts with complex constructions arisen in DLs [44], [45]. On the other hand, researchers in DL tried to exploit the advances of FCA to treat nonstandard inference problems in DLs. For a complete review on the connection between these domains, please refer to [46].

Since our aim is to pick $\mathcal{EL}$-concepts that are not explicit in the subsumption hierarchy, a natural way is to consider as a search space the complete lattice of concepts derived from a given background theory. Hence, we construct such concept lattices using FCA tools. In this paper we rely on the ideas first introduced in [44], and further developed in [47]–[49]. More precisely, tools from FCA are extended to cope with relational structures expressed in a DL language. The connection between FCA and DL is managed through the so-called induced context. This is formally stated as follows.

Definition 6 (Induced Context [48]): The induced context $\mathbb{K}_{\mathcal{T}} := (G, M, I)$ is defined as follows:

$$G := \Delta^{\mathcal{I}}, \text{ a domain of a finite model } \mathcal{I} \tag{3}$$

$$M := \{m_1, \dots, m_n\} \tag{4}$$

$$I := \left\{ (d, m) \mid d \in m^{\mathcal{I}} \right\}. \tag{5}$$

In what precedes, $m_1, \dots, m_n$ denote the concepts defined in a fixed TBox $\mathcal{T}$, and G corresponds to the domain of the model (in the DL sense) of the considered TBox $\mathcal{T}$. Distel [48] proposed a multistep exploration algorithm for checking the possible entailment holding in a given terminological base expressed with the DL $\mathcal{EL}$.

In this paper, we rely on a similar construction. The algorithm is summarized in Fig. 4. Further details can be found in [48].

We consider the free finite model elements as objects and $\mathcal{EL}$ concepts as attributes. A key point then is the generation of this free model. Actually, inference systems, e.g., Hermit system [50] based on semantic-tableaux reasoning can be used to generate such counter-examples in the FCA-based processes described in [28] and [47].

Example 5 (SHD Cont'd): Considering the SHD example, the implication base resulting from Algorithm 4 is depicted in LISP-like syntax in Fig. 6. The corresponding lattice is depicted in Fig. 5. The drawing is performed using Conexp software.[6]

The following subsumption:

$Researcher \sqcap \exists CarriesVirus.\bot \sqsubseteq \exists HasDisease.SHD \sqcap \exists HasSymptom.GoogleLover \sqcap \exists ShakeHands.\exists CarriesVirus.SHDV$

follows from the first implication in the constructed stem base

$Researcher \sqsubseteq_{\mathcal{K}} \exists HasDisease.SHD \sqcap \exists HasSymptom.GoogleLover \sqcap \exists ShakeHands.\exists CarriesVirus.SHDV$

by applying the following rule:

$$\frac{A \sqsubseteq_{\mathcal{K}} B}{A \sqcap C \sqsubseteq_{\mathcal{K}} B}.$$

[6] Available at http://conexp.sourceforge.net/.

IV. Abduction Operators From Mathematical Morphology on Complete Lattices

Mathematical morphology (MM) on logical formulas has been introduced in [51], showing how the basic morphological operations can be expressed in a logical setting and giving some insights into the possible use of morpho-logics to approximation, reasoning, and decision. Bloch *et al.* [52] proposed using morpho-logics to find explanations of observations and performing revision, contraction, fusion in an unified way. In the framework of abduction, the authors showed how to deal with observations that are inconsistent with the background theory, and introduced methods to handle multiple observations. By exploiting the algebraic structure of mathematical morphology, their main idea is to find the most central part of a theory by successive erosions. Two explanatory relations were constructed and their behavior with respect to the postulates of rationality introduced in [39] was analyzed. Here, we propose adapting and introducing new mathematical morphology operators for the purpose of abductive reasoning in DL. In particular, the new framework differs from [52], by several aspects.

1) The most important one is the underlying complete lattice on which the operators are defined. While in [52], the complete lattice is the one of models, here we will consider the complete lattice constructed from one fixed finite model that is constructed offline by formal concept analysis tools. Noticeably, the latter is not the whole powerset $\mathcal{P}(G)$ but the one obtained by the closure operator leading to the complete concept lattice $\mathbb{C}$.
2) Consequently, the erosion operators are definitely new ones. Those based on structuring elements (i.e., local neighborhoods), following the general case, require the definition of a new distance class, which is defined on the lattice $\mathbb{C}$. We will discuss this new distance class that opens up new perspectives for further developments. Furthermore, we will introduce original last erosion operators that are not based on a local neighborhood but defined directly by jumping in the concept lattice. Besides exhibiting more interesting complexity properties these operators are proved to satisfy more rationality postulates than those that are based on a distance.

Let us first recall the basic algebraic framework of mathematical morphology. Let $(L, \preceq)$ and $(L', \preceq')$ be two complete lattices (which do not need to be equal). All the following definitions and results are common to the general algebraic framework of mathematical morphology in complete lattices [53]–[57].

Definition 7: An operator $\delta : L \to L'$ is a dilation if it commutes with the supremum: $\forall (x_i) \in L,\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)$, where $\vee$ denotes the supremum associated with $\preceq$ and $\vee'$ the one associated with $\preceq'$.

An operator $\varepsilon : L' \to L$ is an erosion if it commutes with the infimum: $\forall (x_i) \in L',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)$, where $\wedge$ and $\wedge'$ denote the infimum associated with $\preceq$ and $\preceq'$, respectively.

Here, we will consider operators on the concept lattice $\mathbb{C}$ defined from (G, M, I) [where G, M, and I are defined by (3), (4), and (5)]. As in any complete lattice, we define

Require: finite model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, $\mathcal{N}_{prim}$-the set of primitive concepts
Ensure: $\mathcal{S}$ - a base for the GCIs holding in the model $\mathcal{I}$
$M_0 \coloneqq \mathcal{N}_{prim}$
$\mathbb{K}_0 \coloneqq$ the context induced by M_0 and $\mathcal{I}$
$\mathcal{S}_0 \coloneqq \varnothing$, $\Pi \coloneqq \varnothing$, $P_0 \coloneqq \varnothing$, $k \coloneqq 0$
while $P_k \neq \varnothing$ **do**
$\Pi_{k+1} \coloneqq \Pi_k \cup \{P_k\}$
$M_{k+1} \coloneqq M_k \cup \{\exists r.\text{msc}\left((\sqcap_{U \in P_k} U)^{\mathcal{I}}\right) \mid r \in N_R\}$
$S_{k+1} \coloneqq \{\{C\} \to \{D\} \mid C, D \in M_k,\ C \sqsubseteq D\}$
$k \coloneqq k+1$
if $M_k = M_{k-1} = P_k$ **then**
$P_k \coloneqq \varnothing$
else
$P_k \coloneqq$ lectically next set of attributes that respects all implications in $\mathcal{S}_k$ and $\{P_j \to \beta\alpha_k(P_j) \mid 1 \leq j \leq k\}$ (with $\beta\alpha_k$ meaning that the derivation operators are applied w.r.t the context $\mathbb{K}_k$)
end if
end while

Fig. 4. Algorithm for computing a base for the general concept inclusions holding in a given finite model, adapted from [48].

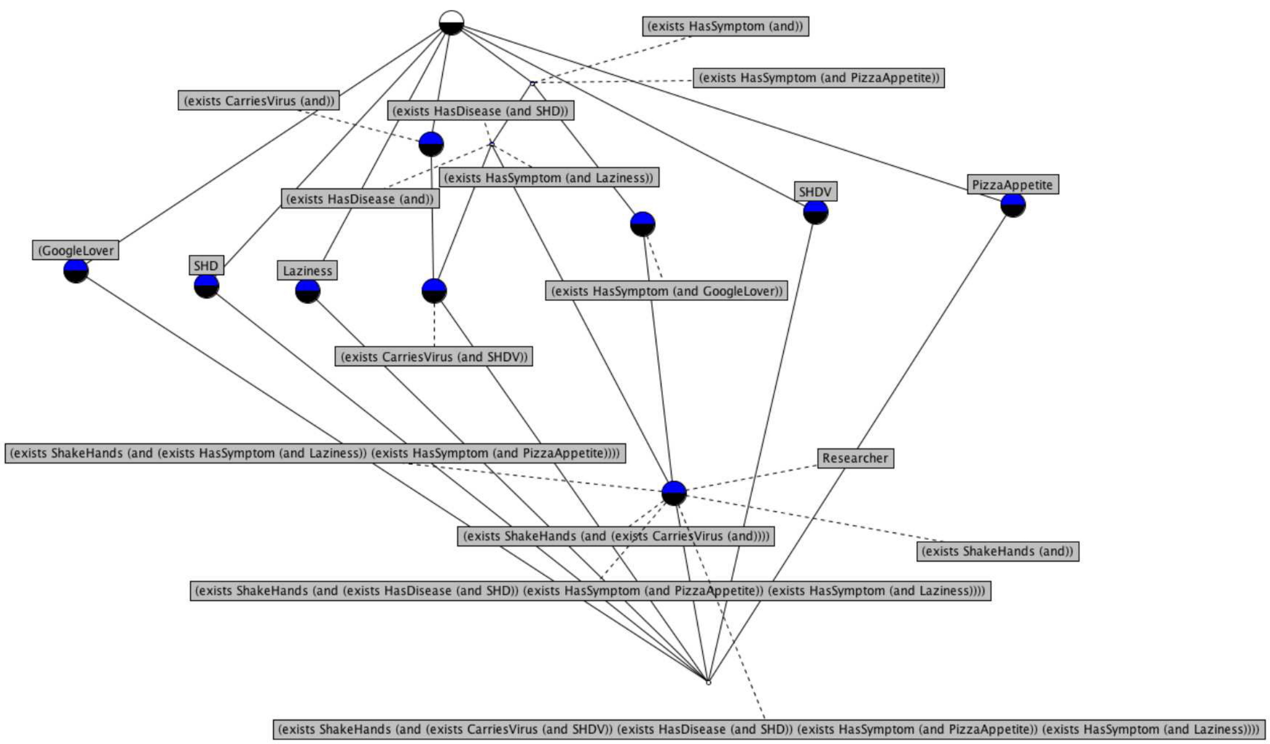


Fig. 5. Concept lattice induced by the SHD ontology.

dilations and erosions in the concept lattice as operations that commute with the supremum and the infimum, respectively. With the aim of performing concept abduction, we would like to reason on subsets of G (via β) in order to find their best explanations (in M). This will be performed by erosions to find a more restricted subset that would explain a subset X. Note that since the partial ordering on the concept lattice can be expressed equivalently as an inclusion on G or on M, the proposed construction on G will directly induce a way of reasoning on M.

In the following, we propose two approaches to concretely define erosions on $\mathbb{C}$.

1) The first one consists of defining morphological erosions, based on the notion of structuring element, defined as an elementary neighborhood of elements of G or as a binary relation between elements of G. Such a neighborhood can be defined as a ball of radius 1 of some distance function. Then finding an explanation will be expressed as performing successive erosions so as to derive what we call last nonempty erosion and last consistent erosion, providing explanations. This follows the line of previous work on propositional logics [52], [58].
2) The second approach consists of directly defining the last erosions that are used for abduction purposes (i.e., directly jump to the last step of the construction proposed in the first approach). This is the way adopted for our examples in the computation.

A. Erosion From Distance and Local Neighborhood

In order to define explicit operations on the concept lattice, we will make use of particular erosions and dilations, called morphological ones [59], which involve the notion of structuring element, i.e., a binary relation b between

((and Researcher) `==>` **(and (exists HasDisease (and SHD)) (exists HasSymptom (and GoogleLover)) (exists ShakeHands (and (exists CarriesVirus (and SHDV))))))**

```
((and PizzaAppetite SHDV) ==> Nothing)
((and SHD SHDV) ==> Nothing)
((and PizzaAppetite SHD) ==> Nothing)
((and Laziness SHDV) ==> Nothing)
((and Laziness PizzaAppetite) ==> Nothing)
((and Laziness SHD) ==> Nothing)
((and GoogleLover SHDV) ==> Nothing)
((and GoogleLover PizzaAppetite) ==> Nothing)
((and GoogleLover SHD) ==> Nothing)
((and GoogleLover Laziness) ==> Nothing)
((and (exists HasDisease (and))) ==> (and (exists HasDisease (and SHD))))
((and (exists HasSymptom (and))) ==> (and (exists HasSymptom (and PizzaAppetite))))
((and (exists CarriesVirus (and)) SHDV) ==> Nothing)
((and (exists CarriesVirus (and)) PizzaAppetite) ==> Nothing)
((and (exists CarriesVirus (and)) SHD) ==> Nothing)
((and (exists CarriesVirus (and)) Laziness) ==> Nothing)
((and (exists CarriesVirus (and)) GoogleLover) ==> Nothing)
((and (exists ShakeHands (and))) ==> (and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) Researcher))
((and (exists CarriesVirus (and SHDV))) ==> (and (exists HasDisease (and SHD))))
((and (exists HasSymptom (and PizzaAppetite)) SHDV) ==> Nothing)
((and (exists HasSymptom (and PizzaAppetite)) PizzaAppetite) ==> Nothing)
((and (exists HasSymptom (and PizzaAppetite)) SHD) ==> Nothing)
((and (exists HasSymptom (and PizzaAppetite)) Laziness) ==> Nothing)
((and (exists HasSymptom (and PizzaAppetite)) GoogleLover) ==> Nothing)
((and (exists CarriesVirus (and)) (exists HasSymptom (and PizzaAppetite))) ==> (and (exists CarriesVirus (and SHDV))))
((and (exists HasSymptom (and SHDV))) ==> Nothing)
((and (exists CarriesVirus (and PizzaAppetite))) ==> Nothing)
((and (exists HasSymptom (and SHD))) ==> Nothing)
((and (exists CarriesVirus (and SHD))) ==> Nothing)
((and (exists HasSymptom (and Laziness))) ==> (and (exists HasDisease (and SHD))))
((and (exists HasDisease (and SHDV))) ==> Nothing)
((and (exists HasDisease (and PizzaAppetite))) ==> Nothing)
((and (exists HasDisease (and Laziness))) ==> Nothing)
((and (exists CarriesVirus (and Laziness))) ==> Nothing)
((and (exists HasDisease (and GoogleLover))) ==> Nothing)
((and (exists HasDisease (and SHD)) (exists HasSymptom (and GoogleLover))) ==> (and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) Researcher))
((and (exists CarriesVirus (and GoogleLover))) ==> Nothing)
((and (exists HasDisease (and (exists HasSymptom (and PizzaAppetite))))) ==> Nothing)
((and (exists HasSymptom (and (exists HasSymptom (and PizzaAppetite))))) ==> Nothing)
((and (exists CarriesVirus (and (exists HasSymptom (and PizzaAppetite))))) ==> Nothing)
((and (exists HasDisease (and (exists CarriesVirus (and))))) ==> Nothing)
((and (exists HasSymptom (and (exists CarriesVirus (and))))) ==> Nothing)
((and (exists CarriesVirus (and (exists CarriesVirus (and))))) ==> Nothing)
((and (exists CarriesVirus (and SHDV)) (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) Researcher) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and SHDV))) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and PizzaAppetite))) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and SHD))) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and Laziness))) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and GoogleLover))) ==> Nothing)
((and (exists ShakeHands (and (exists CarriesVirus (and SHDV)))) (exists ShakeHands (and (exists HasSymptom (and GoogleLover)))) Researcher) ==> Nothing))
```

Where the concept `Nothing` which approximates the $\perp$ concept corresponds to:

```
(and SHDV Researcher PizzaAppetite SHD Laziness GoogleLover (exists HasDisease All) (exists HasSymptom All) (exists CarriesVirus All) (exists ShakeHands All))
```

Fig. 6. Implication base derived from the SHD example.

elements of G. For $g \in G$, we denote by $b(g)$ the set of elements of G in relation with g. For instance b can represent a neighborhood system in G or a distance relation. For a distance d between elements of G, structuring elements can be defined as balls of this distance. Several distances could be used. Let us mention one example.

Relying on notions from the theory of graded lattices [60], we equip $\mathcal{P}(G)$, the powerset of G, with a height function ℓ, defined as the supremum of the lengths of all chains that join the empty set to the considered element. This function is strictly monotonous and satisfies the following property: if Y covers X (i.e., $X \subset Y$ and $\nexists Z$ such that $X \subset Z \subset Y$), then $\ell(Y) = \ell(X)+1$. Hence, this function endows the concept lattice with a graded lattice structure. In a general graded lattice, a pseudometric can be defined as $d(X, Y) = \ell(X)+\ell(Y)-2\ell(X \wedge Y)$, where $\wedge$ denotes the infimum associated with the partial ordering of the lattice [61]. In the particular case where the lattice is the power set of a set equipped with the subsethood partial ordering, the ℓ function is simply the cardinality of each subset, i.e., $\forall X \in \mathcal{P}(G), \ell(X) = |X|$, Y covers X means that Y has exactly one more element than X, and d is a true metric, which can also be expressed as

$$\begin{aligned}\forall (X, Y) \in \mathcal{P}(G)^2,\ d(X, Y) &= |X| + |Y| - 2|X \cap Y| \\ &= |X \cup Y| - |X \cap Y| \\ &= |X \Delta Y| \end{aligned} \tag{6}$$

where Δ is the symmetric set difference operator.

This is one example of distance that can be used on $\mathbb{C}$, among others. One of its drawbacks however is that it strongly depends on the granularity of the concept descriptions in the underlying ontology.

In the following, we assume any distance d, restricted to singletons, and define a neighborhood of each element of G, as a ball of d of radius 1 centered on g:

$$\forall g \in G,\ b(g) = \{g' \in G \mid d(\{g\}, \{g'\}) \leq 1\}.$$

What follows applies whatever the distance, for a structuring element b defined as a ball of the chosen distance.

Definition 8: The morphological dilation of a subset X of G with respect to b is expressed as

$$\delta_b(X) = \{g \in X \mid b(g) \cap X \neq \emptyset\}. \tag{7}$$

The morphological erosion of X is expressed as

$$\varepsilon_b(X) = \{g \in G \mid b(g) \subseteq X\}. \tag{8}$$

Taking b as derived from a distance is particularly interesting in the context of abduction, where the most central parts of X will have to be defined. Erosion is then expressed as follows:

$$\varepsilon^n(X) = \{g \in X \mid d(\{g\}, X^C) > n\} \tag{9}$$

where X^C denotes the complement of X in G. We note $\varepsilon(X) = \varepsilon^1(X)$, and have $\varepsilon^0(X) = X$. Here, G is a discrete finite space, and therefore only integer values of n are considered.

More generally, ε^n denotes the iterative application of ε, n times.

Proposition 2: All classical properties of mathematical morphology hold. The ones that will be important in the following are as follows.

1) Erosion commutes with the infimum, i.e., $\forall (X, X') \in \mathcal{P}(G)^2, \varepsilon(X \cap X') = \varepsilon(X) \cap \varepsilon(X')$.
2) Only an inclusion holds for the supremum: $\forall (X, X') \in \mathcal{P}(G)^2, \varepsilon(X) \cup \varepsilon(X') \subseteq \varepsilon(X \cup X')$.
3) If $g \in b(g)$, then erosion is anti-extensive, i.e., $\forall X \in \mathcal{P}(G), \varepsilon_b(X) \subseteq X$. This property holds in particular for (9).
4) Iterativity property: $\varepsilon^n(\varepsilon^m(X)) = \varepsilon^{n+m}(X)$. Performing successive erosions then leads to smaller and smaller results, equivalent to a direct application of a larger erosion. This property will be used to define an explanation as the most reduced result obtained by erosions.
5) An important notion is the one of adjunction: a pair of operators (ε, δ) forms an adjunction if $\forall x \in L, \forall y \in L', \delta(x) \preceq' y \Leftrightarrow x \preceq \varepsilon(y)$. If (ε, δ) is an adjunction, then ε is an erosion and δ is a dilation. It follows that δ preserves the smallest element and ε preserves the largest element. In the particular case considered here, $(\varepsilon_b, \delta_b)$ is an adjunction. This notion is equivalent to the one of Galois connection, by reversing the order on the second lattice: for a formal concept (X, Y), $X \subseteq \beta(Y) \Leftrightarrow Y \subseteq \alpha(X)$. Hence, the derivation operators in formal concept analysis can also be interpreted in terms of mathematical morphology [53].

B. Last Nonempty Erosion

As shown in [58] in the framework of propositional logic, erosions can be used to find explanations. In this context, the idea was to find the most central part of a formula as the best explanation. This approach was shown to have good properties with respect to rationality postulates of abductive reasoning [39]. In this paper, we propose similar ideas, but adapted to the context of concept lattices, using erosions defined as in (9). For any $X \subseteq G$ such that $\exists Y \in M, (X, Y) \in \mathbb{C}$, we define its last erosion as

$$\varepsilon_\ell(X) = \varepsilon^n(X) \Leftrightarrow \begin{cases} \varepsilon^n(X) \neq \emptyset, \\ \text{and } \forall m > n,\ \varepsilon^m(X) = \emptyset. \end{cases} \tag{10}$$

This last nonempty erosion defines the subsets in G that are the furthest ones from the complement of X (according to the distance d), i.e., the most central in X. In other words, it defines the most specific concept that is subsumed by the concept having as extent X.

Definition 9: Let C be an $\mathcal{EL}$-concept, β the derivation operator, and ε_ℓ the last nonempty erosion operator. A preferred explanation γ of C is defined from the last nonempty erosion as

$$C \rhd^{\ell ne} \gamma \overset{def}{\Leftrightarrow} \beta(\gamma) \subseteq \varepsilon_\ell(\beta(C)). \tag{11}$$

When a hypothesis $\mathcal{H}$ (e.g. a set of concepts belonging to the background theory from which the solution has to be picked) has to be introduced, then this definition is modified as

$$C \rhd^{\ell ne} \gamma \overset{def}{\Leftrightarrow} \beta(\gamma) \subseteq \varepsilon_\ell(\beta(\mathcal{H}) \cap \beta(C)). \tag{12}$$

Note that this actually defines a set of best possible explanations, not necessarily a unique one. This set is robust in the sense that it can be modified while remaining in C. For instance dilating $\beta(\gamma)$ by a ball of the distance d of size less than n always leads to a subset of $\beta(C)$. The central part can then be interpreted as the subset X of G that can be changed the most while $\alpha(X)$ remaining subsumed by C.

The interpretation in the concept lattice is as follows: starting from the subset to be explained, performing successive erosions amounts to going down in the lattice as much as possible in order to find a nonempty set of G (Fig. 7).

C. Last Consistent Erosion

Another idea to introduce the constraint $\mathcal{H}$ is to erode it, as soon as it remains consistent with C. This leads to a second explanatory relation.

Definition 10: Let C be an $\mathcal{EL}$-concept, $\mathcal{H}$ a prior given constraint, and β the derivation operator. A preferred explanation γ of C is defined from the last consistent erosion as

$$C \rhd^{\ell c} \gamma \overset{def}{\Leftrightarrow} \beta(\gamma) \subseteq \varepsilon_{\ell c}(\beta(\mathcal{H}), \beta(C)) \cap \beta(C)$$

where $\varepsilon_{\ell c}$ is the last consistent erosion defined as

$$\varepsilon_{\ell c}(\beta(\mathcal{H}), \beta(C)) = \varepsilon^n(\beta(\mathcal{H}))$$

$$\text{where } n = \max\{k \mid \varepsilon^k(\beta(\mathcal{H})) \cap \beta(C) \neq \emptyset\}.$$

This definition has a different interpretation. Here, we consider erosion of $\beta(\mathcal{H})$ alone, which means that we are looking at the models that are in C while being the most in the constraint.

D. Direct Definition of Last Nonempty Erosion

Let $X \in \mathcal{P}(G)$ be a subset to be explained. If X is not in the concept lattice, then we first compute $\beta\alpha(X)$. Thus $(\beta\alpha(X), \alpha(X))$ is a formal concept (i.e., $\in \mathbb{C}$). The notion of most specific concept can also be used (Definition 1), or any suitable alternative that may depend on the application (for instance we can also consider several X_i such that their union includes X, and define explanations of X from explanations of X_i). In the sequel, we assume that X is in the lattice.

To define the last nonempty erosion of X, we propose to compute the nonempty subsets (ancestors) of X which are in the lattice and which are minimal. This is formalized as follows.

Definition 11: Let X be any element of $\mathcal{P}(G)$ such that $\exists Y \in \mathcal{P}(M), (X, Y) \in \mathbb{C}$. We assume $X \neq \emptyset, X \neq \top$ (and by convention we set $\varepsilon_\ell(\emptyset) = \emptyset$, and $\varepsilon_\ell(\top) = \top$). The last nonempty erosion of X is defined as

$$\varepsilon_\ell(X) = \cup\{X' \in \mathcal{P}(G) \setminus \emptyset \mid \exists Y' \in \mathcal{P}(M), (X', Y') \in \mathbb{C}, X' \subseteq X, X' \textit{ minimal}\}. \tag{13}$$

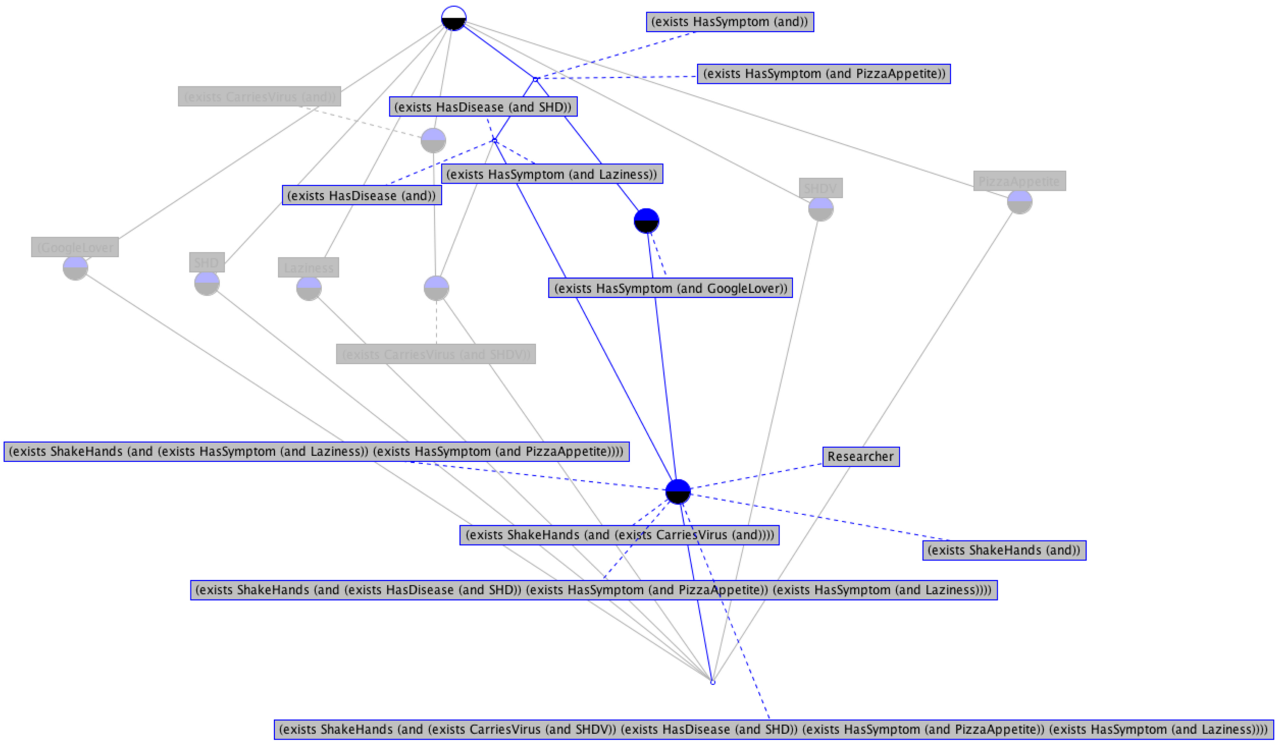


Fig. 7. Erosion path associated with the SHD abduction problem. The concept to be explained is $\exists has\_symptom.(Laziness \sqcap Pizza\_Appetite)$. The solution of the abduction problem is the atom $\exists shake\_hands.(\exists carries\_virus.SHDV \sqcap \exists has\_disease.SHD \sqcap \exists has\_symptom.Pizza\_Appetite \sqcap has\_symptom.Laziness)$, which is not a named concept in the original ontology.

Note that the subsets X' involved in (13) are atoms (i.e., successors of the smallest element $\perp$). The minimality notion in this equation can be defined in various ways, allowing hence for more modularity and flexibility in the definition. For instance, one can consider the following two constraints:

1) cardinality minimality, denoted as $|\cdot| - minimality$. It is a strong constraint that excludes a large number of solution candidates. It presents however the drawback of making the erosion operator dependent on the model;
2) $\subseteq -minimality$ that is less restrictive than the cardinality based constraint and that is less sensitive to the change of the model if the latter is not a free one.

Now defining an explanation from $\varepsilon_\ell(X)$ can be performed using one of the following ways.

1) Choose γ such that $\beta(\gamma) \subseteq \varepsilon_\ell(X)$ ($\beta(\gamma) \in \mathcal{P}(G)$ but $\beta(\gamma)$ is not necessarily in the concept lattice since the union of elements of $\mathbb{C}$ is not always in $\mathbb{C}$ and taking the most specific concept including this union by $\beta\alpha$ could be too large). Moreover, we may want to impose a constraint on minimal cardinality.
2) $\beta(\gamma) \subseteq \varepsilon_\ell(X)$ such that $\exists \Upsilon \in \mathcal{P}(M), (\beta(\gamma), \Upsilon)$ is a formal concept.
3) $\beta(\gamma) = f(\varepsilon_\ell(X))$ where f is a choice function among the subsets X' involved in (13) (thus guaranteeing the minimality constraint).

Theorem 1: The following properties hold:

1) ε_ℓ is an increasing operator;
2) ε_ℓ is an anti-extensive operator;
3) ε_ℓ commutes with the infimum (note that since reasoning is performed on G, the infimum is the intersection);
4) ε_ℓ preserves the largest element.

Let us consider the simple concept lattice illustrated in Fig. 3. Let $X_1 = \{4, 6, 8, 9, 10\}$ and $X_2 = \{1, 9\}$. We have the following.

1) $\varepsilon_\ell(X_1) = \{4, 9\}$ (the subsets X' involved in Definition 11 are $\{4\}$ and $\{9\}$ as nonempty ancestors of minimal cardinality). In this case $\{4, 9\}$ is an element of the lattice, but not of minimal cardinality, and if we want to reduce explanations to be elements of $\mathbb{C}$ with minimal cardinality, we have to choose between $\{4\}$ and $\{9\}$. If this restriction is not imposed, $\{4, 9\}$ or any of its subsets can be considered an explanation of X_1.
2) $\varepsilon_\ell(X_2) = \{9\}$ since the only nonempty ancestor of X_2 is $\{9\}$.
3) $\varepsilon_\ell(X_1) \cap \varepsilon_\ell(X_2) = \{9\}$.
4) $\varepsilon_\ell(X_1 \cap X_2) = \varepsilon_\ell(\{9\}) = \{9\}$.

This shows how the proposed definition works, and also illustrates the commutativity with infimum (last two items).

E. Direct Last Consistent Erosion

Definition 12: Let $X \in \mathcal{P}(G)$ be explained ($X \neq \emptyset$ and $X \neq \top$), and let $\mathcal{H}$ be a constraint. The last consistent erosion of $\mathcal{H}$ is defined as

$$\varepsilon_{\ell c}(\beta(\mathcal{H})) = \cup\{X' \cap X, X' \in Cons(\mathcal{H})\} \tag{14}$$

where

$$Cons(\mathcal{H}) = \{X' \in \mathcal{P}(G) \setminus \emptyset \mid \exists Y' \in \mathcal{P}(M), (X', Y') \in \mathbb{C}, X' \subseteq \beta(\mathcal{H}), X' \ minimal, X' \cap X \neq \emptyset\}. \tag{15}$$

Then, explanations γ are defined from $\varepsilon_{\ell c}(\beta(\mathcal{H}))$, as for ε_ℓ.

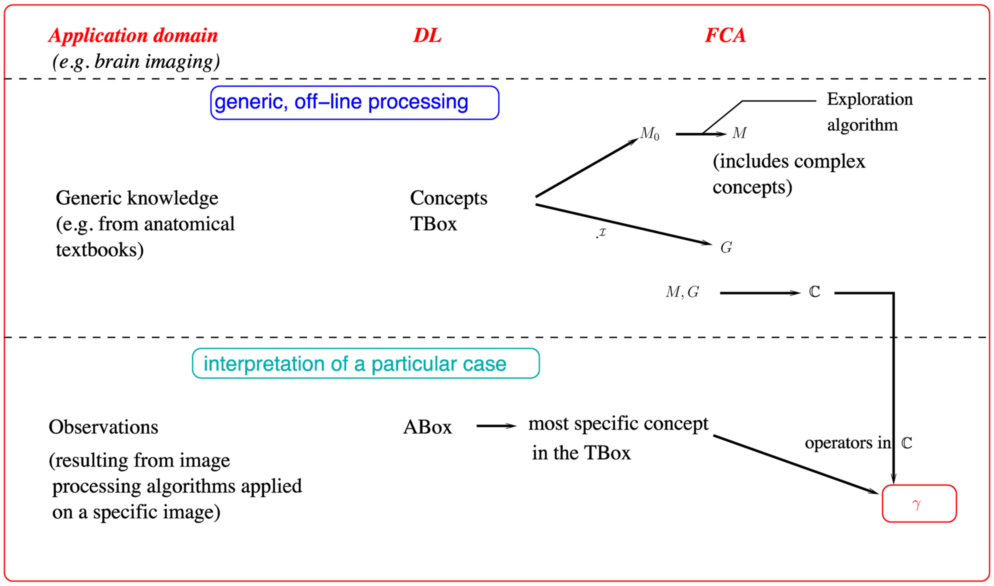


Fig. 8. Scheme describing the whole framework and the connection between the involved theories and the particular problem of image interpretation. The horizontal dashed lines separate the main modules of the framework. The first module that is generic, off-line, and once and for all processed allows for the construction of the concept lattice $\mathbb{C}$ from the background knowledge on the application domain (e.g., brain imaging). First, the generic knowledge is formalized as a TBox using a given description logic ($\mathcal{EL}$ in this paper). The named concepts constitute then the initial attributes set of the formal context in FCA. The objects set is the free model. The exploration algorithm 4 is then used to construct the final induced context $\mathbb{K} = (G, M, I)$ leading to the concept lattice $\mathbb{C}$. The second module is executed for each image to be interpreted. The results from the image processing algorithms applied to the considered image are stored as assertions in the ABox of the underlying DL. The latter, after consistency check, is then rewritten as the conjunction of the involved concepts. This is then the complex concept to explain in $\mathbb{C}$. This abduction process is performed by applying on $\mathbb{C}$ the erosion operators defined in Section IV, leading to the preferred solution γ.

Let us consider again the example in Fig. 3. Let $\beta(\mathcal{H}) = \{2, 4, 6, 8, 10\}$ and $X = \{2, 3, 5, 7\}$. We have $\varepsilon_{\ell c}(\beta(\mathcal{H})) = \{2\}$, since the minimal ancestors are $\{2\}$ and $\{4\}$, and $\{4\}$ is not in X.

F. Properties and Interpretations

A first important property is that reasoning on G actually amounts to reasoning on the whole formal context. Here, explanations are defined from $\mathcal{EL}$-concepts leading to erosions of subsets of G. Let (X, Y) be a formal concept, with $X \subseteq G$ and $Y \subseteq M$, according to the formal context definition. From the definitions of explanations of X, we can derive directly the corresponding concepts for Y, using the derivation operator, i.e., $\alpha(\beta(\gamma)) = \{m \in M \mid \forall g \in \beta(\gamma), (g, m) \in I\}$.

In Fig. 7, the erosion process leading to compute the explanation set is depicted. Note that eroding X amounts to dilating Y, which is in accordance with the correspondence between the Galois connection property between derivation operators and the adjunction properties of dilation and erosion (Section IV-A).

Let us now consider the rationality postulates introduced in [39] for explanation relations. It has been proved that most of them hold for explanations derived from last nonempty erosion and last consistent erosion [58]. These results extend to the DL context as follows.

Theorem 2: The following rationality postulates hold for definitions derived from successive erosions.

1) LLE and RLE: Both $\rhd^{\ell ne}$ and $\rhd^{\ell c}$ are independent of the syntax (since they are computed on the domain of a finite model).
2) E-Reflexivity: A reflexivity property holds for both definitions: if $C \rhd \gamma$, then $\gamma \rhd \gamma$.
3) E-CM: For conjunctions, we have a monotony property for $\rhd^{\ell c}$: if $C \rhd^{\ell c} \gamma$ and $\gamma \sqsubseteq D$, then $(C \sqcap D) \rhd^{\ell c} \gamma$. For $\rhd^{\ell ne}$, only a weaker form holds: if $C \rhd^{\ell ne} \gamma$ and $D \rhd^{\ell ne} \gamma$, then $(C \sqcap D) \rhd^{\ell ne} \gamma$. Note that this weaker form is also very natural and interesting.
4) RS holds for both definitions.
5) E-R-Cut holds for both definitions.
6) E-C-Cut holds for $\rhd^{\ell c}$. For $\rhd^{\ell ne}$, only a weaker form holds, by replacing $\delta \sqsubseteq D$ by $D \rhd \delta$.

Concerning the minimality constraint, it also naturally derives from the definition of last erosion (10).

Theorem 3: For the explanations derived from the direct last nonempty erosion (Definition 11), the following rationality postulates hold:

1) LLE and RLE: independence on the syntax;
2) E-CM (monotony): $\forall (X, X') \in \mathcal{P}(G)^2$, $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma$ and $X' \subseteq \beta(\gamma) \Rightarrow X \sqcap X' \rhd \gamma$;
3) E-Reflexivity (reflexivity): $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma \Rightarrow \gamma \rhd \gamma$;
4) RS: $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma, \gamma' \sqsubseteq \gamma, \beta(\gamma') \neq \emptyset \Rightarrow \bigsqcap_{U \in \alpha(X)} U \rhd \gamma'$;
5) E-R-Cut and E-C-Cut.

Note that E-CM holds here while it does not hold for the last erosion derived from successive erosions based on a distance, since we do not have anymore the centrality property (looking at the most central part for finding an explanation), this constraint being replaced by a minimality constraint.

Theorem 4: For the explanations derived from the direct last consistent erosion (Definition 12), the following rationality postulates hold:

$$
\begin{array}{rcl}
Brain & \sqsubseteq & HumanOrgan \\
CerebralHemisphere & \sqsubseteq & BrainAnatomicalStructure \\
PeripheralCerebralHemisphere & \sqsubseteq & CerebralHemisphereArea \\
SubCorticalCerebralHemisphere & \sqsubseteq & CerebralHemisphereArea \\
GreyNuclei & \sqsubseteq & BrainAnatomicalStructure \\
LateralVentricle & \sqsubseteq & BrainAnatomicalStructure \\
BrainTumor & \sqsubseteq & Disease \sqcap \exists hasLocation.Brain \\
SmallDeformingTumor & \equiv & BrainTumor \sqcap \exists hasBehavior.Infiltrating \\
 & & \sqcap \exists hasEnhancement.NonEnhanced \\
SubCorticalSmallDeformingTumor & \equiv & SmallDeformingTumor \sqcap \\
 & & \exists hasLocation.SubCorticalCerebralHemisphere \\
 & & \sqcap \exists closeTo.GreyNuclei \\
PeripheralSmallDeformingTumor & \equiv & BrainTumor \sqcap \\
 & & \exists hasLocation.PeripheralCerebralHemisphere \\
 & & \sqcap \exists farFrom.LateralVentricle \\
LargeDeformingTumor & \equiv & BrainTumor \sqcap \\
 & & \exists hasLocation.CerebralHemisphere \\
 & & \sqcap \exists hasComponent.Edema \\
 & & \sqcap \exists hasComponent.Necrosis \\
 & & \sqcap \exists hasEnhancement.Enhanced \\
DiseasedBrain & \equiv & Brain \sqcap \exists isAlteredBy.Disease \\
TumoralBrain & \equiv & Brain \sqcap \exists isAlteredBy.BrainTumor \\
SmallDeformingTumoralBrain & \equiv & Brain \sqcap \exists isAlteredBy.SmallDeformingTumor \\
LargeDeformingTumoralBrain & \equiv & Brain \sqcap \exists isAlteredBy.LargeDeformingTumor \\
PeripheralSmallDeformingTumoralBrain & \equiv & Brain \sqcap \exists isAlteredBy.PeripheralSmallDeformingTumor \\
SubCorticalSmallDeformingTumoralBrain & \equiv & Brain \sqcap \exists isAlteredBy.SubCorticalSmallDeformingTumor \\
 & \ldots &
\end{array}
$$

Fig. 9. Background ontology on brain tumor spatial characteristics.

1) LLE and RLE: independence on the syntax;
2) E-CM (monotony): $\forall (X, X') \in \mathcal{P}(G)^2$, $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma$ and $X' \subseteq \beta(\gamma) \Rightarrow X \sqcap X' \rhd \gamma$;
3) E-Reflexivity (reflexivity): $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma \Rightarrow \gamma \rhd \gamma$;
4) E-Cons (consistency);
5) RS: $\bigsqcap_{U \in \alpha(X)} U \rhd \gamma$, $\gamma' \sqsubseteq \gamma$, $\beta(\gamma') \neq \emptyset \Rightarrow \bigsqcap_{U \in \alpha(X)} U \rhd \gamma'$;
6) E-R-Cut and E-C-Cut.

Finally, two fundamental properties in DL and logic in general are soundness and completeness. In the following, we give a sketch of their proofs by exploiting the algebraic properties of erosion, which hold for all proposed definitions.

a) *Soundness:* Informally, a procedure is said to be sound if whenever it proves that a concept γ can be derived from a set of axioms in $\mathcal{K}$, then it is also true that γ is satisfiable with respect to $\mathcal{K}$. Since all proposed explanatory operators perform erosion in the concept lattice constructed from a finite model of the TBox, any solution extracted from this lattice is satisfiable with respect to $\mathcal{K}$. We can hence state the following theorem.

Theorem 5 (Soundness): If $\exists \gamma \mid C \rhd \gamma$ then γ is satisfiable with respect to $\mathcal{K}$.

Proof: The proof is a direct corollary of the anti-extensivity property of erosion (which holds for the proposed definitions). Let us detail the proof of the $\rhd^{\ell ne}$ operator. By definition we have $\beta(\gamma) \subseteq \varepsilon_\ell(\beta(C))$, and from the anti-extentivity we have $\varepsilon_\ell(\beta(C)) \subseteq \beta(C)$. It follows that $\beta(\gamma) \subseteq \beta(C)$ and, since $C \sqsubseteq \top$, $\gamma \sqsubseteq \top$. We then have that γ is satisfiable with respect to $\mathcal{K}$ which completes the proof. The proof for $\rhd^{\ell c}$ is similar. ■

b) *Completeness:* A procedure is said complete if whenever a concept γ is satisfiable with respect to $\mathcal{K}$, then it proves that γ can be derived from $\mathcal{K}$ (i.e., $\exists C$ satisfiable with respect to $\mathcal{K} : C \rhd \gamma$).

Theorem 6 (Completeness): If γ is satisfiable with respect to $\mathcal{K}$ then $\exists C \mid \mathcal{K} \models (\gamma \sqsubseteq C)$.

Proof: Since ε preserves the largest element, we have $\varepsilon(\beta(\top)) = \beta(\top)$, and $\varepsilon_\ell(\beta(\top)) = \beta(\top)$. It follows that any subset of $\beta(\top)$ is an interpretation of a preferred explanation for $\rhd^{\ell ne}$. Hence $\mathcal{K} \rhd^{\ell ne} \gamma$. Let us now take $C = \gamma$. Then C is satisfiable with respect to $\mathcal{K}$ and $C \rhd^{\ell ne} \gamma$ from the reflexivity property. ■

Example 6 (SHD Cont'd): The set of all admissible solutions to the SHD abduction problem as well as the erosion path is depicted in Fig. 7. The preferred solution

$$
\begin{aligned}
\exists shake\_hands\ .\ (&\exists carries\_virus.SHDV \\
&\sqcap \exists has\_disease.SHD \\
&\sqcap \exists has\_symptom.Pizza\_Appetite \\
&\sqcap has\_symptom.Laziness)
\end{aligned}
$$

belongs, as explained earlier, to an atom and is the one that is $|\cdot|$-minimal.

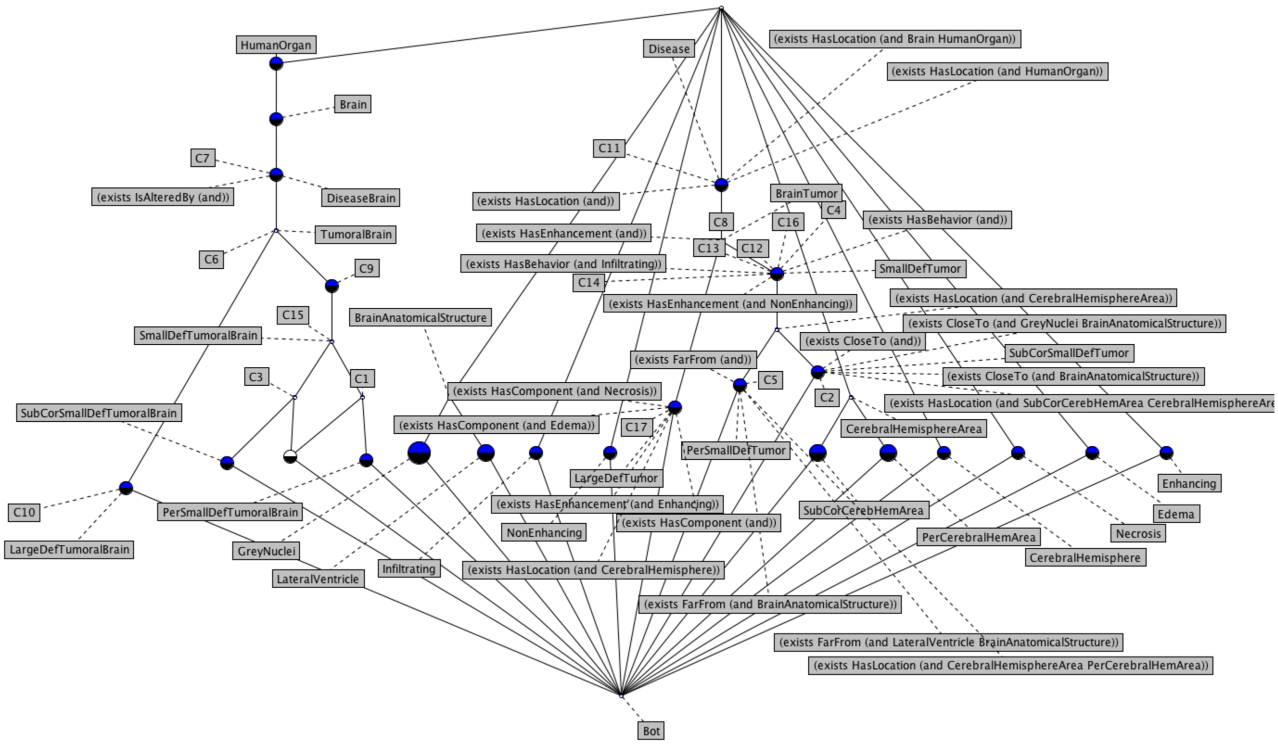


Fig. 10. Concept lattice induced by the formal context $\mathbb{K}_{Brain}$.

V. Brain Image Interpretation

In this section, we show how our theoretical framework applies in the challenging field of pathological brain interpretation. The whole process is summarized and depicted in Fig. 8.

The prior anatomical and pathological knowledge on the brain is formalized using the DL language $\mathcal{EL}$ introduced in Section III-A. The background theory, i.e., the Tbox $\mathcal{T}$, defining the terminological part of the knowledge, is depicted in Fig. 9. Spatial relations, which are important components in spatial reasoning and image understanding, are represented as roles.

For the cerebral image interpretation of Fig. 1 and from image processing analysis, and specialized recognition processes such as those developed in [5], [12], and [62], we derive the following ABox:

$$\begin{aligned} t_1 &: BrainTumor\\ e_1 &: NonEnhanced\\ l_1 &: LateralVentricle\\ p_1 &: PeripheralCerebralHemisphere\\ (t_1, e_1) &: hasEnhancement\\ (t_1, l_1) &: farFrom\\ (t_1, p_1) &: hasLocation \end{aligned}$$

from which we can derive the following most specific concept of t_1:[7]

$$\begin{aligned} BrainTumor \sqcap \exists hasEnhancement.NonEnhanced \sqcap\\ \exists farFrom.LateralVentricle \sqcap\\ \exists hasLocation.PeripheralCerebralHemisphere. \end{aligned}$$

The interpretation task seen as a concept abduction problem $\langle \mathcal{K}, C\rangle$ can be formulated as follows: $\gamma \sqsubseteq_K C$, where C stands for $BrainTumor \sqcap \exists hasEnhancement.NonEnhanced \sqcap \exists farFrom.LateralVentricle \sqcap \exists hasLocation.PeripheralCerebralHemisphere$, denoted as $C7$ in the lattice (see Fig. 10). A possible explanation set is $\{DiseasedBrain, \exists isAlteredBy.\top, SmallDeformingTumoralBrain, PeripheralSmallDeformingTumoralBrain, C1, C6, C9, C15\}$, where C_i are complex cyclic concepts that are too large to be expanded here.

A preferred solution with respect to some minimality and rationality postulates could be $\gamma \equiv PeripheralSmallDeformingTumoralBrain$.

We then construct a possible finite model $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. The domain $\Delta^{\mathcal{I}}$ corresponds to the set $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, gn_1, gn_2, gn_3, gn_4, lv_1, lv_2, t_1, t_2, t_3, t_4, inf_1, ne_1, ne_2, ch_1, a_1, a_2, a_3, a_4, e_1, n_1, d_1\}$, and an excerpt of the assignment function is as follows:[8]

1) $HumanOrgan^{\mathcal{I}} := \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, c_1\}$;
2) $CerebralHemisphere^{\mathcal{I}} := \{ch_1\}$;
3) $BrainAnatomicalStructure^{\mathcal{I}} := \{gn_1, gn_2, gn_3, gn_4, lv_1, lv_2\}$;
4) $\cdots$;

[7] Most specific concept is used here in the classical description logics sense [1]. Distinction should be made here with model most specific concept used earlier to construct the concept lattice in Algorithm 4.

[8] The complete finite model, the induced context, and the concept lattice in conexp format can be downloaded from www.lri.fr/ atif/SMC-2012/.

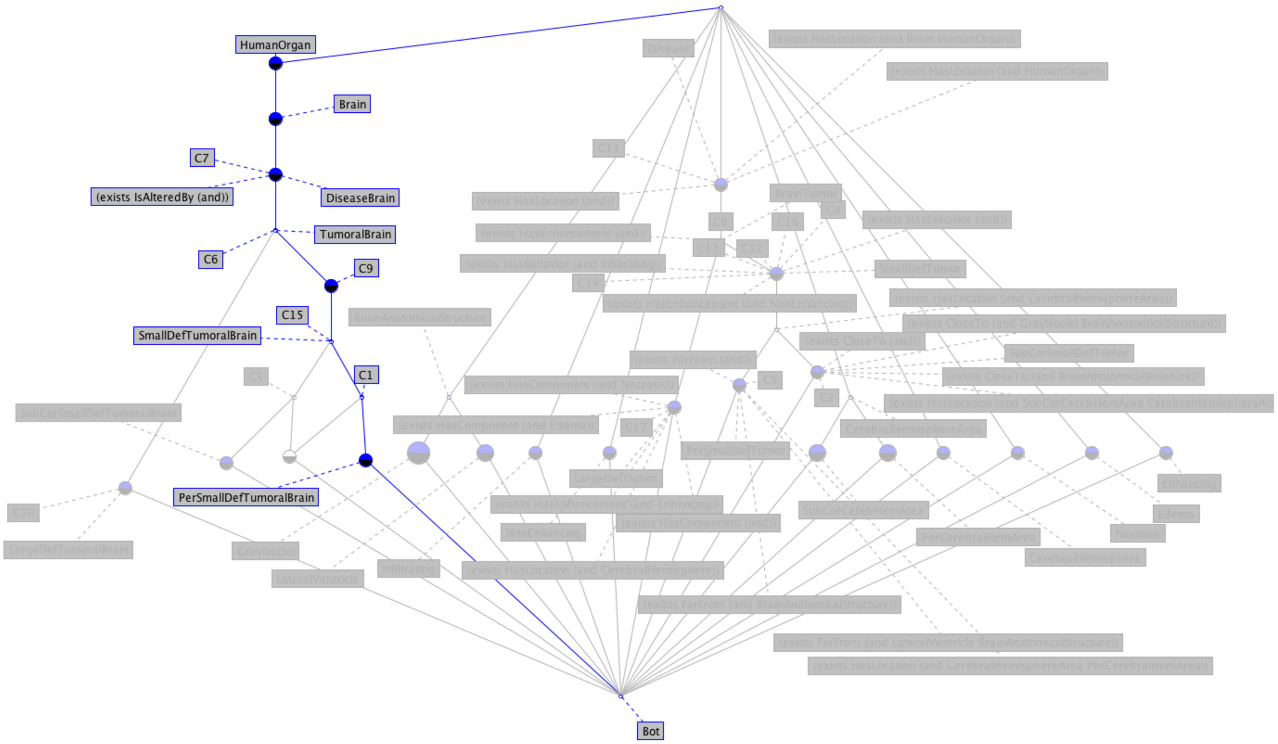


Fig. 11. Erosion path leading to compute the preferred explanation of our image interpretation abduction problem. The concept to be explained is $BrainTumor \sqcap \exists hasEnhancement.NonEnhanced \sqcap \exists farFrom.LateralVentricle \sqcap \exists hasLocation.PeripheralCerebralHemisphere$, denoted as $C7$ in the lattice. The solution here is the named concept: *PeripheralSmallDeformingTumoralBrain*.

5) $isAlteredBy^{\mathcal{I}} := \{(b_3, t_3), (b_4, t_1), (b_4, t_2), (b_4, t_3), (b_5, t_1), (b_6, t_2), (b_7, t_4)\}$;
6) $\cdots$.

The associated concept lattice is shown in Fig 10. Nodes correspond to formal concepts, i.e., pairs (X, Y) where X is a set of domain elements and Y a set of $\mathcal{EL}$-concepts.

In Fig. 11, the erosion process leading to compute the explanation set is depicted. We can see that this process leads to the expected preferred explanation *PeripheralSmallDeformingTumoralBrain*.

In this case, a solution is a named concept. A simple backward chaining on the classification tree would have led to the same result. This is not surprising since the result depends on the expressivity of the knowledge base and in this case a named concept satisfies the minimality constraints as well as the rationality postulates. However, to demonstrate the generality of our approach, and in particular its ability to extract solutions that are complex concepts when necessary, we depict in Fig. 12 an abduction process involving spatial relations. The observation C corresponds to the following complex concept that is not specified in the ontology: $\exists HasLocation.(Brain \sqcap HumanOrgan)$. The preferred solutions are in this case:

1) $\exists FarFrom.(LateralVentricle \sqcap BrainAnatomicalStructure)$;
2) $\exists HasLocation.(CerebralHemisphere \sqcap PerCerebralHemArea)$.

These concepts are complex ones. One should also remark that other complex concepts would have been solutions to this abductive problem but were not chosen since they involve less atomic concepts, and are hence less minimal than the two chosen so far.

The proposed interpretation problem is a very simplified brain cerebral image interpretation problem that aims at illustrating the benefits of our proposed abductive inference services on a real case. A more realistic problem would have implied more anatomical structures and more spatial relations between the different anatomical structures and tumor components. In particular, the presence of a certain kind of tumor can significantly alter the spatial organization of the brain, leading to observations that are not consistent with the expert knowledge. We will study this complex scenario in our future work.

VI. Discussion

A. Related Work

From the image understanding standpoint, our approach differs significantly from classical ontology-based approaches, since it formalizes the interpretation task as a concept abduction problem. However, a close work can be found in [37], where Möller and Neumann discussed ontology based reasoning techniques for multimedia interpretation and reasoning. The main ingredient in this approach is the notion of aggregates that explicitly materializes the relationship between high-level concepts and relations between low level data. Formally, aggregates are concepts defined by: 1) inheritance from parent concepts; 2) roles relating the aggregate to parts; and 3) constraints relating each part to others. Interpretation is then seen as instantiating the aggregates, by explaining

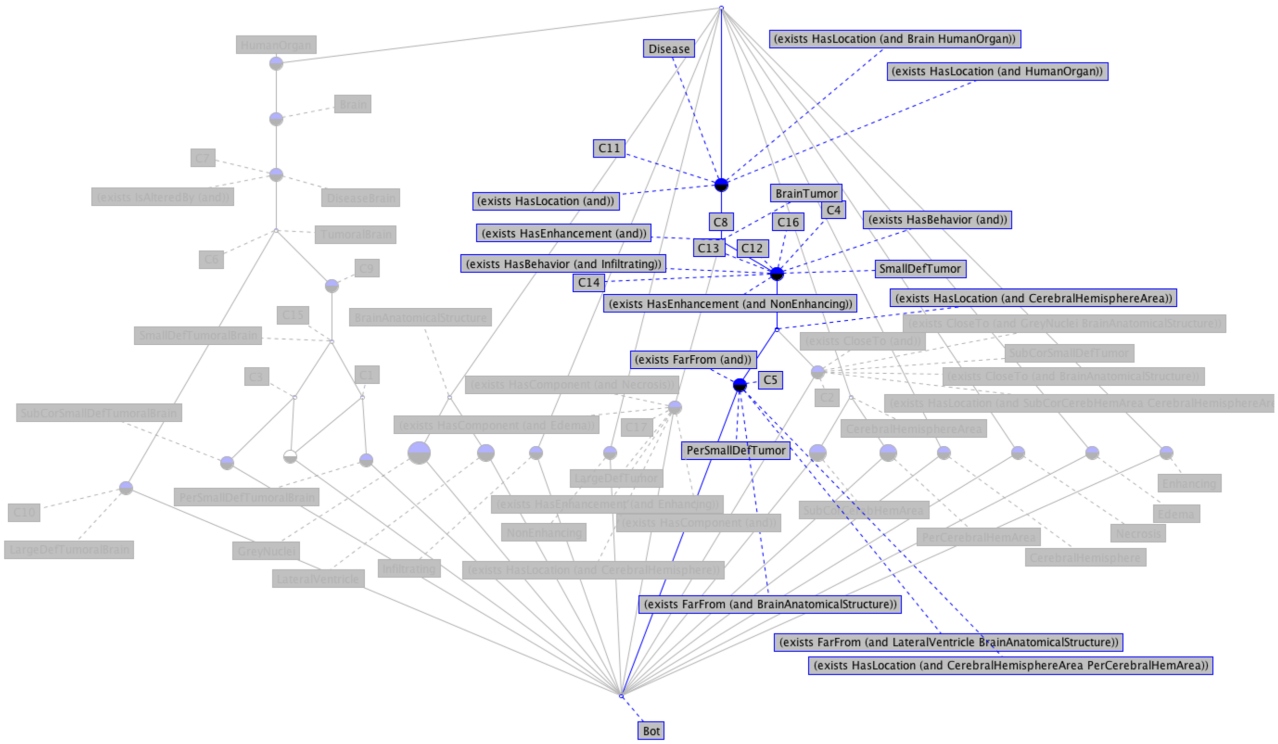


Fig. 12. Erosion path leading to compute a preferred explanation that is not a named concept. The concept to be explained is $\exists HasLocation.(Brain \sqcap HumanOrgan)$. The preferred solutions are in this case the following complex $\mathcal{EL}$-concepts: $\{\exists FarFrom.(LateralVentricle \sqcap BrainAnatomicalStructure), \exists HasLocation.(CerebralHemisphere \sqcap PerCerebralHemArea)\}$.

the individuals in the ABox resulting from low level image analysis. However, description logics are not expressive enough to represent aggregates since they involve at least three objects (for decidability reasons, DLs are restricted to the two-variable fragment of first order logic). Hence, the ontology is extended with the so-called DL-safe rules (rules applied to ABox individuals only) to represent and capture aggregate parts. Abduction is then performed by applying the rules in a backward chaining way to the query derived from the initial ABox. Our approach differs from the method explained so far by several aspects.

1) We consider a concept abduction problem while the authors in [37] considered an ABox abduction problem. In this sense our approach is more general since ABox individuals can be represented as nominals in the TBox (the DL $\mathcal{EL}^{++}$ extends $\mathcal{EL}$ with nominals).
2) The approach in [37] is based on aggregates and hence requires extending the DL with rules while our approach does not require such extensions.
3) Our abduction operators are sound and complete and are proved to satisfy rationality postulates and minimality constraints.
4) Last, but not least, our abduction service can compute complex concepts that are not explicit in the ontology (thanks to the concept lattice), while the approach defined in [37] is restricted to named individuals and concepts.

B. Choice of Morphological Operators

Other morphological operators can be defined as well, see for instance [58]. Defining dilation either from distances or directly is possible and can lead to interesting knowledge revision/negotiation/fusion operators. However, this is out of the scope of abductive reasoning since explaining a concept or GCI amounts, in our view, to filtering the most central concept of the observation. Hence, operations that are anti-extensive, such as the proposed erosions, are appropriate, while dilations are not. Within the context of abduction, opening has the required property of anti-extensivity and can lead to filter concepts belonging to the admissible solution sets, but it does not necessarily provide the most minimal ones (with respect to their cardinality). Closing is not appropriate for abduction since it is extensive. Other operators from mathematical morphology such as thinning or skeleton could be investigated.

C. Implementation Details

Our approach is based on the output of the exploration algorithm proposed in [49] and implemented using clojure, a LISP-like language based on JVM.[9] The implementation details along with the complexity analysis can be found in [63]. Since the total number of pseudointents of a given context $\mathbb{K} := (G, M, I)$ can be exponential in $|G|.|M|$, the stem base cannot be computed in a polynomial time. Furthermore, it has been proved in [64] that pseudointents cannot be enumerated in the lectic order with polynomial delay, i.e., the time between the output of one formal concept and the next one is polynomial in the size of the context.

However, within the context of image analysis, unlike semantic web mining, the number of concepts and roles in

[9] Java virtual machine.

the ontology is small. Hence, the approach is computationally tractable, *a fortiori* with new generation computers. Besides the concept lattice can be constructed and stored offline. The distance based morphological operators are of linear complexity with respect to the sum of the cardinalities of the attribute and of object sets: $\mathcal{O}(|G| + |M|)$, and the direct erosions are of linear complexity with respect to the cardinality of G.

VII. Conclusion

With the aim of image interpretation, we have proposed in this paper abductive inference services in description logics based on mathematical morphology over concept lattices. The construction of these lattices is based on exploiting the advances of using formal concept analysis in description logics. The properties and interpretations of the introduced explanatory operators were analyzed, and the rationality postulates of abductive reasoning were stated and extended to our context. Future work will concern the complexity analysis of these operators and associated algorithms, and a deeper investigation of their applications to image interpretation.

Acknowledgment

The authors would like to thank F. Distel for fruitful discussions, his criticism, and comments on a preliminary version of this paper and D. Borchmann for providing the $\mathcal{EL}$-exploration tool and for his kind support on its manipulation.

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