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A New Fuzzy Connectivity Measure for Fuzzy Sets

And Associated Fuzzy Attribute Openings

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Abstract Fuzzy set theory constitutes a powerful representation framework that can lead to more robustness in problems such as image segmentation and recognition. This robustness results to some extent from the partial recovery of the continuity that is lost during digitization. In this paper we deal with connectivity measures on fuzzy sets. We show that usual fuzzy connectivity definitions have some drawbacks, and we propose a new definition that exhibits better properties, in particular in terms of continuity. This definition leads to a nested family of hyperconnections associated with a tolerance parameter. We show that corresponding connected components can be efficiently extracted using simple operations on a max-tree representation. Then we define attribute openings based on crisp or fuzzy criteria. We illustrate a potential use of these filters in a brain segmentation and recognition process.

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1 Introduction

Connectivity is a key concept in image segmentation, filtering and pattern recognition, where objects of interest are often constrained to be connected according to some definition of connectivity. This definition depends on the selected object representation. Binary representation on a discrete grid remains the most widespread, and the connectivity is then generally derived from an elementary connectivity, such as 4- or 8-connectivity in 2D (for a square grid).

In [38], Serra introduced the notion of connection as an axiomatization of the classical definitions of connectivity. A connection (also referred to as connectivity class) on a space E is a family of subsets of E that are said connected. Among others the connected subsets of a binary image, the arc-connected subsets of $\mathbb{R}^n$ or the connected subsets of a graph can be represented by a connection. Another equivalent axiomatization was proposed by Ronse in [29]. Based on connections derived from usual notions of connectivity, more complex connections can be obtained. For instance we can assume that a subset is connected if its dilation is [14, 29], consider connections in attribute spaces [48] or make use of nested connections to represent multiscale connectivity [5].

Connections are closely related to connected operators (i.e. operators that manipulate only connected components according to a definition of connectivity). For example, considering a usual connection on the digital space, connected filters for binary images [12, 18, 37] modify only connected components of the object or of the background, without creating new boundaries nor moving existing ones.

In [40] the framework of connections was extended to general complete lattices and to the notion of hyperconnectivity (i.e. based on a more general definition of overlap), in particular to represent connectivity on grey scale images [7, 41]. Further properties of connections on complete lattices were given in [4, 31]. Connected filters for grey-level images [36, 46] were proposed independently, generally relying on the notion of flat-zones (i.e. the largest connected regions with constant grey level). Those filters were recently generalized to connective segmentation [30, 32, 42].

From a computational point of view, connected filtering is generally efficiently performed using a tree representation of the image. The well known max-tree representation (also referred to as the component tree [16, 25] or opening tree [47]) was introduced in [35] to compute attribute openings [8] and can be built using efficient algorithms [1, 19, 23, 25, 49]. The max-tree representation has also been extended to handle second-generation connectivity [28]. The tree of shapes proposed in [24] introduces a contrast-invariant tree representation of the image that allows the computation of more complex filters.

In this paper we deal with connectivity of fuzzy sets defined on the digital space and with associated connected attribute openings. Object representation using fuzzy sets [51] enables to model various types of imperfections, in particular related to image imprecision. Considering fuzzy representations of objects (for instance in an image segmentation and recognition process) can lead to more robustness that results, to some extent, from the partial recovery of the continuity that is lost during the digitization process. However the use of fuzzy representations leads also to more complexity in the definition of filters and their numerical implementation. Extending filters for binary sets to filters for fuzzy sets is sometimes possible, for instance using the extension principle [50]. The result of that extension is generally quite different from extensions to grey-level images. For instance when dealing with connected filters, the binary definition can be extended to the grey-level case stating that the connected components of a grey-level image are the largest regions that present a constant grey-level. Such filters are known as flat-zones connected filters [36, 46]. Obviously this extension does not make sense for fuzzy sets, since the semantics of pixels values is quite different. The values of a fuzzy set refer to a membership degree to a set while there is in general no obvious relation between grey-levels and their variations and membership degrees. Since in the binary case the connected components of a set are sets, the connected components of a fuzzy set should in the same manner be defined as fuzzy sets. Connectivity for fuzzy sets and connected operators thus need a specific definition that really takes into account the semantics of the membership values.

Table 1 Main notations

Notation	Definition
X	Spatial domain, i.e. a bounded subset of $\mathbb{Z}^n$
μ	A fuzzy set on X, i.e. a mapping from X to $[0, 1]$
$(\mu)_\alpha$	α-cut of μ
$\mathcal{F}$	Set of all fuzzy sets on X
$\mathcal{H}^1$	Set of connected fuzzy sets according to (1)
$\mathcal{H}^1_\tau$	Set of connected fuzzy sets according to (2)
$\mathcal{H}^2_\tau$	Set of connected fuzzy sets according to (4)
$c^1_\mu(x, y)$	Point to point connectivity degree with respect to a fuzzy set μ (cf. Definition 1)
$c^1(\mu)$	Connectivity degree of μ as defined by (3)
$c^2(\mu)$	Connectivity degree of μ according to Definition 6
$\delta^t_x(y)$	Impulse function: $\delta^t_x(y) = t$ if $y = x$ and 0 otherwise
$\perp^1, \perp^1_\tau, \perp^2_\tau$	Overlap mappings associated to $\mathcal{H}^1$, $\mathcal{H}^1_\tau$ and $\mathcal{H}^2_\tau$

In [33, 34], Rosenfeld proposed a first definition of fuzzy connectivity between points in the digital space according to a fuzzy set. Based on this definition, he derived a characterization of the connectivity of a fuzzy set, known as topographic connectivity. According to this characterization, a fuzzy set is connected if it presents a unique regional maximum or equivalently if all its α-cuts are connected. A similar definition for fuzzy sets on continuous spaces was proposed in [20].

Its later extension [6] leads to a characterization of the connectivity as a degree, defined as the membership degree of the lowest saddle point. This degree is however not continuous with respect to the membership function.

Therefore we propose a new definition that exhibits better properties, in particular in terms of continuity. We will show that this definition can be appropriately represented by a hyperconnection.

We first recall in Sect. 2 some preliminary definitions on fuzzy sets, connections and fuzzy connectivity. We illustrate in particular some of their drawbacks. In Sect. 3, we introduce a new connectivity measure, and we show that it leads to a nested family of hyperconnections indexed by a tolerance parameter, with nice continuity properties. Hyperconnected components are then defined, and an extraction scheme based on a max-tree representation is proposed. These notions lead to the definition of filters over hyperconnected components and in particular to attribute openings that may be used in a combined segmentation and recognition process. A generic formulation of such filters is given in Sect. 4, and in Sect. 5 two practical examples are presented. The first one illustrates the use of a fuzzy marker, while the second one makes use of a fuzzy volume prior. Both filters are illustrated in a recognition process on a brain magnetic resonance image (MRI). Proofs of all propositions are provided in Appendix. Table 1 brings together the main notations.

2 Background

2.1 Fuzzy Sets

Let X be a bounded subset of the digital space $\mathbb{Z}^n$ endowed with a discrete connectivity c_d. A fuzzy set on X will be denoted by its membership function $\mu : X \to [0, 1]$ which quantifies the membership degree of $x \in X$ to the fuzzy set. We only consider fuzzy sets having a bounded support (which is always the case if X is bounded). A fuzzy set μ is entirely characterized by the set of its α-cuts, denoted by $(\mu)_\alpha$: $(\mu)_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$. We denote by $\mathcal{F}$ the set of fuzzy sets defined on X. The binary relation $\leq$ on $\mathcal{F}$, defined by $\mu_1 \leq \mu_2 \Leftrightarrow \forall x \in X, \mu_1(x) \leq \mu_2(x)$, is a partial order, and $(\mathcal{F}, \leq)$ is a complete lattice. The supremum $\bigvee$ and infimum $\bigwedge$ over any family I of fuzzy sets are defined respectively as $\forall x \in X, (\bigvee_{\mu_i \in I} \mu_i)(x) = \sup_{\mu_i \in I}(\mu_i(x))$ and $\forall x \in X, (\bigwedge_{\mu_i \in I} \mu_i)(x) = \inf_{\mu_i \in I}(\mu_i(x))$. The smallest element is denoted by $0_\mathcal{F}$ and the largest element by $1_\mathcal{F}$. They are fuzzy sets with constant membership functions, equal to 0 and 1, respectively.

A family Δ of fuzzy sets on X is said to be sup-generating if $\forall \mu \in \mathcal{F}, \mu = \bigvee\{\delta \in \Delta \mid \delta \leq \mu\}$. We will consider in particular the family $\{\delta_x^t\}$ defined as $\delta_x^t(y) = t$ if $y = x$ and $\delta_x^t(y) = 0$ otherwise, which is sup-generating in the lattice $(\mathcal{F}, \leq)$.

As a metric on $\mathcal{F}$, inducing a definition of continuity, we use: $d_\infty(\mu_1, \mu_2) = \sup_{x \in X} |\mu_1(x) - \mu_2(x)|$, for which $(\mathcal{F}, d_\infty)$ is a metric space.

2.2 Fuzzy Connectivity

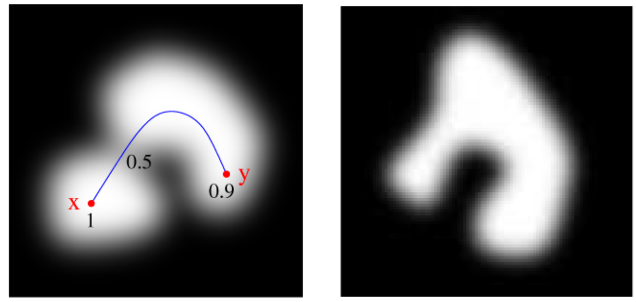
The first definition of fuzzy connectivity was proposed by Rosenfeld [33]. More precisely, a degree of connectivity between two points in a fuzzy set was defined, from which the connectivity of a fuzzy set was derived.

Definition 1 [33] The degree of connectivity between two points x and y of X in a fuzzy set μ ($\mu \in \mathcal{F}$) is defined as:

$$c_\mu^1(x, y) = \max_{\substack{l \in L_{x,y} \\ l = \{x_0 = x, x_1, \ldots, x_n = y\}}} \min_{0 \leq i \leq n} \mu(x_i)$$

where $L_{x,y}$ denotes the set of digital paths from x to y, according to the underlying digital connectivity defined on X.

This degree of connectivity is symmetrical in x and y (i.e. $\forall (x, y) \in X^2, c_\mu^1(x, y) = c_\mu^1(y, x)$), weakly reflexive (i.e. $\forall (x, y) \in X^2, c_\mu^1(x, x) \geq c_\mu^1(x, y)$), and max-min transitive (i.e. $\forall (x, y, z) \in X^3, c_\mu^1(x, z) \geq \min(c_\mu^1(x, y), c_\mu^1(y, z))$). It is thus a similitude relation over X. We also have $\forall x \in X$, $c_\mu^1(x, x) = \mu(x)$ and $\forall (x, y) \in X^2$, $c_\mu^1(x, y) \leq \min(\mu(x), \mu(y))$. Moreover, the following monotony property holds: $\forall (\mu_1, \mu_2) \in \mathcal{F}^2, \mu_1 \leq \mu_2 \Rightarrow \forall (x, y) \in X^2$, $c_{\mu_1}^1(x, y) \leq c_{\mu_2}^1(x, y)$.



(a) (b)

Fig. 1 (**a**) A non-connected fuzzy set according to Definition 2, and membership values on the path defining the degree of connectivity between two points x and y. (**b**) A connected fuzzy set

This definition was incorporated in segmentation processes, based on markers [9, 15, 43]. The idea was to extend this definition by defining an affinity measure between image points based on adjacency and grey level similarity.

Definition 2 [33] A fuzzy set μ is said connected if

$$\forall (x, y) \in X^2, \quad c_\mu^1(x, y) = \min(\mu(x), \mu(y)).$$

Proposition 1 [33, 34] *A fuzzy set is connected iff all its α-cuts are connected* (*in the sense of the digital connectivity c_d on X*).

Proposition 2 [33, 34] *A fuzzy set μ is connected iff it has a unique regional maximum.*[1]

These definitions are illustrated in Fig. 1. One of the optimal paths between x and y (achieving the max-min criterion of the definition) is displayed in (a), and the minimal value on this path is 0.5, which provides the degree of connectivity between x and y. The fuzzy set in (a) is non-connected since $c_\mu^1(x, y) = 0.5$, which is strictly less than the membership degrees of x and y ($\mu(x) = 1$ and $\mu(y) = 0.9$). On the contrary, the fuzzy set in Fig. 1(b) is connected.

2.3 Connections and Hyperconnections

Definition 2 provides a crisp definition of the connectivity of a fuzzy set. However, if a set is fuzzy, it may be intuitively more satisfactory to consider that its connectivity is also a matter of degree. The notions of connection and hyperconnection [21, 22, 29, 38, 40] provide an appropriate

[1] A regional maximum $R \subseteq X$ of a fuzzy set μ is a connected component (according to the discrete connectivity c_d) of an α-cut μ_α, such that $\forall x \in R, \mu(x) = \alpha$.

framework to this aim. We consider here the axiomatization of connectivity classes with canonical markers proposed in Sect. 2.3 of [40], which was also considered in [4, 31].

Definition 3 [40] Let $(L, \leq)$ be a complete lattice with sup-generating family S and 0_L its smallest element. A connected class, or connection, $\mathcal{C}$ is a family of elements of L such that:

1. $0_L \in \mathcal{C}$,
2. $S \subseteq \mathcal{C}$,
3. for any family $\{C_i\}$ of elements of $\mathcal{C}$ such that $\bigwedge_i C_i \neq 0_L$, then $\bigvee_i C_i \in \mathcal{C}$.

This definition provides an abstract framework for manipulating connectivity notions. Generic properties can be derived, without referring explicitly to the considered space and the underlying connectivity. As will be seen in Sect. 3.3, connected components of a set A can be simply defined as the greatest elements of $\mathcal{C}$ which are smaller than A according to the spatial ordering of the lattice.

Let us first consider the lattice $(\mathcal{P}(X), \subseteq)$. On this lattice, we use the usual connection $\mathcal{C}_d$ [39] induced by a digital connectivity c_d on X (in the sense of the graph of digital points). An element of $\mathcal{C}_d$ is then simply a subset A of X that is connected in the sense of c_d (i.e. $\forall (x, y) \in A^2, \exists x_0 = x, x_1, \ldots, x_n = y, \forall i < n, x_i \in A$, and $c_d(x_i, x_{i+1}) = 1$). It is easy to check that the class $\mathcal{C}_d$ satisfies all conditions of Definition 3:

1. $\emptyset \in \mathcal{C}_d$,
2. points constitute a sup-generating family for $\mathcal{P}(X)$ and belong to $\mathcal{C}_d$,
3. if a family $\{A_i\}$ of elements of $\mathcal{C}_d$ satisfies $\bigcap_i A_i \neq \emptyset$, we get $\bigcup_i A_i \in \mathcal{C}_d$ since $\forall (x, y) \in \bigcup_i A_i$ it is possible to find a path from x to y in $\bigcup_i A_i$ that meets the intersection $\bigcap_i A_i$.

Other examples of connections defined on continuous spaces or on graphs can be found e.g. in [6].

A connection $\mathcal{C}$ defined on a complete lattice L and a sup-generating family S for L can be characterized by a family of openings $\{\gamma_x, x \in S \setminus \{0_L\}\}$, which are called connectivity openings, satisfying the following conditions [40]:

1. $\forall x \in S \setminus \{0_L\}, \gamma_x(x) = x$,
2. $\forall A \in L, \forall (x, y) \in (S \setminus \{0_L\})^2, (\gamma_x(A) = \gamma_y(A))$ or $(\gamma_x(A) \wedge \gamma_y(A) = 0_L)$,
3. $\forall A \in L, \forall x \in S \setminus \{0_L\}, x \leq A$ or $\gamma_x(A) = 0_L$.

These openings are defined as:

$$\gamma_x(A) = \bigvee \{C \in \mathcal{C} \mid x \leq C \leq A\}.$$

Conversely, if $\mathcal{C}$ is a sup-generating family of L which corresponds to the invariant elements of a family $\{\gamma_x, x \in S\}$ satisfying the previous conditions, then $\mathcal{C}$ is a connection. The element $\gamma_x(A)$ is then the connected component of A (according to the connection $\mathcal{C}$) which contains x.

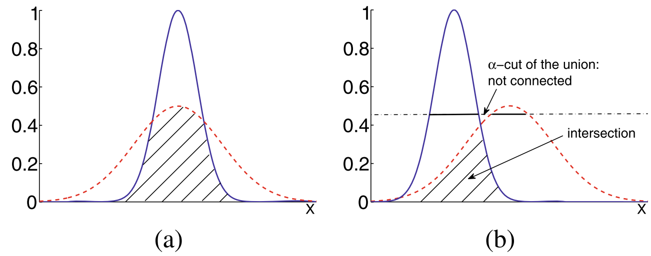


Fig. 2 Examples of 1D fuzzy sets. (**a**) The union is connected in the sense of Definition 2. (**b**) The union is not connected

Let us again consider the lattice $(\mathcal{P}(X), \subseteq)$. For point $x \in X$ and a set $A \subseteq X$, $\gamma_{\{x\}}(A)$ is the connected component of A containing x in the usual sense. This is illustrated in Fig. 8 in Sect. 3.3.

An equivalent axiomatization, based on the notion of separation, has been proposed in [29].

Now, on the lattice $(\mathcal{F}, \leq)$, let us consider the crisp definition of connectivity in Definition 2, and the 1D examples in Fig. 2. In (a), each fuzzy set is connected, and so is their union (defined as the point-wise maximum of membership functions). However, in (b), the union is not connected, although each fuzzy set is connected and their intersection is not equal to $0_{\mathcal{F}}$. Therefore Definition 3 cannot account for this type of situation on the lattice of fuzzy sets. Dealing with such cases require to replace the infimum ($\bigwedge$) in condition 3 by another overlap mapping $\perp$ [40], leading to the notion of hyperconnection.

Definition 4 [6, 40] Let $(L, \leq)$ be a complete lattice. A hyperconnection $\mathcal{H}$ is a family of elements of L such that:

1. $0_L \in \mathcal{H}$,
2. $\mathcal{H}$ contains a sup-generating family S of L,
3. for any family $\{H_i\}$ of elements of $\mathcal{H}$ such that $\perp_i H_i \neq 0_L$, then $\bigvee_i H_i \in \mathcal{H}$.

As for connections, hyperconnectivity openings associated with $\mathcal{H}$ can be defined:

$$\eta_x(A) = \bigvee \{h \in \mathcal{H} \mid x \leq h \leq A\},$$

with $x \in S$ and $A \in L$. However, some properties may not hold anymore in the case of hyperconnections. In particular, the property $\eta_x(A) \in \mathcal{H}$ may not hold, and it does not for hyperconnections associated with fuzzy sets.

On the lattice $(\mathcal{F}, \leq)$, let us consider the following hyperconnection:

$$\mathcal{H}^1 = \{\mu \in \mathcal{F} \mid \forall (x, y) \in X^2, c^1_\mu(x, y) = \min(\mu(x), \mu(y))\}, \quad (1)$$

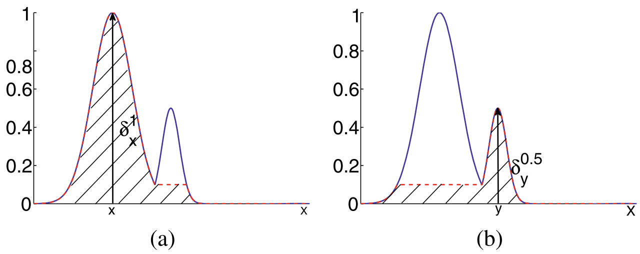


Fig. 3 A 1D fuzzy set μ (*plain*). x and y two points that belong to its regional maxima. Corresponding connected openings: (**a**) $\eta^1_{\delta_x^{\mu(x)}}(\mu)$ (*dashed*) and (**b**) $\eta^1_{\delta_y^{\mu(y)}}(\mu)$ (*dashed*)

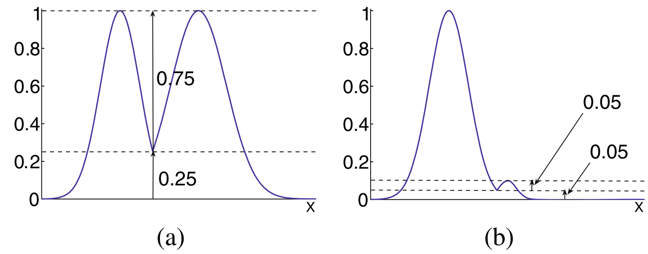


Fig. 4 (**a**) The degree of connectivity of the 1D fuzzy set according to c^1 is equal to 0.25. (**b**) The degree of connectivity according to c^1 is equal to 0.05, although this fuzzy set seems to be more connected than the one in (**a**). According to c^2 (see Sect. 3.1), we obtain a connectivity degree of 0.25 (**a**) and 0.95 (**b**)

which contains the connected fuzzy sets according to Definition 2. It is obtained for the overlap mapping $\perp^1$ defined as [6]:

$$\perp^1(\{\mu_i\}) = \begin{cases} 1 & \text{if } \forall\alpha \in [0,1],\ \bigcap_i \{(\mu_i)_\alpha \mid (\mu_i)_\alpha \neq \emptyset\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the sake of simplicity, we denote the values taken by $\perp^1$ as 1 and 0 (instead of $1_{\mathcal{F}}$ and $0_{\mathcal{F}}$). It is easy to check that the union of connected fuzzy sets such that their non empty α-cuts intersect is connected in the sense of Definition 2. For instance in Fig. 2, the two fuzzy sets in (a) belong to $\mathcal{H}^1$ since they are connected according to Definition 2. All their non empty α-cuts intersect, we thus have $\perp^1(\mu_1,\mu_2) = 1$ and their union belongs to $\mathcal{H}^1$. The two fuzzy sets in (b) are also connected but some of their non empty α-cuts do not intersect and their union do not belong to $\mathcal{H}^1$.

For $\nu \in \mathcal{F}$, let $\eta^1_\nu(\mu)$ denote the connected opening with origin ν associated with this hyperconnection:

$$\eta^1_\nu(\mu) = \bigvee\{h \in \mathcal{H}^1 \mid \nu \leq h \leq \mu\}.$$

Proposition 3 *Let y be a point of a regional maximum of μ. Then $\eta^1_{\delta_y^{\mu(y)}}(\mu)$ belongs to $\mathcal{H}^1$ and $\forall x \in X$, $\eta^1_{\delta_y^{\mu(y)}}(\mu)(x) = c^1_\mu(x,y)$.*

For instance in Fig. 3, the 1D fuzzy set μ presents two regional maxima and x and y are two points that belong to those maxima. $\eta^1_{\delta_x^{\mu(x)}}(\mu)$ (a) and $\eta^1_{\delta_y^{\mu(y)}}(\mu)$ (b) belong to $\mathcal{H}^1$ since all their α-cuts are connected. Moreover the max-min criterion of the connectivity degree $c^1_\mu(x,y)$ between x and y is reached for the saddle point (whose membership degree is 0.1) between the regional maxima. We check that $\eta^1_{\delta_x^{\mu(x)}}(\mu)(y) = 0.1 = c^1_\mu(x,y)$ and $\eta^1_{\delta_y^{\mu(y)}}(\mu)(x) = 0.1 = c^1_\mu(y,x)$.

The overlap mapping $\perp^1$ was extended in [6] to the following family indexed by a parameter τ:

$$\perp^1_\tau(\{\mu_i\}) = \begin{cases} 1 & \text{if } \forall\alpha \leq \tau,\ \bigcap_i \{(\mu_i)_\alpha \mid (\mu_i)_\alpha \neq \emptyset\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

If the set of subsets over which the intersection is taken is empty, we use the classical lattice rule $\bigwedge\emptyset = 1_L$ (and $\bigvee\emptyset = 0_L$).

Let us define:

$$\mathcal{H}^1_\tau = \{\mu \in \mathcal{F} \mid \forall\alpha \leq \tau, (\mu)_\alpha \in \mathcal{C}_d\}. \tag{2}$$

Proposition 4 [6] *Each $\mathcal{H}^1_\tau$ is a hyperconnection, i.e. verifies all items of Definition 4, for the overlap mapping $\perp^1_\tau$. It contains in particular the sup-generating family $\Delta = \{\delta^t_x, x \in X, t \in [0,1]\}$. The family $\{\mathcal{H}^1_\tau, \tau \in [0,1]\}$ is decreasing with respect to τ: $\tau_1 \leq \tau_2 \Rightarrow \mathcal{H}^1_{\tau_2} \subseteq \mathcal{H}^1_{\tau_1}$.*

For $\tau = 1$, we have $\perp^1_\tau = \perp^1$ and $\mathcal{H}^1_\tau = \mathcal{H}^1$. The family $\{\mathcal{H}^1_\tau, \tau \in [0,1]\}$ is then an extension of the hyperconnection $\mathcal{H}^1$ (which is associated with Rosenfeld's notion of fuzzy connectivity).

Now the connectivity of a fuzzy set can be defined as a degree, instead of a crisp notion, as follows:

$$\begin{aligned} c^1(\mu) &= \sup\{\tau \in [0,1] \mid \mu \in \mathcal{H}^1_\tau\} \\ &= \sup\{\tau \in [0,1] \mid \forall\alpha \leq \tau, (\mu)_\alpha \in \mathcal{C}_d\}. \end{aligned} \tag{3}$$

As an illustration, the fuzzy sets in Fig. 4(a) and (b) have a degree of connectivity of 0.25 and 0.05, respectively. However, intuitively we would rather say that the example in (b) is more connected than the one in (a), which seems to have two distinct parts. The degree of connectivity depends on the height of the lowest minimum or saddle point, and not on its depth. A small modification in (b) would make the fuzzy set fully connected, illustrating that this definition is not continuous.

The contribution of this paper aims at overcoming this drawback.

3 A New Class of Connectivity

3.1 Connectivity Measure

In this section, we introduce a new extension of the fuzzy connectivity introduced by Rosenfeld [33]. In the sense of Definition 2, a fuzzy set μ is connected iff $\forall(x,y)\in X^2$, $c^1_\mu(x,y)=\min(\mu(x),\mu(y))$. We propose to define the degree of connectivity of a fuzzy set μ as a degree of satisfaction of this equality.

Let us consider two fixed points x and y, then the degree of satisfaction of the equality $c^1_\mu(x,y)=\min(\mu(x),\mu(y))$ can be characterized from the Lukasiewicz equality operator [10] defined as: $\forall(a,b)\in[0,1]^2$, $\mu_=(a,b)=1-|a-b|$. Rewriting this expression for $a=\min(\mu(x),\mu(y))$ and $b=c^1_\mu(x,y)$ leads to the following definition.

Definition 5 The connectivity degree between two points x and y in a fuzzy set μ is defined as:

$$\begin{aligned} c^2_\mu(x,y) &= 1-|\min(\mu(x),\mu(y))-c^1_\mu(x,y)| \\ &= 1-\min(\mu(x),\mu(y))+c^1_\mu(x,y), \end{aligned}$$

since by definition $c^1_\mu(x,y)\le\min(\mu(x),\mu(y))$.

The degree $c^2_\mu(x,y)$ is obtained as 1 minus the difference between the membership degrees of x and y on the one hand and the reached minimum through the optimal path on the other hand.

This measure takes its values in $[0,1]$, is symmetrical and reflexive ($c^2_\mu(x,x)=1$). It is not transitive, but satisfies the following weaker property:

Proposition 5 *Let x_m such that $\forall x\in X$, $\mu(x_m)\ge\mu(x)$ (the global maximum of μ is always reached since μ is assumed to have a bounded support and X is discrete). The following inequality is satisfied*:

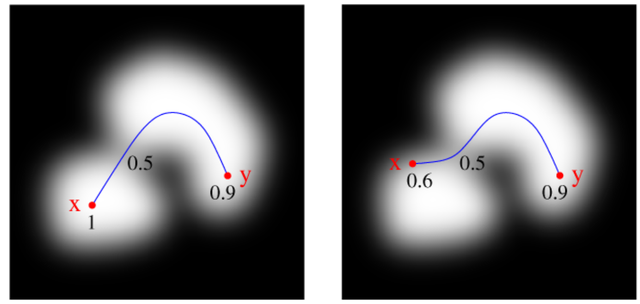
$$\forall(x,y)\in X^2,\quad c^2_\mu(x,y)\ge\min(c^2_\mu(x,x_m),c^2_\mu(x_m,y)).$$

The following property can then be derived.

Proposition 6 *$c^2_\mu(x,y)$ reaches its minimum for x being a point for which a global maximum of μ is reached and y belonging to a regional maximum of μ.*

Note that this minimum can also be reached for other values for x and y.

Figure 5 illustrates this property. The connectivity degree $c^2_\mu(x,y)$ between two points x and y belonging to the two regional maxima of the fuzzy set μ (a) is equal to 0.6 $(1-\min(1,0.9)+c^1_\mu(x,y)=1-0.9+0.5=0.6)$. One



(a) (b)

Fig. 5 (Color online) (**a**) The connectivity degree $c^2_\mu(x,y)$ is equal to 0.6. One of the paths achieving this connectivity degree is depicted in *blue*. (**b**) The connectivity degree $c^2_\mu(x,y)$ is equal to 0.9

of the optimal paths between x and y (achieving the max-min criterion in the definition of c^1_μ) is depicted in blue. The minimal value through this path is equal to 0.5. In (b) x does no longer belong to a regional maximum. The connectivity degree $c^2_\mu(x,y)$ is now equal to 0.9 $(1-\min(1,0.6)+0.5)$, which is indeed higher than the connectivity degree obtained for x and y belonging to the two regional maxima of μ.

From this degree of connectivity between two points we derive the following definition of the connectivity degree of a fuzzy set.

Definition 6 The connectivity degree of a fuzzy set μ is defined as : $c^2(\mu)=\min_{(x,y)\in X^2}c^2_\mu(x,y)$.

For any two given points x and y, $c^1_\mu(x,y)$ and $c^2_\mu(x,y)$ are achieved for the same point on the same path from x to y, but $c^1(\mu)$ and $c^2(\mu)$ are not achieved for the same points. Roughly speaking, the connectivity degree of a fuzzy set depends now on the depth of the deepest saddle point in the fuzzy set.[2] On the examples illustrated in Fig. 4, it can be observed that the fuzzy set in (a) is 0.25-connected $(1-1+0.25)$, while the fuzzy set in (b) is 0.95-connected. In both cases, the minimum $(c^2(\mu)=\min_{(x,y)\in X^2}c^2_\mu(x,y))$

[2] Note that there exists some links between $c^2(\mu)$ and the notion of grey-level dynamics defined in [2, 11]. Indeed we can rewrite the dynamic of a point x_i that belongs to a regional maximum as: $\Delta_\mu(x_i)=\mu(x_i)-\max\{c^1_\mu(x_i,x_j)\mid\mu(x_i)<\mu(x_j)\text{ and }x_j\in\mathcal{M}(\mu)\}$ (where $\mathcal{M}(\mu)$ is the set of points that belong to a regional maximum) and $\Delta_\mu(x_i)=+\infty$ if x_i belongs to a global maximum. If we compute the maximal dynamic over all regional maxima x_i (except the global maxima), we obtain the expression $\max_{x_i}(\mu(x_i)-\max\{c^1_\mu(x_i,x_j)\mid\mu(x_i)<\mu(x_j)\text{ and }x_j\in\mathcal{M}(\mu)\})$. On the other hand as the minimum of $c^2_\mu(x,y)$ is reached for x and y belonging to regional maxima (cf. Proposition 6), we can rewrite $c^2(\mu)=1-\max_{x_i\in\mathcal{M}(\mu)}(\mu(x_i)-\min\{c^1_\mu(x_i,x_j)\mid\mu(x_i)\le\mu(x_j)\text{ and }x_j\in\mathcal{M}(\mu)\})$. For some examples the connectivity degree c^2 is thus linked to the maximal dynamic, but this is generally not the case, especially when the minimum for c^2 is reached for two points belonging to global maxima.

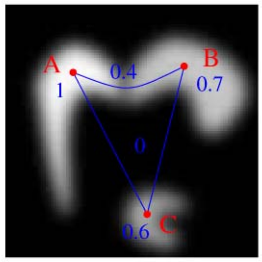


Fig. 6 A fuzzy set μ with a connectivity degree $c^2(\mu)$ equal to 0.4

is reached for x and y corresponding to the two local maxima. On these examples, if one mode progressively shrinks to 0, the connectivity degree c^2 will evolve smoothly towards 1. This property is expressed formally by the following result, using as a distance between fuzzy sets μ_1 and μ_2: $d_\infty(\mu_1, \mu_2) = \sup_{(x,y)\in X^2} |\mu_1(x, y) - \mu_2(x, y)|$.

Proposition 7 *For fixed x and y, the mapping associating μ and $c^1_\mu(x, y)$ is Lipschitz and therefore continuous, and the mapping associating μ and $c^2_\mu(x, y)$ is 2-Lipschitz with respect to the d_∞ metric on $\mathcal{F}$.*

Proposition 8 *The mapping associating μ and $c^2(\mu)$ is 2-Lipschitz.*

Figure 6 illustrates these definitions on a 2D example. The fuzzy set presents three regional maxima whose height are respectively 1, 0.7 and 0.6, and containing the points A, B et C (A is a global maximum). One of the optimal paths between A and B (achieving the max − min criterion in the definition of $c^2_\mu(A, B)$) is provided. It has a minimum value of 0.4. We obtain $c^2_\mu(A, B) = 1 - \min(1, 0.7) + 0.4 = 0.7$. All paths between A and C present a minimum value of 0. The connectivity degree according to c^1 between A and C is hence equal to 0. The connectivity degree according to c^2 corresponds to $c^2_\mu(A, C) = 1 - \min(1, 0.6) + 0 = 0.4$.[3] Similarly, we obtain $c^2_\mu(B, C) = 1 - \min(0.7, 0.6) + 0 = 0.4$. Since according to Proposition 6, the connectivity degree $c^2(\mu)$ can be computed from the regional maxima of μ, we thus obtain $c^2(\mu) = 0.4$.

3.2 Link with a hyperconnection

We define the family $\mathcal{H}^2_\tau$ indexed by the connectivity degree τ as follows:

$$\mathcal{H}^2_\tau = \{\mu \in \mathcal{F} \mid c^2(\mu) \geq \tau\}. \quad (4)$$

[3]The value of $c^2_\mu(A, C)$ is not equal to zero even if there does not exist any path between A a C with a minimum different from 0. In fact the degree $c^2_\mu(A, C)$ is obtained as 1 minus the membership degrees difference between A and C on the one hand and the reached minimum through the optimal path on the other hand. It would therefore be zero if the membership degrees of A and C would be equal to 1 and if the minimum of all paths between A and C would be equal 0.

Each set $\mathcal{H}^2_\tau$ contains all the fuzzy sets with a connectivity degree higher than or equal to τ, in the sense of c^2. We will show in the sequel that this family is a family of hyperconnections and specify its associated overlap measure.

If we consider the example in Fig. 7, the connectivity degree, according to c^2, of the fuzzy set μ_A depicted in (a) is equal to 0.6 (the minimum is in fact achieved for x and y corresponding to the two regional maxima; in this case $c^1_{\mu_A}(x, y) = 0.1$ and thus $c^2(\mu_A) = 1 - \min(1, 0.5) + 0.1 = 0.6$). Let us consider now a second fuzzy set μ_B depicted in red dashed line in (b) with a connectivity degree in the sense of c^2 equal to 1. We are interested in the family $\mathcal{H}^2_{0.6}$. Our aim is to derive an overlap measure that will be satisfied for at least 0.6-connected fuzzy sets if it is satisfied for their union as well. In Fig. 7(b) one can easily check that the union of μ_A and μ_B is 0.4-connected ($\mu_A \vee \mu_B \notin \mathcal{H}^2_{0.6}$) and that the α-cuts of the two fuzzy sets intersect for α lower than or equal to 0.5. However, for the configuration in Fig. 7(c) the union is 0.7-connected ($\mu_A \vee \mu_B \in \mathcal{H}^2_{0.6}$) whereas the two sets also intersect only for the levels lower than or equal to 0.5.

Conversely to the case in Fig. 7(c), in the case exhibited in (b), the fuzzy set μ_B intersects a "secondary mode" (i.e. corresponding to a regional maximum that is not a global maximum) of μ_A and does not intersect its "principal mode" (i.e. corresponding to a global maximum). The overlap measure associated with the hyperconnection $\mathcal{H}^2_\tau$ relies therefore on the overlap of the "principal modes" of the considered fuzzy sets. In the following definition these modes can be defined as the connected openings with origin $\delta^{h_i}_{x_i}$: $\eta^1_{\delta^{h_i}_{x_i}}(\mu_i)$, where μ_i is the fuzzy set, x_i a point where the global maximum of μ_i is reached and $h_i = \mu_i(x_i)$.

We therefore propose a new overlap measure, considering that two fuzzy sets do not overlap if they "do not significantly overlap", as follows:

$$\perp^2_\tau(\{\mu_i\}) = \begin{cases} 1 & \text{if } \forall \alpha \in [0, 1], \\ & \bigcap_i \{(\eta^1_{\delta^{h_i}_{x_i}}(\mu_i))_\alpha \mid \alpha \leq h_i - 1 + \tau\} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where x_i is any point for which the global maximum of μ_i is reached.

A family of fuzzy sets $\{\mu_i\}$ overlaps according to this measure if the α-cuts with $\alpha \leq h_i - 1 + \tau$ of their principal modes overlap. The value $h_i - 1 + \tau$ guarantees that saddle points in $\bigvee\{\mu_i\}$ (if $\forall i$, $c^2_{\mu_i} \geq \tau$) have a maximal "depth" of $1 - \tau$. If the fuzzy set $\bigvee\{\mu_i\}$ does not reveal any saddle point with a "depth" higher than $1 - \tau$, its connectivity degree according to c^2 will then be higher than τ.

Let us again consider the examples in Fig. 7 with $\tau = 0.6$. The "principal modes" $\eta^1_{\delta^{h_i}_{x_i}}(\mu_i)$ of the two fuzzy sets in (c) are depicted in (d). The height h_A of μ_A (in blue) is equal to 1. The α-cuts of $\eta^1_{\delta^{h_A}_{x_A}}(\mu_A)$ used in the calculus of the overlap measure verify $\alpha \leq h_A - 1 + 0.6 = 0.6$. Likewise the height h_B of μ_B (in red dashed line) is equal to 0.8 and the α-cuts to consider verify $\alpha \leq h_B - 1 + 0.6 = 0.4$. For α comprised between 0 and 0.4, the α-cuts $(\eta^1_{\delta^{h_i}_{x_i}}(\mu_i))_\alpha$ associated with the two fuzzy sets intersect. For α comprised between 0.4 and 0.6, the condition $\alpha \leq h_i - 1 + \tau$ is not satisfied by μ_B and therefore we consider only the α-cuts associated with μ_A. The intersection is thus not zero. For α higher than 0.6, the condition $\alpha \leq h_i - 1 + \tau$ is never satisfied. The intersection of an empty family being non-zero, the overlap condition is satisfied. Therefore we have $c^2(\mu_A) \geq 0.6$, $c^2(\mu_B) \geq 0.6$ and $\perp^2_{0.6}(\{\mu_A, \mu_B\}) = 1$ and we can verify that $c^2(\mu_A \vee \mu_B) \geq 0.6$.

Let us consider now $\tau = 0.8$ and μ_A and μ_B the fuzzy sets depicted respectively in blue and in red in Fig. 7(d). The α-cuts of μ_B used in the calculus of the overlap measure verify $\alpha \leq 0.6$. However the α-cuts of $\eta^1_{\delta^{h_A}_{x_A}}(\mu_A)$ and $\eta^1_{\delta^{h_B}_{x_B}}(\mu_B)$ do not intersect between the levels 0.5 and 0.6; the overlap measure is therefore zero. We can also verify that the connectivity degree of the union of the two fuzzy sets is equal to 0.7. We thus have in (d): $c^2(\mu_A) \geq 0.8$, $c^2(\mu_B) \geq 0.8$, $\perp^2_{0.8}(\{\mu_A, \mu_B\}) = 0$ and $c^2(\mu_A \vee \mu_B) < 0.8$.

Proposition 9 *$\mathcal{H}^2_\tau$ defines a hyperconnection for the overlap measure $\perp^2_\tau$.*

For $\tau = 1$, the hyperconnection $\mathcal{H}^2_\tau$ contains the fuzzy sets that are connected according to Definition 2. We then have $\mathcal{H}^2_1 = \mathcal{H}^1_1 = \mathcal{H}^1$. The family $\{\mathcal{H}^2_\tau, \tau \in [0, 1]\}$ is decreasing with respect to τ: $\tau_1 \leq \tau_2 \Rightarrow \mathcal{H}^2_{\tau_2} \subseteq \mathcal{H}^2_{\tau_1}$. These definitions lead us to a connected component definition that is more interesting than the one associated with $\mathcal{H}^1_\tau$, as shown next.

3.3 Connected Components

In the general framework of connections, connected components of an element A of a lattice $(L, \leq)$, relatively to a connection $\mathcal{C}$ on L, are the elements C_i of $\mathcal{C}$ such that: $C_i \leq A$ and $\nexists C \in \mathcal{C}, C_i < C \leq A$ (i.e. C_i are the largest elements of $\mathcal{C}$ that are smaller than A) [40]. The connected components can be extracted using the following connected openings: $\gamma_x(A)$ corresponds to the connected component of A including the element $x \in \mathcal{C}$ (see Sect. 2.3).

If we consider for instance the lattice $(\mathcal{P}(X), \subseteq)$, the set A presented in Fig. 8 contains five connected components. The connected component $\gamma_{\{x\}}(A)$ extracted using the connected opening associated with the point x is depicted in red.

This definition extends naturally to hyperconnections. Let $\mathcal{H}$ be a hyperconnection on a lattice $(L, \leq)$. The hyperconnected components of $A \in L$ are the elements H_i of

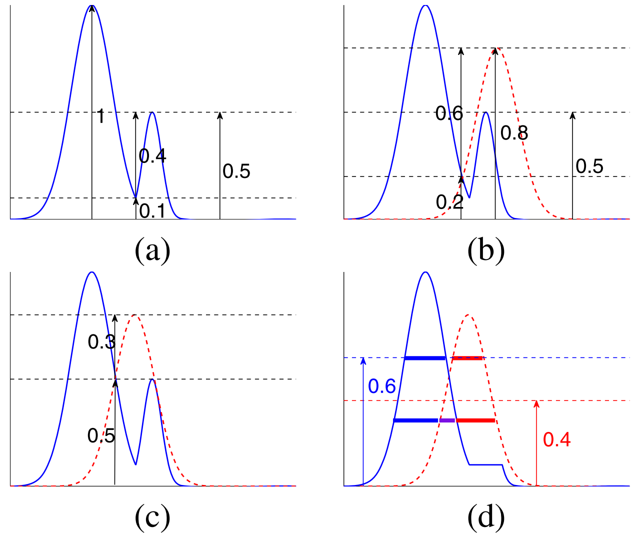
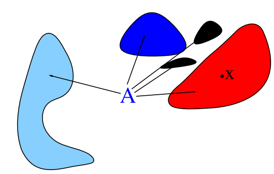


Fig. 7 (Color online) (**a**) The fuzzy set μ_A contains one principal mode and one secondary mode. (**b**) The fuzzy set μ_B (in *red dashed line*) does not overlap with μ_A according to $\perp^2_{0.6}$. (**c**) Fuzzy sets overlap with respect to $\perp^2_{0.6}$ due to the fact that the α-cuts of their "principal modes" (**d**) overlap at least for all levels lower than 0.4

Fig. 8 Connected components of the object $A \subseteq X$



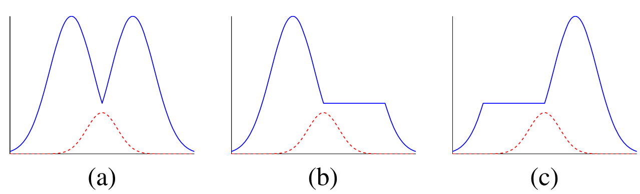


Fig. 9 (Color online) (**a**) Two 1D fuzzy sets, μ in blue and ν in *red dashed line*. The opening result $\eta^1_\nu(\mu)$ does not belong to $\mathcal{H}^1$: the fuzzy sets μ_1 (**b**) and μ_2 (**c**) verify $\nu \leq \mu_i \leq \mu$ and $\mu_i \in \mathcal{H}^1$, hence $\eta^1_\nu(\mu) = \mu$ but $\mu \notin \mathcal{H}^1$

$\mathcal{H}$ such that: $H_i \leq A$ and $\nexists H \in \mathcal{H}, H_i < H \leq A$. However $\eta_x(A)$ (where $x \in L$) does not necessarily correspond to the hyperconnected component of A containing x (in contrary to the case of connections). In fact, nothing ensures that $\eta_x(A)$ would belong to $\mathcal{H}$. Figure 9 presents an example where the opening result $\eta^1_\nu(\mu)$ (associated with the hyperconnection $\mathcal{H}^1$) does not belong to $\mathcal{H}^1$. By definition $\eta^1_\nu(\mu) = \bigvee\{h \in \mathcal{H}^1 \mid \nu \leq h \leq \mu\}$. Since $\nu \leq \mu_1 \leq \mu$ and $\mu_1 \in \mathcal{H}^1$, we then deduce that $\mu_1 \leq \eta^1_\nu(\mu)$. Similarly, we derive $\mu_2 \leq \eta^1_\nu(\mu)$. Moreover, $\mu = \mu_1 \vee \mu_2$ and therefore $\eta^1_\nu(\mu) = \mu$. But $\mu \notin \mathcal{H}^1$.

Proposition 10 [6] *Let $x \in \mathcal{H}$ and $A \in L$. If $\eta_x(A) \in \mathcal{H}$ then $\eta_x(A)$ is a hyperconnected component of A in the sense of $\mathcal{H}$.*

If $\perp$ is an overlap measure associated with a hyperconnection, then two hyperconnected components H_i and H_j of A verify either $H_i = H_j$, or $H_i \perp H_j = 0$. Furthermore $\bigvee_i H_i = A$, where the supremum is computed over all the hyperconnected components of A. For the hyperconnection $\mathcal{H}^2_\tau$, we will speak of τ-hyperconnected component and will denote by $\mathcal{H}^2_\tau(\mu)$ the set of τ-hyperconnected components of μ. Similarly, we will denote by $\mathcal{H}^1(\mu)$ $(= \mathcal{H}^2_1(\mu) = \mathcal{H}^1_1(\mu))$ the set of hyperconnected components of μ according to $\mathcal{H}^1$.

Proposition 11 *If x_i belongs to a regional maximum of μ and $h_i = \mu(x_i)$, then $\eta^1_{\delta^{h_i}_{x_i}}(\mu)$ is a hyperconnected component of μ according to $\mathcal{H}^1$. If $\{R_i\}$ denotes the set of regional maxima of μ, then $\mathcal{H}^1(\mu)$ is isomorphic to $\{R_i\}$.*

The 1-hyperconnected components are therefore exactly the geodesic reconstructions in μ of its regional maxima.

Proposition 12 *Let $\mathcal{H}^1(\mu) = \{\mu_i\}$ be the set of hyperconnected components of μ according to $\mathcal{H}^1$ and $\{x_i\}$ the set of points that belong to the associated regional maxima. It follows that $c^1_\mu(x_i, x_j) = \max_{x \in X} \min(\mu_i(x), \mu_j(x))$.*

These notions are illustrated in Fig. 10, for the hyperconnection $\mathcal{H}^2_\tau$. Let μ be the fuzzy set in (a). It has four 1-hyperconnected components, corresponding to each regional maximum of μ (b–e), two 0.5-hyperconnected components (f–g), and one 0.1-hyperconnected component (a), equal to μ. The computation of the hyperconnected components will be explained in Sect. 3.4. The degree of connectivity of μ is $c^2(\mu) = 0.2$, hence μ is a connected component in the sense of $\mathcal{H}^2_\tau$ for $\tau \leq 0.2$. If we denote by μ_1 and μ_2 the two 0.5-hyperconnected components in (f) and (g), it is easy to check that $c^2(\mu_1) = c^2(\mu_2) = 0.5$. They are hence elements of $\mathcal{H}^2_{0.5}$ ($\tau = 0.5$). Let x_1 and x_2 be points at which the global maxima of μ_1 and μ_2 are reached. We have $h_1 = \mu_1(x_1) = h_2 = \mu_2(x_2) = 1$. The hyperconnected openings $\eta^1_{\delta^{h_1}_{x_1}}(\mu_1)$ and $\eta^1_{\delta^{h_2}_{x_2}}(\mu_2)$ (h) overlap only for levels lower than or equal to $\alpha = 0.2$, which is less than $h_i - 1 + \tau = 0.5$. This shows that actually μ_1 and μ_2 do not overlap in the sense of $\perp^2_\tau$.

3.4 Tree-Based Representation

From an algorithmical point of view, computing the τ-hyperconnected components and further processing them can benefit from an appropriate representation. Since the manipulation of α-cuts plays a critical role in the proposed concepts and definitions, we can rely on a classical max-tree [35] representation. From now on, we assume that membership degrees are uniformly quantified. For each level α of this quantification, vertices of the tree representation are associated with the connected components (in the sense of $\mathcal{C}_d$) of the α-cut of the considered fuzzy set. Edges are induced by the inclusion relation between connected components for successive values of α. A fuzzy set μ can therefore be exactly represented by a tree $T(\mu)$, with:

- $\mathcal{V}$ the set of vertices of the tree,
- $h(v)$ (with $v \in \mathcal{V}$) the height of v, i.e. the value of α associated with this vertex,
- R the root of the tree,
- $\mathcal{L}$ the set of leaves,
- $Pt(v)$ the set of points associated with the vertex v (i.e. the set of points of the connected component of the α-cut associated with v),
- $\mathcal{E}$ the set of edges of the tree ($\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$), derived from the inclusion relation between associated sets of points,
- $P^v(h)$, for $v \in \mathcal{V}$, the set of vertices of $T(\mu)$ linking the root to the vertex v and having a height smaller than or equal to h.

A subtree of $T(\mu)$ is represented by a subset $G \subseteq \mathcal{V}$ and we write $h(G)$ the maximal height of its vertices. Edges of this subtree are induced by the inclusion relation for the set of points associated with the vertices, for successive heights.

Several algorithms have been proposed to compute the tree [1, 17, 19, 23, 35, 46, 49], a recent one [25] with a quasi-linear time complexity.

Those notations are illustrated in Fig. 11 for a 1D fuzzy set. In this example, the membership degrees are quantified with a step of 0.1. Each connected component of an α-cut is associated with a vertex of the tree. For instance, the red vertex v_1 and the blue vertex v_2 are associated with the binary regions drawn in red $Pt(v_1)$ and blue $Pt(v_2)$ which are connected components of α-cuts at levels 0.6 and 0.3, respectively. The root R is associated with the whole space. Leaves $\mathcal{L} = \{l_1, l_2\}$ are associated with regional maxima of the fuzzy set. The set $P^{l_1}(0.3)$ of vertices linking the root to the leaf l_1 and whose height is lower than or equal to 0.3 is circled in red. The initial fuzzy set can be recovered from its tree representation as: $\mu(x) = \max_{v \in \mathcal{V} \mid x \in Pt(v)} h(v)$, assuming that the quantification steps for membership degrees in the fuzzy set and in the tree are the same. This assumption is made in the rest of this paper.

As described below, we make use of this representation to extract efficiently the τ-hyperconnected components of a fuzzy set μ and to compute the filters proposed in Sect. 4 (since this extraction is the core component for the computation of these filters).

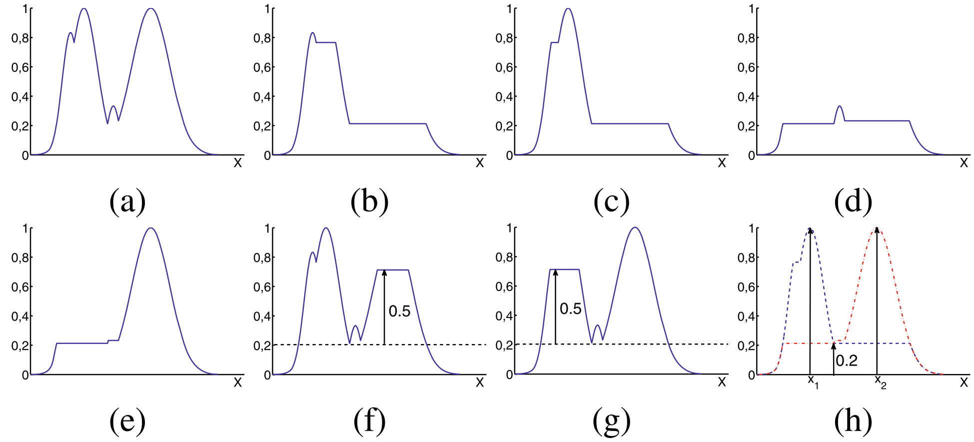


Fig. 10 (Color online) (**a**) Fuzzy set μ with its τ-hyperconnected component for $\tau \leq 0.2$. (**b–e**) Four 1-hyperconnected components. (**f**, **g**) The two 0.5-hyperconnected components μ_1 and μ_2. (**h**) $\eta^1_{\delta^{h_1}_{x_1}}(\mu_1)$ in *blue* and $\eta^1_{\delta^{h_2}_{x_2}}(\mu_2)$ in *red*

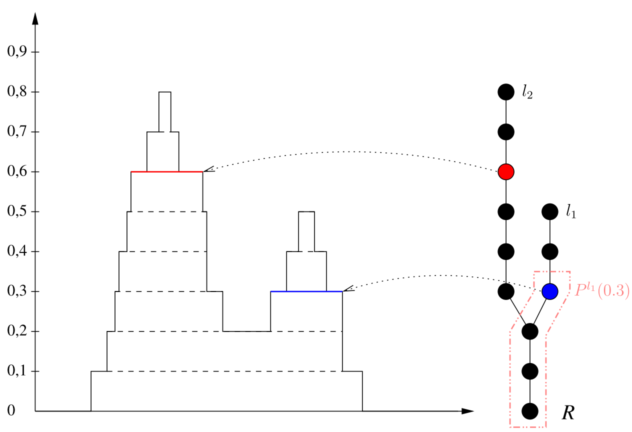


Fig. 11 1D fuzzy set and associated max-tree representation

Let us first consider $\tau = 1$. Proposition 2 ensures that a fuzzy set is connected if it has a unique regional maximum. As the regional maxima of an image are isomorphic to the set of leaves of its tree representation, a fuzzy set is 1-hyperconnected if its associated tree has only one leaf. We can also show that the 1-hyperconnected components of a fuzzy set are exactly the branches of its tree representation.

Proposition 13 *The set $\mathcal{H}^1(\mu) = \{\mu_i\}$ of 1-hyperconnected components of μ is isomorphic to the set of leaves $\mathcal{L}$, and $T(\mu_i) = P^{l_i}(h(l_i))$, where l_i is the leaf associated with μ_i.*

The 1-hyperconnected components of a fuzzy set can therefore be efficiently obtained via simple operations on its associated tree representation.

Extraction of τ-hyperconnected components for $\tau < 1$ requires the extraction of more complex subtrees. These subtrees are "complete" in the sense that when a vertex belongs to a subtree, all its parents also belong to that subtree. Let us consider the set $S_{T(\mu)}$ of subtrees of $T(\mu)$ so that if $S \in S_{T(\mu)}$, then $\forall v \in S$, $P^v(h(v)) \subseteq S$. The set of subtrees $S_{T(\mu)}$ is endowed with the following partial order relation:

$$\forall (S_1, S_2) \in {S_T}^2, \quad S_1 \leq S_2$$
$$\Leftrightarrow \quad \forall v \in \mathcal{V}, \quad v \in S_1 \Rightarrow v \in S_2.$$

$(S_{T(\mu)}, \leq)$ is a complete lattice. The associated infimum $\bigwedge$ and supremum $\bigvee$ are respectively defined as the intersection and union of subtrees defined as sets of vertices. We have $\forall l \in \mathcal{L}, \forall h \in [0, 1]$, $P^l(h) \in S_{T(\mu)}$ and the family $\{P^l(h) \mid l \in \mathcal{L}, h \in [0, 1]\}$ is sup-generating in $(S_{T(\mu)}, \leq)$. Any subtree $S \in S_{T(\mu)}$ can be written as: $S = \bigvee_{l\in\mathcal{L}} P^l(h^l_S)$, where h^l_S corresponds to the maximal height of the subtree S on the branch corresponding to leaf l. The supremum $\bigvee$ and infimum $\bigwedge$ can therefore be respectively reformulated as:

$$\bigvee_i S_i = \bigvee_{l\in\mathcal{L}} P^l(\max_i h^l_{S_i}),$$

$$\bigwedge_i S_i = \bigvee_{l\in\mathcal{L}} P^l(\min_i h^l_{S_i}).$$

In addition we denote by $i^{l_1,l_2} = h(P^{l_1}(1) \wedge P^{l_2}(1))$ the height of the common subtree associated with both leaves l_1 and l_2.

Figure 12 illustrates these definitions. The tree $T(\mu)$ (b) representing the fuzzy set (a) (quantified with a step of 0.2) has four leaves $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ and we have for instance $i^{l_1,l_2} = 0$ and $i^{l_2,l_3} = 0.6$. A subtree $S_1 \in S_{T(\mu)}$ is represented by its set of vertices (c) (in red). This subtree can be expressed as $S_1 = \bigvee_{l\in\mathcal{L}} P^l(h^l_{S_1})$. The set of vertices $P^l(h^l_{S_1})$ associated with the leaves l_1, l_2, l_3 and l_4 are respectively circled in red, yellow, blue and green and we have $h^{l_1}_{S_1} = 0.4$, $h^{l_2}_{S_1} = 0.8$, $h^{l_3}_{S_1} = 0.6$ and $h^{l_4}_{S_1} = 0.4$. Another subtree S_2 is displayed in (d). The supremum $S_1 \vee S_2$ (e) is obtained either as the union of S_1 and S_2 (considered as sets of vertices), or as $\bigvee_{l\in\mathcal{L}} P^l(\max(h^l_{S_1}, h^l_{S_2}))$. Sets $P^l(\max(h^l_{S_1}, h^l_{S_2}))$ are circled in (e). Similarly the infimum

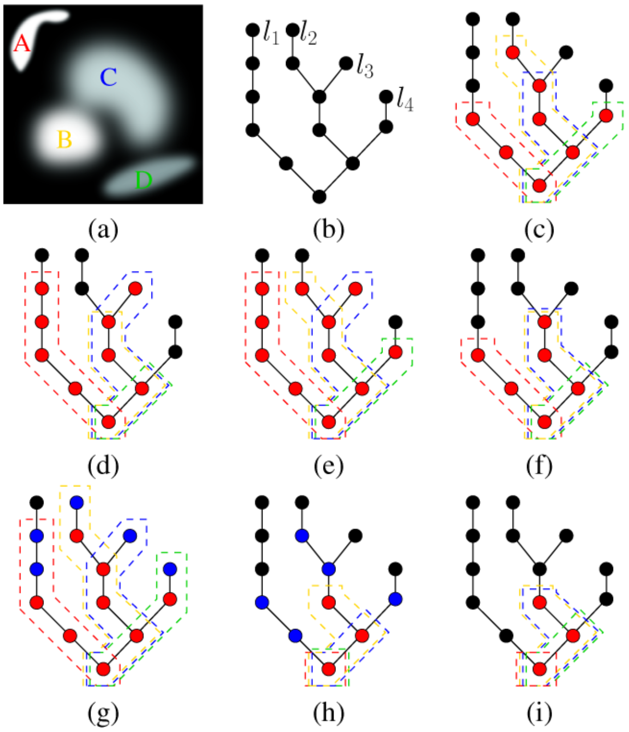


Fig. 12 (Color online) (**a**) Fuzzy set μ. (**b**) Associated tree $T(\mu)$ (α-cuts are quantified with a step of 0.2). (**c**) A subtree $S_1 \in S_{T(\mu)} : S_1 = \bigvee_{l\in\mathcal{L}} P^l(h^l_{S_1})$. Sets of vertices $P^l(h^l_{S_1})$ associated with the leaves l_1, l_2, l_3 and l_4 are circled respectively in *red*, *yellow*, *blue* and *green*. (**d**) Another subtree $S_2 \in S_{T(\mu)}$. (**e**) $S_1 \vee S_2$. (**f**) $S_1 \wedge S_2$. (**g**) $\delta_{T(\mu)}(S_1, 0.4)$ (added vertices are plotted in *blue*). (**h**) $\varepsilon_{T(\mu)}(S_1, 0.4)$ (deleted vertices are plotted in *blue*). Sets $P^l(\max(0, h^l_{S_1} - 0.4))$ involved in the computation are highlighted in this figure and can be compared to the sets $P^l(h^l_\varepsilon)$ in (**i**) obtained as in (**c**)

$S_1 \wedge S_2$ (f) is obtained either as $\bigvee_{l\in\mathcal{L}} P^l(\min(h^l_{S_1}, h^l_{S_2}))$, or as the intersection of vertices sets.

Proposition 14 *The degree of connectivity $c^2(\mu)$ of a fuzzy set μ can be computed from its tree representation, as*:

$$c^2(\mu) = 1 - \max_{(l_1,l_2)\in\mathcal{L}^2} (\min(h(l_1), h(l_2)) - i^{l_1,l_2}).$$

Proposition 15 *Let $\mu \in \mathcal{F}$, $G \in S_{T(\mu)}$ and ν the fuzzy set associated with G. We have*:

$$c^2(\nu) = \min\Big(1, 1 - \max_{(l_1,l_2)\in\mathcal{L}^2} (\min(h^{l_1}_G, h^{l_2}_G) - i^{l_1,l_2})\Big).$$

We define the following operators on $S_{T(\mu)} \times \mathbb{R}^+$:

$$\varepsilon_{T(\mu)}(S, r) = \bigvee_{l\in\mathcal{L}} P^l(\max(0, h^l_S - r)),$$

$$\delta_{T(\mu)}(S, r) = \bigvee_{l\in\mathcal{L}} P^l(\min(h(l), h^l_S + r)).$$

The first one intuitively corresponds to a contraction of the subtree S whose branch lengths are reduced by size r.[4] The second operator corresponds to a dilation[5] of size r of the subtree S.

Let us consider again the example in Fig. 12. The result of the dilation $\delta = \delta_{T(\mu)}(S_1, 0.4)$ of S_1 (c) is displayed in (g). It is obtained by increasing S_1 height by 0.4 on all branches. Added vertices are displayed in blue, and the sets $P^l(\min(h(l), h^l_{S_1} + r))$ for $l \in \mathcal{L}$ are circled. We have exactly $h^l_\delta = \min(h(l), h^l_{S_1} + r)$ (see proof of Proposition 16). The contraction $\varepsilon = \varepsilon_{T(\mu)}(S_1, 0.4)$ is illustrated in (h) (with vertices plotted in red). It is computed by reducing the height of all branches by 0.4. The sets $P^l(\max(0, h^l_{S_1} - 0.4))$ are circled in (h). The contraction result is then obtained as the supremum of these sets. As opposed to the dilation operator, the equality $h^l_\varepsilon = \max(0, h^l_{S_1} - 0.4)$ is not true in general and we have $h^l_\varepsilon \geq \max(0, h^l_{S_1} - 0.4)$: for instance $h^{l_4}_\varepsilon = 0.2$ and $\max(0, h^{l_4}_{S_1} - 0.4) = 0$. The sets $P^l(h^l_\varepsilon)$ are shown in (i) and can be compared to those in (h).

Proposition 16 *If $G \in S_{T(\mu)}$ represents a τ-hyperconnected fuzzy subset of μ, then $\delta_{T(\mu)}(G, r)$ represents a $\max(0, \tau - r)$-hyperconnected fuzzy set and $\varepsilon_{T(\mu)}(G, r)$ a $\min(1, \tau + r)$-hyperconnected fuzzy set.*

Proposition 17 *The set of τ-hyperconnected components $\mathcal{H}^2_\tau(\mu)$ of a fuzzy set μ is isomorphic to the set of leaves of $\varepsilon_{T(\mu)}(T(\mu), 1 - \tau)$. A τ-hyperconnected component of μ can be obtained by the dilation of size $(1 - \tau)$ of a 1-hyperconnected component of $\varepsilon_{T(\mu)}(T(\mu), 1 - \tau)$.*

If a fuzzy set presents k τ-hyperconnected components, their extraction is therefore performed by one contraction and k dilations. Since both contraction and dilation have a complexity in $O(|\mathcal{V}|)$, the extraction of all connected components can be performed in $O((k + 1)|\mathcal{V}|)$ (once the tree is built). The filters defined in Sect. 4 are based on this extraction and their complexity will include this cost.

Figure 13 shows in (b) the tree $T(\mu)$ associated with the fuzzy set (a) (with a quantification step of 0.2). There are four 1-hyperconnected components (c–j) corresponding to the four regional maxima of (a). Results for a contraction of size 0.4 applied to $T(\mu)$ and for a dilation of size 0.4 of one

[4]This is not an erosion since this operator does not commute with the infimum.

[5]To prove that this operator commutes with the supremum, we can check that the heights of the subtrees $\delta^1 = \delta_{T(\mu)}(\bigvee_i S_i, r)$ and $\delta^2 = \bigvee_i \delta_{T(\mu)}(S_i, r)$ are equal on every branch of $T(\mu)$. We have $h^l_{\bigvee_i S_i} = \max_i(h^l_{S_i})$ and therefore $h^l_{\delta^1} = \min(h(l), \max_i(h^l_{S_i}) + r) = \max_i \min(h(l), h^l_{S_i} + r)$. Moreover, we have $h^l_{\delta_{T(\mu)}(S_i,r)} = \min(h(l), h^l_{S_i} + r)$ and therefore $h^l_{\delta^2} = \max_i \min(h(l), h^l_{S_i} + r)$.

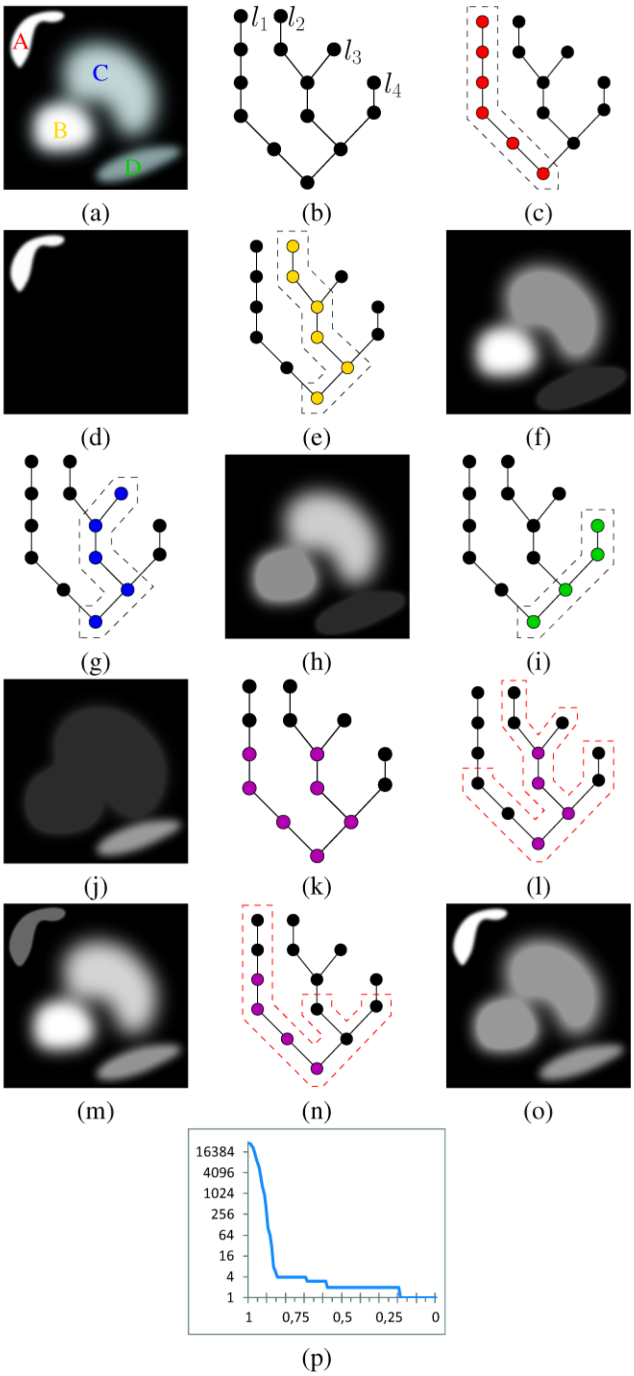


Fig. 13 (Color online) (**a**) Fuzzy set. (**b**) Associated tree (the α-cuts are quantified with a 0.2 step). The four 1-hyperconnected components (**d**), (**f**), (**h**) and (**j**) and associated subtrees (**c**), (**e**), (**g**) and (**i**). (**k**) Subtree corresponding to the contraction of size 0.4 (in magenta). (**l**) A 0.6-hyperconnected component (circled in *red*) obtained as the dilation of a 1-hyperconnected component of the contraction (in magenta) and the corresponding fuzzy set (**m**). Another 0.6-hyperconnected component (**n**) and the associated fuzzy set (**o**). (**p**) Number of τ-hyperconnected components of the fuzzy set (**a**) corrupted with white Gaussian noise (of variance 0.005) as function of τ

of the 1-hyperconnected components are illustrated respectively in (k) et (l). The resulting subtree corresponds exactly to a 0.6-hyperconnected component (m) of μ. The second 0.6-hyperconnected component is illustrated in (o) and its associated subtree in (n). If the fuzzy set (a) is corrupted with white Gaussian noise of variance 0.05, we obtain about 20,000 1-hyperconnected components. The evolution of the number of τ-hyperconnected components as a function of τ is plotted in (p), illustrating the grouping effect.

4 Attribute Openings Applied to Fuzzy Sets

4.1 Motivation

We focus here on segmentation and recognition tasks. In this context we want to extract a connected object A represented by its membership function μ_A. For this purpose, we rely on prior structural knowledge expressed as spatial relations between structures [3], and on grey-level information. The segmentation and recognition process can thus be formalized as an iterative reduction of an over-estimation $\overline{A}$ of μ_A [27]. Considering a current over-estimation $\overline{A}$ and a prior on A represented as a characteristic function f on $\mathcal{F}$, we can obtain a new upper-bound $\overline{A}'$ as the supremum of the fuzzy sets that fulfill the prior and that are smaller than $\overline{A}$: $\overline{A}' = \bigvee\{\nu \in \mathcal{F} \mid \nu \leq \overline{A} \text{ and } f(\nu) = 1\}$. As illustrated in the sequel, such reductions can be performed based on grey-level prior or on an approximate spatial location. In this framework, we can also take advantage of connectivity priors about objects of interest. Considering the connectivity constraint and some available prior information about the object, we want to obtain a reduction $\overline{A}'$ of an over-estimation $\overline{A}$ such that $\mu_A \leq \overline{A}' \leq \overline{A}$.

Figure 14 provides an example on a brain MRI segmentation and recognition task. Suppose for instance that we have already extracted the brain surface and that we want to extract the right lateral ventricle LVr delineated in red (a) on one axial slice of a 3D T1 weighted brain MRI. From grey-level prior we can obtain a first over-estimation $\overline{LVr}_{Gl}$, since the lateral ventricles have darker intensities than other brain tissues, including white matter and grey matter, on this type of images. Lateral ventricles are also always quite central in the brain. This prior can be translated into a fuzzy set $\overline{LVr}_{Sp}$ (c), so as to guarantee $\mu_{LVr} \leq \overline{LVr}_{Sp}$. The conjunctive fusion $\overline{LVr} = \overline{LVr}_{Gl} \wedge \overline{LVr}_{Sp}$ is shown in (d), and satisfies $\mu_{LVr} \leq \overline{LVr}$. Although the over-estimation has been strongly reduced, $\overline{LVr}$ still exhibits several connected components. The circled ones, for instance, correspond to brain sulci and present a volume smaller than the typical range of volume for the lateral ventricle in normal cases. We can thus remove the connected components of $\overline{LVr}$ that do not fulfill a minimal volume criterion (based on a prior volume information for the lateral ventricle).

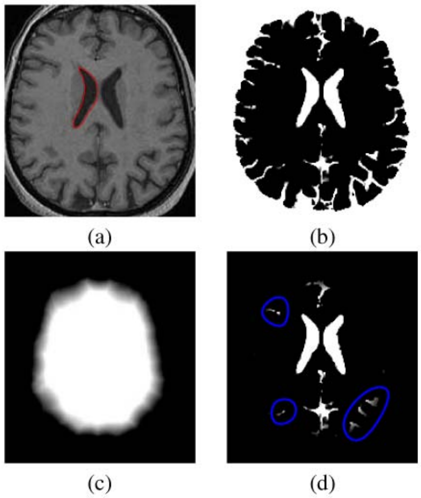


Fig. 14 (**a**) One axial slice of a 3D brain MRI. (**b**) Grey level information: $\overline{LVr_{Gl}}$. (**c**) Central location inside the brain: $\overline{LVr_{Sp}}$. (**d**) Conjunctive fusion

- computation of the tree $T(\overline{A})$,
- computation of the contraction $\varepsilon_{T(\overline{A})}(T(\overline{A}), 1-\tau)$,
- initialization of the resulting subtree *Res* as the root node R of $T(\overline{A})$,
- for each leaf ε_i of $\varepsilon_{T(\overline{A})}(T(\overline{A}), 1-\tau)$,
  - extraction of the τ-hyperconnected component G_i as the dilation of size $1-\tau$ of $\varepsilon_{T(\overline{A})}(T(\overline{A}), 1-\tau)$,
  - computation of the binary criterion $f_C(G_i)$,
  - if the criterion is satisfied, add G_i to the resulting subtree *Res*,
- return the fuzzy set $\xi(\overline{A})$ associated with *Res*.

Fig. 15 Algorithm used to compute $\xi(\overline{A})$

4.2 Attribute Openings Based on a Crisp Criterion

We first suppose that some prior knowledge about the connected object A is expressed as a crisp criterion function: $f_C : \mathcal{F} \to \{0, 1\}$ such that $f_C(\mu_A) = 1$. Based on f_C, we define the following operator on $\mathcal{F}$:

$$\xi(\overline{A}) = \bigvee\{\nu \in \mathcal{H}^2_\tau | \nu \leq \overline{A} \text{ and } f_C(\nu) = 1\}. \quad (5)$$

This operator satisfies the property $\mu_A \leq \xi(\overline{A}) \leq \overline{A}$, since μ_A is supposed to be connected, to be smaller than $\overline{A}$ and to fulfill the criterion. The resulting fuzzy set is thus a better approximation of μ_A than $\overline{A}$. It is important to note that this operator is increasing, idempotent and anti-extensive and is thus a morphological opening.

However without any condition on f_C, the computation of this operator requires to evaluate the criterion over all elements of $\mathcal{H}^2_\tau$ smaller than $\overline{A}$ and has an exponential complexity. To overcome this, we take advantage of the following property: $\forall \nu \in \mathcal{H}^2_\tau, \nu \leq \overline{A} \Rightarrow \exists \mu_i \in \mathcal{H}^2_\tau(\overline{A}), \nu \leq \mu_i$. Therefore if the criterion f_C is increasing, the computation of $\xi(\overline{A})$ can be performed over the τ-hyperconnected components of $\overline{A}$:

$$\xi(\overline{A}) = \bigvee\{\mu_i \in \mathcal{H}^2_\tau(\overline{A}) | f_C(\mu_i) = 1\}. \quad (6)$$

This filter only processes connected components of $\overline{A}$ and corresponds intuitively to an extension of attribute openings as defined for binary images [8, 13]. Note that the increasingness of the criterion is required to obtain a tractable computation of (5), which prevents using non increasing criteria such as shape criteria as recently proposed in [44].

This filter can be easily computed using the tree representation. Proposition 17 ensures an isomorphism between the leaves of $\varepsilon_{T(\overline{A})}(T(\overline{A}), 1-\tau)$ and the τ-hyperconnected components of $\overline{A}$. The process can thus be efficiently performed with the algorithm described in Fig. 15, where the most time consuming operation is the tree computation [25].

More precisely, the complexity of this filter is in $O((k+1)|\mathcal{V}| + kC_{f_C} + C_T)$, where C_T is the cost of the tree computation, C_{f_C} is the cost of the criterion computation and k is the cardinality of $\mathcal{H}^2_\tau(\overline{A})$. Computing the criterion can be computationally expensive since it has to be performed k times and it generally involves a computation over all vertices of the τ-hyperconnected components (a vertex can belong to more than one τ-hyperconnected component and a pixel to more than one vertex). However, a preliminary partial computation of the criterion can sometimes be performed during the tree computation. For instance if we consider an area criterion (the area of a fuzzy set μ is defined as $S(\mu) = \sum_{x \in X} \mu(x)$), it can be advantageous to compute, during the tree initialization, the area associated with every vertex of the tree. The area of the τ-hyperconnected components is then obtained as the sum of vertices area, that were previously computed.

Figure 16 presents an example, where the criterion is defined as a minimal area of 10,000 pixels and the chosen hyperconnection is $\mathcal{H}^2_{0.6}$. During the computation of the tree (b) associated with the fuzzy set (a), we compute for each vertex its associated area. We then compute $\varepsilon_{T(\mu)}(T(\mu), 0.4)$ to obtain the 0.6-hyperconnected components (c) and (d). Their areas are respectively 8612 and 11,520 pixels and can be easily obtained from the vertices area. The first one does not satisfy the criterion. The second one does and is the resulting subtree in this case (d) which represents the fuzzy set (e).

4.3 Extension to a Fuzzy Criterion

The filter proposed in the previous section only handles crisp criteria. It follows that it is not continuous since a small modification of the input set may result in the modification of a complete connected component. To overcome this and achieve more robustness in the filtering process, we extend in this section the previous definition to fuzzy criteria. For

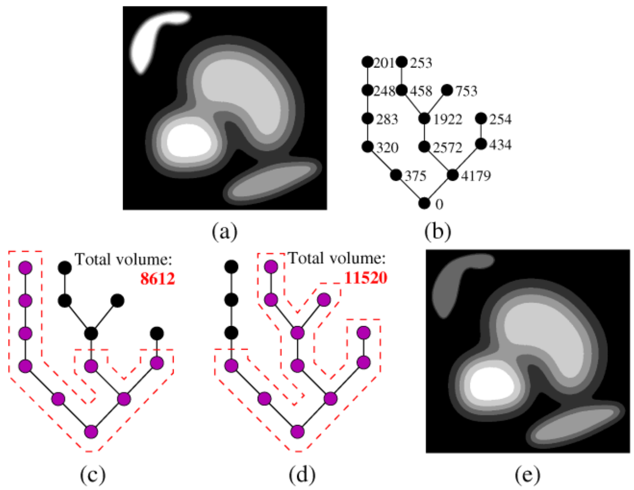


Fig. 16 Fuzzy set (**a**) and its representation as a tree (**b**) (the vertices are labeled with their area). Two 0.6-hyperconnected components (**c**) and (**d**). The area of the component (**c**) is smaller than the minimal threshold. This component does not belong to the result, whereas the component (**d**) does. (**e**) Resulting fuzzy set

instance, a minimal area criterion can be represented by a membership function (corresponding for instance to a linguistic value such as "large"). The satisfaction of the criterion by a fuzzy set ν is thus defined as a degree $\mu_C(\nu)$.

We propose to preserve connected fuzzy subsets whose maximum membership degree is less than or equal to the satisfaction degree of the criterion via the following operator (which guarantees the idempotence of the associated filter):

$$\xi_{\mu_C}(\overline{A}) = \bigvee \Big\{ \nu \in \mathcal{H}^2_\tau \mid \nu \leq \overline{A} \text{ and } \max_{x \in X} \nu(x) \leq \mu_C(\nu) \Big\}. \tag{7}$$

This operator is also a morphological opening and reduces to (5) if μ_C is crisp. If it is not crisp but Lipschitz, it presents nice regularity properties expressed by the following proposition (regularity properties given in the following make the assumption that the membership degrees are not quantified).

Proposition 18 *If μ_C is Lipschitz, then the mapping associating $\overline{A}$ with $\xi_{\mu_C}(\overline{A})$ is Lipschitz.*

For computational purposes, we also assume that μ_C is increasing which leads to a simplification of (7).[6]

Proposition 19 *If μ_C is increasing, $\xi_{\mu_C}(\overline{A})$ rewrites as:*

[6]Note that if $\mu_C(\nu)$ is increasing for fixed $\max_{x\in X} \nu(x)$, Proposition 19 still holds.

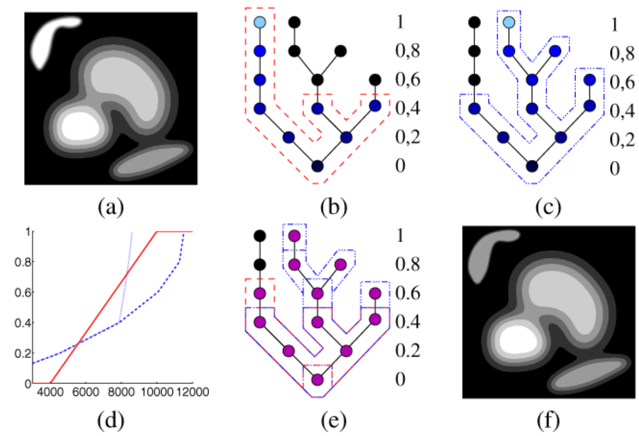


Fig. 17 (Color online) (**a**) Fuzzy set. Two 0.6-hyperconnected components (**b**) and (**c**). (**d**) μ_S in *red plain*, in *dashed blue* the values $(S(\min(\mu_i, m)), m)$. Resulting subtree (**e**) and associated fuzzy set (**f**). See Table 2 for detailed values involved in this filtering process

$$\xi_{\mu_C}(\overline{A}) = \bigvee_{\substack{\mu_i \in \\ \mathcal{H}^2_\tau(\overline{A})}} \bigvee_{m \in [0,1]} \{\min(\mu_i, m) \mid m \leq \mu_C(\min(\mu_i, m))\}.$$

This leads to a fast computation of $\xi_{\mu_C}(\overline{A})$ since we only have to handle the τ-hyperconnected components "leveled" at m (i.e. $\min(\mu_i, m)$). Therefore this filter has a complexity in $O((k+1)|\mathcal{V}| + \frac{kC_{f_C}}{s} + C_T)$, where C_T is the cost of the tree computation, C_{f_C} is the cost of the criterion computation, k is the cardinal of $\mathcal{H}^2_\tau(\overline{A})$ and s is the quantification step of the membership degrees.

We illustrate this definition in Fig. 17. The criterion is here defined as a membership function $\mu_S : \mathbb{R}^+ \to [0, 1]$ (d) representing a fuzzy minimal threshold on the area (computed as the cardinality of the fuzzy set: $S(\mu) = \sum_{x \in X} \mu(x)$). First we extract from the tree (b) (which represents the fuzzy set μ (a)) the two 0.6-hyperconnected components μ_1 (b) and μ_2 (c). These components are then progressively leveled from 1 to 0: $\nu = \min(\mu_i, m)$. The satisfaction degree of the criterion $\mu_S(S(\nu))$ is evaluated for each leveled subtree. If the level is less than or equal to this degree we add the leveled subtree to the resulting subtree (e). The associated fuzzy set is shown in (f). Table 2 shows for the two 0.6-hyperconnected components and all levels m, the area of leveled subtree, the satisfaction degree of the criterion μ_S and finally whether the leveled subtree has to be added to the result or not.

5 Filtering

We propose in this section two connected filters, based on the definitions introduced in Sect. 4, that can be used in a segmentation and recognition process, implementing the idea of deriving a finer estimation of μ_A from a first rough over-estimation $\overline{A}$ and a criterion.

Table 2 For each connected component presented in Fig. 17 and for each level m, the area of the thresholded component: $\min(\mu_i, m)$, the satisfaction degree $\mu_S(S(\nu))$ and the satisfaction of the criterion C: $\max_{x \in X} \nu(x) \le \mu_S(S(\nu))$ are provided

	Connected component (b)			Connected component (c)		
Level	$S(\nu)$	$\mu_S(S(\nu))$	C	$S(\nu)$	$\mu_S(S(\nu))$	C
1	8612	0.7687	no	11520	1	yes
0.8	8411	0.7352	no	11267	1	yes
0.6	8163	0.6938	yes	10056	1	yes
0.4	7880	0.6467	yes	7880	0.6467	yes
0.2	4554	0.0923	no	4554	0.0923	no
0	0	0	yes	0	0	yes

5.1 Marker Inclusion

We define a criterion from another estimation $\underline{A}$ of μ_A, such that $\underline{A} \le \mu_A$ ($\underline{A}$ can be seen as a marker[7] of the target object):

$$\xi^1_{\underline{A}}(\overline{A}) = \bigvee \{\nu \in \mathcal{H}^2_\tau | \nu \le \overline{A} \text{ and } \underline{A} \le \nu\}. \tag{8}$$

Figure 18 illustrates this filter on a 1D example and for $\tau = 1$. The fuzzy set $\overline{A}$ (a) in blue is filtered by various markers $\underline{A}$ (b–f) in dashed red. In (b) only one hyperconnected component fulfills the inclusion constraint and is preserved. In (c) and (d), two hyperconnected components satisfy the inclusion constraint. A small modification of the marker leads in (e) to the satisfaction of the constraint with the four connected components. As the height of the marker decreases, the connectivity degree of the result also decreases. This property will be formally expressed by Proposition 20. In addition, discontinuities appear as $\underline{A}$ changes (consider for instance (d) and (e)), which illustrates that $\xi^1_{\underline{A}}$ is not continuous with respect to $\underline{A}$. In (f) the marker $\underline{A}$ cannot be included in any hyperconnected component of $\overline{A}$ (since the chosen connectivity degree $\tau = 1$ is too strict) and the filter thus returns an empty set.

Proposition 20 *Let $\alpha = \max_{x \in X} \underline{A}(x)$. The result of the filter defined in (8) is $\max(0, (\alpha - (1 - \tau)))$-hyperconnected.*

Instead of considering a strict inclusion, we can rely on a fuzzy one, based on Lukasiewicz operator [10]:

$$\mu_{\le}(\mu_A, \mu_B) = \min_{x \in X} \min(1, 1 - \mu_A(x) + \mu_B(x)).$$

The filter defined by (7) then writes:

$$\xi^2_{\underline{A}}(\overline{A}) = \bigvee \left\{ \nu \in \mathcal{H}^2_\tau \mid \nu \le \overline{A} \text{ and } \max_{x \in X} \nu(x) \le \mu_{\le}(\underline{A}, \nu) \right\}. \tag{9}$$

[7] Note that this filter does not behave as a reconstruction filter.

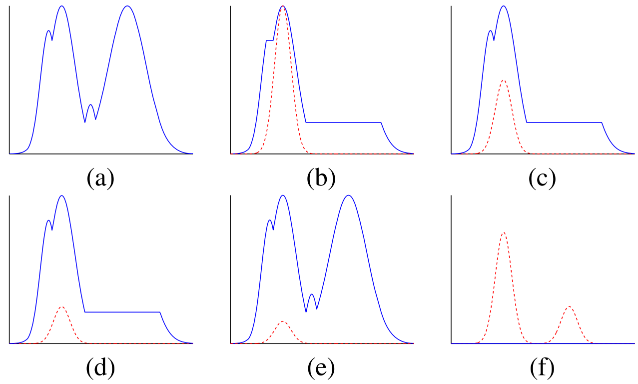


Fig. 18 (Color online) Filtering of a fuzzy set $\overline{A}$ (a) according to (8) using markers $\underline{A}$ (in *red*) of decreasing height (**b–e**) or a marker $\underline{A}$ that presents two distinct modes (**f**). The result is displayed in *blue*. In (**f**), the result is $0_{\mathcal{F}}$ (meaning that there is actually no solution satisfying the constraints)

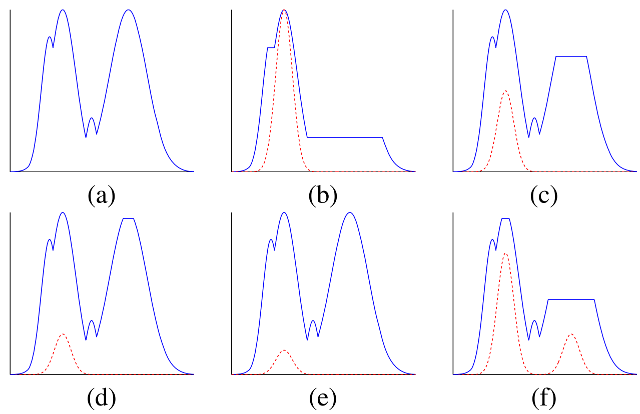


Fig. 19 (Color online) Filtering of a fuzzy set $\overline{A}$ (**a**) according to (9) (for $\tau = 1$) using markers $\underline{A}$ (in *red*) of decreasing height (**b–e**) or a marker $\underline{A}$ that presents two distinct modes (**f**). The result is displayed in blue

The results of this filter are presented in Fig. 19 (for $\tau = 1$) and can be compared to the results in Fig. 18. We can notice that the input fuzzy set is now progressively filtered when the marker gets larger and larger. Intuitively, hyperconnected components verifying the inclusion constraint are kept, while the other ones are reduced to a level corresponding to the degree of satisfaction of the constraint.

Proposition 21 *The mapping associating $\underline{A}$ with $\xi^2_{\underline{A}}(\overline{A})$ is Lipschitz, as well as the mapping that associates $\overline{A}$ to $\xi^2_{\underline{A}}(\overline{A})$.*

Proposition 22 *Let $\alpha = \max_{x \in X} \underline{A}(x)$. The result of the connected filter defined in (9) is $\max(0, (\alpha - (1 - \tau)))$-hyperconnected.*

We illustrate now this filter on a brain recognition task in Fig. 20. As in Sect. 4.1, we want to extract the right lateral ventricle from a brain MRI (a). We will refine the overestimation $\overline{LVr}$ (b) obtained in Fig. 14(d) based on a marker

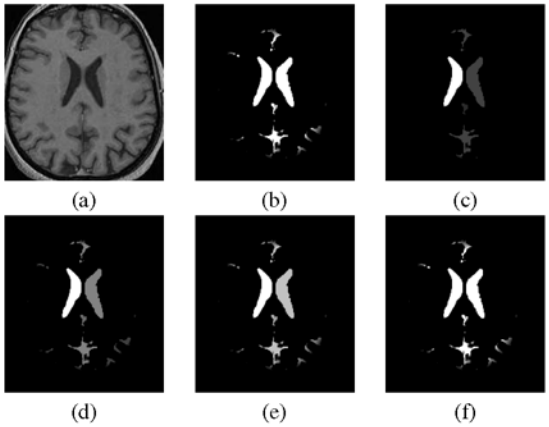


Fig. 20 (**a**) One axial slice of a 3D brain MRI. (**b**) $\overline{LVr}$. (**c–f**) Results of the connected filter specified by (9) using a marker centered in the right ventricle, with maximal value 1, 0.75, 0.5 and 0, respectively

$\underline{LVr}$ defined as a fuzzy set having a support reduced to one point centered in the right lateral ventricle, with a membership value taking values 1 (which mostly selects the right ventricle), 0.75, 0.5 and 0 (which does not filter), respectively (c–f). In (c) the right lateral ventricle is clearly distinguished from the other parts of the fuzzy set in (b). This distinction decreases with the maximal value of the marker and the other parts of the fuzzy set have higher and higher membership degrees in the filtering result. A potential application of this approach is to perform a filter, preserving connectivity properties, and being more or less selective depending on the confidence we may have in the marker.

5.2 Fuzzy Area Opening

Area opening is a well known connected operators [45]. It filters connected components based on a minimal area criterion, and can be formulated as:

$$\xi_{S_{\min}}(A) = \bigvee \{c \in \mathcal{C} \mid c \leq A \text{ and } S(c) \geq S_{\min}\},$$

where $\mathcal{C}$ is a connection over X and S a function computing the area. In the examples developed in Sect. 4, we used a criterion based on the area defined as the cardinality of the fuzzy objects. We denote by $\xi^1_{S_{\min}}$ the filter derived from (5) using a crisp minimal area criterion, and by $\xi^2_{\mu_{S_{\min}}}$ the filter derived from (7) using a fuzzy minimal area criterion.

However modeling the area of the fuzzy set by its cardinality is quite simplistic (think for instance of a fuzzy set that has a large support and a small kernel) and it may be more appropriate to consider a criterion on the area of each α-cut. To this aim we make use of the following fuzzy measure of the area [10]: $\mu_S(\mu)(s) = \sup_{S(\mu_\alpha) \geq s} \alpha$, which represents the highest level such that all α-cuts of μ until this level have an area larger than s. The membership function $\mu_S(\mu)$ is decreasing. If we consider a minimal area $S_{\min}$, the inequality $\max_{x \in X} \nu(x) \leq \mu_S(\nu)(S_{\min})$ is therefore fulfilled if all non empty α-cuts of ν satisfy the area criterion. Deriving the filter of (7), we obtain:

$$\xi^2_{S_{\min}}(\overline{A}) = \bigvee \Big\{ \nu \in \mathcal{H}^2_\tau \mid \nu \leq \overline{A} \text{ and } \max_{x \in X} \nu(x) \leq \mu_S(\nu)(S_{\min}) \Big\}.$$

For more flexibility in recognition tasks, it is more appropriate to represent the criterion by a membership function $\mu_{S_{\min}} : \mathbb{R}^+ \to [0,1]$ (for instance a ramp function replacing the crisp threshold). We will now keep fuzzy sets whose height is smaller than the satisfaction degree $\mu_{S_{\min}}(s_m)$, where s_m is the area of its highest non empty α-cut. This property can be expressed as the satisfaction of the inequality

$$\max_{x \in X} \nu(x) \leq \max_{s \in \mathbb{R}^+} \min(\mu_S(\nu)(s), \mu_{S_{\min}}(s)).$$

The filter defined by (7) then rewrites:

$$\xi^2_{\mu_{S_{\min}}}(\overline{A}) = \bigvee \Big\{ \nu \in \mathcal{H}^2_\tau \mid \nu \leq \overline{A} \text{ and } \max_{x \in X} \nu(x) \leq \max_{s \in \mathbb{R}^+} \min(\mu_S(\nu)(s), \mu_{S_{\min}}(s)) \Big\}.$$

Figure 21 illustrates these filters for $\tau = 1$. The fuzzy set (a) contains 7 objects of increasing area. Their fuzzy area $\mu_S(\mu)$ is represented in (b) and their area (defined as the cardinality of the fuzzy set) is respectively 60.77, 281.57, 669.44, 1202.12, 1896.76, 2696.83 and 3702.12. We first apply the filter $\xi^1_{S_{\min}}$ (based on a minimal value of the cardinality) with $S_{\min} = 1202$ (c) and $S_{\min} = 1203$ (d). In (c) three objects are filtered whereas in (d) there are four. This illustrates that this filter is not continuous according to the parameter $S_{\min}$. In (e) we apply $\xi^2_{S_{\min}}$ with $S_{\min} = 1202$. All α-cuts of the 1-hyperconnected components that satisfy the criterion are selected and the third object is now partially filtered. The use of a membership function $\mu_{S_{\min}}$ (b) (in dashed red) as criterion leads to more robustness of the filter. The result $\xi^2_{\mu_{S_{\min}}}$ is presented in (f).

Proposition 23 *The mapping that associates $\overline{A}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$ is Lipschitz, as well as the mapping that associates $\mu_{S_{\min}}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$.*

Figure 22 illustrates this filter on a brain MRI example. We filter an over-estimation $\overline{LVr}$ of the lateral ventricles (b) according to a minimal volume prior represented by a membership function $\mu_{S_{\min}}$. The result $\xi^2_{\mu_{S_{\min}}}(\overline{LVr})$ is shown in (c). Some components corresponding in particular to the sulci are efficiently removed and we thus obtain a better approximation of the lateral ventricles.

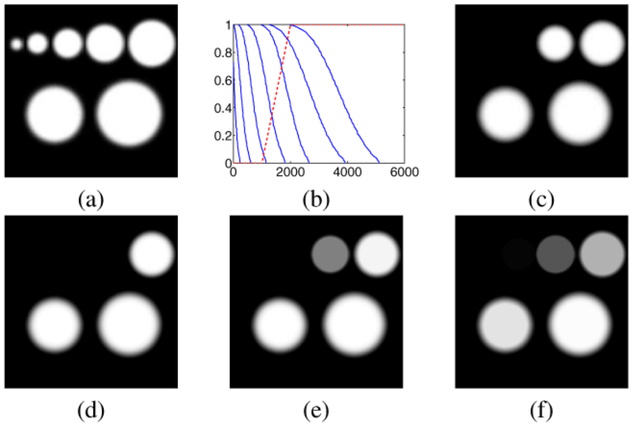


Fig. 21 (Color online) (**a**) Fuzzy set that contains 7 objects of increasing size. (**b**) For each object, fuzzy area $\mu_S(\mu)(s)$ in *blue* and a minimal fuzzy threshold $\mu_{S_{\min}}$ in *dashed red*. Filtering of (**a**) by ξ^1_{1202} (**c**), ξ^1_{1203} (**d**), ξ^2_{1202} (**e**) and $\xi^2_{\mu_{S_{\min}}}$ (**f**)

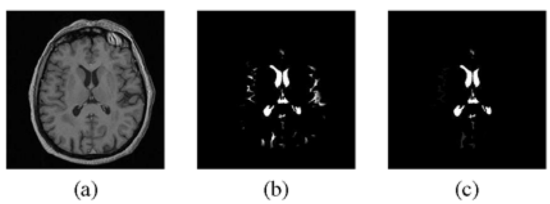


Fig. 22 (**a**) One axial slice of a 3D MRI. (**b**) $\overline{LVr}$. (**c**) $\xi^2_{\mu_{S_{\min}}}(\overline{LVr})$

6 Conclusion

In this paper, we focused on the notion of connectivity of fuzzy sets. The contributions are three-fold. From a theoretical point of view, we proposed a new definition of a measure of connectivity, represented as a family of hyperconnections and adapted to the semantics of fuzzy sets. We derived a number of properties showing that some drawbacks of previously proposed measures are overcome. From an algorithmical point of view, an efficient computational framework was developed, based on a max-tree representation of the considered fuzzy set. In this framework the hyperconnected-components can be easily identified and extracted in linear time with respect to the number of vertices, once the tree is built. This leads to derived attribute openings, based on crisp or fuzzy criteria that presents also nice regularity properties. Finally, from an application point of view, some hints on the potential of the proposed approach are illustrated on a brain imaging segmentation and recognition task. The proposed connected openings based on markers or on volume criteria lead to interesting gradual filtering and are used to refine a first approximation of specific brain structures.

Future work aims at further developing such filters, and integrating them in a segmentation framework in both normal and pathological cases, as suggested in [26]. From a more conceptual and theoretical point of view, another perspective of this work concerns the combination of different structural criteria, including connectivity ones, in a constraint network in order to drive a segmentation and recognition process [27].

Appendix: Proofs of the Main Results

Proof of Proposition 3 We will first prove that if x_i belongs to a regional maximum R_μ of μ, then $\eta^1_{\delta_{x_i}^{\mu(x_i)}}(\mu) \in \mathcal{H}^1$. By definition $\eta^1_{\delta_{x_i}^{\mu(x_i)}}(\mu) = \bigvee\{\nu \in \mathcal{H}^1 \mid \delta_{x_i}^{\mu(x_i)} \leq \nu \leq \mu\}$. Let us show that all elements $\nu \in \mathcal{H}^1$ satisfying $\delta_{x_i}^{\mu(x_i)} \leq \nu \leq \mu$ overlap according to $\perp^1$.

First let us show that $\max_{x\in X} \nu(x) = \mu(x_i)$. According to Proposition 2, if $\nu \in \mathcal{H}^1$, ν has a unique regional maximum. Let x_m be a point belonging to this regional maximum. Since this maximum is unique and X is bounded and discrete, for all points $x \in X$, there exists an increasing discrete path in ν from x to x_m. Thus there exists an increasing path l_{x_i,x_m} in ν from x_i to x_m. Since $\nu(x_i) = \mu(x_i)$ and $\nu \leq \mu$, we obtain $\forall x_k \in l_{x_i,x_m}$, $\mu(x_k) \geq \nu(x_k) \geq \mu(x_i)$. Since l_{x_i,x_m} is increasing, $\forall x_k \in l_{x_i,x_m}$, $x_k \in R_\mu$ (otherwise the first point x_k not belonging to R_μ would satisfy $\mu(x_k) < \mu(x_i)$). The point x_m thus belongs to R_μ and $\mu(x_i) = \mu(x_m)$. Therefore ν reaches its global maximum at x_i and all elements ν of $\mathcal{H}^1$ such that $\delta_{x_i}^{\mu(x_i)} \leq \nu \leq \mu$ satisfy $\nu(x_i) = \mu(x_i)$ and $\max_{x\in X} \nu(x) = \mu_{x_i}$. They overlap according to $\perp^1$ (since whatever the level α, their α-cuts are either empty or contain x_i and so intersect in x_i) and we obtain $\eta^1_{\delta_{x_i}^{\mu(x_i)}}(\mu) \in \mathcal{H}^1$.

Let us now prove that $\forall x \in X$, $\eta^1_{\delta_{x_i}^{\mu(x_i)}}(\mu)(x) = c^1_\mu(x, x_i)$. Let c be a fuzzy set such that $\forall x \in X$, $c(x) = c^1_\mu(x, x_i)$. Then c belongs to $\mathcal{H}^1$. Indeed, by definition, $\forall \alpha \in [0, 1]$, $(c)_\alpha = \{x \in X \mid c^1_\mu(x, x_i) \geq \alpha\}$. Let x be a point in $(c)_\alpha$ and l_{x,x_i} be a discrete path that maximizes the criterion:

$$\max_{\substack{l\in L_{x,x_i}\\ l=\{x_0=x,x_1,\ldots,x_n=y\}}} \min_{0\leq k\leq n} \mu(x_k).$$

We have $\forall x_k \in l_{x,x_i}$, $x_k \in (c)_\alpha$ (since $\min_{x_k\in l_{x,x_i}} \mu(x_k) \geq \alpha$). For all pairs of points belonging to $(c)_\alpha$, there exists a discrete path in $(c)_\alpha$ between those points that contains the point x_i. We thus have $\forall \alpha \in [0, 1]$, $(c)_\alpha \in \mathcal{C}_d$ and according to Proposition 1, $c \in \mathcal{H}^1$. In addition $c(x_i) = c^1_\mu(x_i, x_i) = \mu(x_i)$. Therefore c satisfies $c \in \mathcal{H}^1$ and $\delta_{x_i}^{\mu(x_i)} \leq c \leq \mu$. We derive $\eta^1_{\delta_{x_i}^{\mu(x_i)}}(\mu) \geq c$.

Let ν be in $\mathcal{H}^1$ such that $\delta_{x_i}^{\mu(x_i)} \leq \nu \leq \mu$. Thus we have $\forall (x, y) \in X^2$, $c^1_\nu(x, y) = \min(\nu(x), \nu(y))$. Since $c^1_h(x, y)$ is increasing according to h and $\nu \le \mu$, we obtain $c^1_\mu(x, y) \ge \min(\nu(x), \nu(y))$. If we choose $y = x_i$, we obtain $c^1_\mu(x, x_i) \ge \min(\nu(x), \nu(x_i))$ and since $\nu(x) \le \nu(x_i)$ (see the first part of this proof), the property $c \ge \nu$ is fulfilled for all elements ν that belong to $\mathcal{H}^1$ and such that $\delta^{\mu(x_i)}_{x_i} \le \nu \le \mu$. Therefore we have $\eta^1_{\delta^{\mu(x_i)}_{x_i}}(\mu) \le c$, which completes the proof. □

Proof of Proposition 5 Let x_m be a point for which the global maximum of μ is reached. Since X is bounded and finite, the existence of x_m is guaranteed. We want to prove that:

$$\forall (x, y) \in X^2, \quad c^2_\mu(x, y) \ge \min(c^2_\mu(x, x_m), c^2_\mu(y, x_m)).$$

Let x and y be any two points in X. By definition:

$$\begin{aligned}
c^2_\mu(x, y) &= 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, y)\\
&\ge 1 - \min(\mu(x), \mu(y))\\
&\quad + \min(c^1_\mu(x, x_m), c^1_\mu(x_m, y))\\
&\quad \text{(since } c^1_\mu \text{ is max-min transitive)}\\
&\ge \min(1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, x_m),\\
&\qquad 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x_m, y))\\
&\ge \min(1 - \min(\mu(x), \mu(x_m)) + c^1_\mu(x, x_m),\\
&\qquad 1 - \min(\mu(x_m), \mu(y)) + c^1_\mu(x_m, y))\\
&\quad \text{(since } \mu(x_m) \ge \max(\mu(x), \mu(y)))\\
&\ge \min(c^2_\mu(x, x_m), c^2_\mu(x_m, y)).
\end{aligned}$$
□

Proof of Proposition 6 Let $\mu \in \mathcal{F}$. We want to show that $c^2_\mu(x, y)$ reaches its minimum for x being a point for which the global maximum of μ is reached and y belonging to a regional maximum of μ. Let x_m be a point for which the global maximum of μ is reached. Proposition 5 guarantees that:

$$\forall (x, y) \in X^2, \quad c^2_\mu(x, y) \ge \min(c^2_\mu(x, x_m), c^2_\mu(y, x_m)).$$

We thus have:

$$\min_{(x,y)\in X^2} c^2_\mu(x, y) \ge \min_{x\in X} c^2_\mu(x, x_m),$$

and since $x_m \in X$:

$$\min_{(x,y)\in X^2} c^2_\mu(x, y) = \min_{x\in X} c^2_\mu(x, x_m).$$

This proves that $c^2_\mu(x, y)$ reaches its minimum for x or y being a point for which the global maximum of μ is reached.

Let us now show that if x_m is a point for which the global maximum of μ is reached, then $c^2_\mu(x_m, x)$ reaches its minimum for x belonging to a regional maximum of μ. Let x be any point in X. Let x_i be a point that belongs to a regional maximum of μ such that there exists an increasing digital path from x to x_i in μ (we have in particular $\mu(x) \le \mu(x_i)$). The existence of this local maximum is guaranteed since X is bounded and finite. We have to prove that:

$$\begin{aligned}
&c^2_\mu(x, x_m) \ge c^2_\mu(x_i, x_m), \quad \text{or equivalently}\\
&c^1_\mu(x, x_m) - \mu(x) \ge c^1_\mu(x_i, x_m) - \mu(x_i), \quad \text{or}\\
&c^1_\mu(x, x_m) + \mu(x_i) \ge c^1_\mu(x_i, x_m) + \mu(x).
\end{aligned}$$

Let us prove the last inequality. Since c^1_μ is max-min transitive, we have:

$$c^1_\mu(x, x_m) \ge \min(c^1_\mu(x, x_i), c^1_\mu(x_i, x_m)). \tag{10}$$

- If $c^1_\mu(x_i, x_m) \ge c^1_\mu(x, x_i)$:

$$\begin{aligned}
10 \quad &\Rightarrow \quad c^1_\mu(x, x_m) \ge c^1_\mu(x, x_i)\\
&\Rightarrow \quad c^1_\mu(x, x_m) \ge \mu(x)\\
&\qquad \text{(there exists an increasing path}\\
&\qquad \text{ in } \mu \text{ from } x \text{ to } x_i)\\
&\Rightarrow \quad c^1_\mu(x, x_m) + \mu(x_i) \ge c^1_\mu(x_i, x_m) + \mu(x)\\
&\qquad \text{(since } \mu(x_i) \ge c^1_\mu(x_i, x_m)).
\end{aligned}$$

- If $c^1_\mu(x, x_i) \ge c^1_\mu(x_i, x_m)$:

$$10 \quad \Rightarrow \quad c^1_\mu(x, x_m) \ge c^1_\mu(x_i, x_m).$$

As c^1_μ is max-min transitive, we have:

$$\begin{aligned}
c^1_\mu(x_i, x_m) &\ge \min(c^1_\mu(x_i, x), c^1_\mu(x, x_m))\\
&\ge \min(\mu(x), c^1_\mu(x, x_m))\\
&\ge c^1_\mu(x, x_m).
\end{aligned}$$

Therefore we obtain:

$$\begin{aligned}
&c^1_\mu(x, x_m) = c^1_\mu(x_i, x_m)\\
&\quad \Rightarrow \quad c^1_\mu(x, x_m) + \mu(x_i) \ge c^1_\mu(x_i, x_m) + \mu(x)\\
&\qquad \text{since } \mu(x_i) \ge \mu(x).
\end{aligned}$$

The property $c^1_\mu(x, x_m) + \mu(x_i) \ge c^1_\mu(x_i, x_m) + \mu(x)$ is therefore always fulfilled and thus:

$$\forall x \in X, \quad c^2_\mu(x, x_m) \ge c^2_\mu(x_i, x_m),$$

where x_i belongs to a regional maximum of μ and x_m to a global maximum of μ. □

Proof of Proposition 7 We first show that the mapping associating $c^1_\mu(x, y)$ with μ is Lipschitz for any fixed points

x and y. Let μ_A and μ_B be two fuzzy sets and $\eta = d_{\mathcal{F}}(\mu_A, \mu_B)$. We want to prove that $d_{\mathcal{F}_{X^2}}(c^1_{\mu_B}, c^1_{\mu_A}) \le \eta$ (where $d_{\mathcal{F}_{X^2}}(f_1, f_2) = \sup_{(x,y)\in X^2} |f_1(x,y) - f_2(x,y)|$).

Let x and y be any two points in X. Let LA and LB be two paths from x to y such that $c^1_{\mu_A}(x,y) = \min_{x_i\in LA} \mu_A(x_i)$ and $c^1_{\mu_B}(x,y) = \min_{x_i\in LB} \mu_B(x_i)$ (since X is bounded and finite, the existence of those paths is guaranteed). In addition $d_{\mathcal{F}}(\mu_A, \mu_B) = \eta \Rightarrow \forall z \in X, |\mu_A(z) - \mu_B(z)| \le \eta$. Let x_a be a point in LA such that $\min_{x_i\in LA} \mu_A(x_i) = \mu_A(x_a)$ and x_b be a point in LA such that $\min_{x_i\in LA} \mu_B(x_i) = \mu_B(x_b)$. As $d_{\mathcal{F}}(\mu_A, \mu_B) = \eta$, we have:

$$\begin{cases} |\mu_A(x_a) - \mu_B(x_a)| \le \eta, \\ |\mu_A(x_b) - \mu_B(x_b)| \le \eta \end{cases}$$

$$\Rightarrow \quad \begin{cases} -\eta + \mu_B(x_a) \le \min_{x_i\in LA} \mu_A(x_i) \le \eta + \mu_B(x_a), \\ -\eta + \mu_A(x_b) \le \min_{x_i\in LA} \mu_B(x_i) \le \eta + \mu_A(x_b) \end{cases}$$

$$\Rightarrow \quad \begin{cases} -\eta + \min_{x_i\in LA} \mu_B(x_i) \le \min_{x_i\in LA} \mu_A(x_i), \\ -\eta + \min_{x_i\in LA} \mu_A(x_i) \le \min_{x_i\in LA} \mu_B(x_i) \end{cases}$$

$$\Rightarrow \quad \left| \min_{x_i\in LA} \mu_A(x_i) - \min_{x_i\in LA} \mu_B(x_i) \right| \le \eta.$$

In the same way we derive:

$$\left| \min_{x_i\in LB} \mu_A(x_i) - \min_{x_i\in LB} \mu_B(x_i) \right| \le \eta.$$

So:

$$\begin{cases} |\min_{x_i\in LB} \mu_A(x_i) - \min_{x_i\in LB} \mu_B(x_i)| \le \eta, \\ |\min_{x_i\in LA} \mu_A(x_i) - \min_{x_i\in LA} \mu_B(x_i)| \le \eta \end{cases}$$

$$\Rightarrow \quad \begin{cases} |\min_{x_i\in LB} \mu_A(x_i) - c^1_{\mu_B}(x,y)| \le \eta, \\ |c^1_{\mu_A}(x,y) - \min_{x_i\in LA} \mu_B(x_i)| \le \eta \end{cases}$$

$$\Rightarrow \quad \begin{cases} c^1_{\mu_B}(x,y) \le \min_{x_i\in LB} \mu_A(x_i) + \eta, \\ c^1_{\mu_A}(x,y) \le \min_{x_i\in LA} \mu_B(x_i) + \eta. \end{cases}$$

The definition of c^1_μ guarantees that:

$$\begin{cases} \min_{x_i\in LB} \mu_A(x_i) \le c^1_{\mu_A}(x,y), \\ \min_{x_i\in LA} \mu_B(x_i) \le c^1_{\mu_B}(x,y), \end{cases}$$

from which we derive, $\forall (x,y) \in X^2$:

$$\begin{cases} c^1_{\mu_B}(x,y) \le c^1_{\mu_A}(x,y) + \eta, \\ c^1_{\mu_A}(x,y) \le c^1_{\mu_B}(x,y) + \eta \end{cases}$$

$$\Rightarrow \quad |c^1_{\mu_B}(x,y) - c^1_{\mu_A}(x,y)| \le \eta.$$

Therefore we obtain $d_{\mathcal{F}_{X^2}}(c^1_{\mu_B}, c^1_{\mu_A}) \le \eta$ and the mapping associating $c^1_\mu(x,y)$ with μ is Lipschitz.

Let us now prove that the mapping associating $c^2_\mu(x,y)$ with μ is 2-Lipschitz for given x and y. Let μ_A and μ_B be two fuzzy sets, $\eta = d_{\mathcal{F}}(\mu_A, \mu_B)$ and $(x,y) \in X^2$. Since c^1_μ is Lipschitz:

$$|c^1_{\mu_B}(x,y) - c^1_{\mu_A}(x,y)| \le \eta$$

$$\Rightarrow \quad -\eta + c^1_{\mu_B}(x,y) \le c^1_{\mu_A}(x,y) \le \eta + c^1_{\mu_B}(x,y). \quad \text{(I1)}$$

In addition:

$$\forall z \in X, \quad |\mu_A(z) - \mu_B(z)| \le \eta$$

$$\Rightarrow \quad \forall z \in X, -\eta + \mu_B(z) \le \mu_A(z) \le \eta + \mu_B(z)$$

$$\Rightarrow \quad \begin{cases} -\eta + \mu_B(x) \le \mu_A(x) \le \eta + \mu_B(x), \\ -\eta + \mu_B(y) \le \mu_A(y) \le \eta + \mu_B(y) \end{cases}$$

(for $z = x$ and $z = y$ respectively)

$$\Rightarrow \quad \begin{cases} -\eta + \min(\mu_B(x), \mu_B(y)) \le \min(\mu_A(x), \mu_A(y)), \\ \min(\mu_A(x), \mu_A(y)) \le \eta + \min(\mu_B(x), \mu_B(y)) \end{cases}$$

$$\Rightarrow \quad \begin{cases} 1 - \eta - \min(\mu_B(x), \mu_B(y)) \\ \quad \le 1 - \min(\mu_A(x), \mu_A(y)), \\ 1 - \min(\mu_A(x), \mu_A(y)) \\ \quad \le 1 + \eta - \min(\mu_B(x), \mu_B(y)) \end{cases}$$

$$\Rightarrow \quad \begin{cases} -2\eta + 1 - \min(\mu_B(x), \mu_B(y)) + c^1_{\mu_B}(x,y) \\ \quad \le 1 - \min(\mu_A(x), \mu_A(y)) + c^1_{\mu_A}(x,y), \\ 1 - \min(\mu_A(x), \mu_A(y)) + c^1_{\mu_A}(x,y) \\ \quad \le 2\eta + 1 - \min(\mu_B(x), \mu_B(y)) + c^1_{\mu_B}(x,y) \end{cases}$$

(adding (I1) and (I2))

$$\Rightarrow \quad |c^2_{\mu_A}(x,y) - c^2_{\mu_B}(x,y)| \le 2\eta. \quad \text{(I2)}$$

Consequently the mapping associating μ with $c^2_\mu(x,y)$ is 2-Lipschitz. □

Proof of Proposition 8 We show that the mapping $c^2(\mu)$ is 2-Lipschitz. Let μ_A and μ_B be two fuzzy sets and $\eta = d_{\mathcal{F}}(\mu_A, \mu_B)$. Let x_A and y_A be two points in X such that $c^2(\mu_A) = c^2_{\mu_A}(x_A, y_A)$ and $(x_B, y_B) \in X^2$ such that $c^2(\mu_B) = c^2_{\mu_B}(x_B, y_B)$.

As c^2_μ is 2-Lipschitz:

$$\begin{cases} |c^2_{\mu_A}(x_A, y_A) - c^2_{\mu_B}(x_A, y_A)| \le 2\eta, \\ |c^2_{\mu_A}(x_B, y_B) - c^2_{\mu_B}(x_B, y_B)| \le 2\eta \end{cases}$$

$$\Rightarrow \quad \begin{cases} c^2_{\mu_B}(x_A, y_A) \le 2\eta + c^2_{\mu_A}(x_A, y_A), \\ c^2_{\mu_A}(x_B, y_B) \le 2\eta + c^2_{\mu_B}(x_B, y_B) \end{cases}$$

$$\Rightarrow \quad \begin{cases} c^2_{\mu_B}(x_A, y_A) \le 2\eta + c^2(\mu_A), \\ c^2_{\mu_A}(x_B, y_B) \le 2\eta + c^2(\mu_B) \end{cases}$$

$$\Rightarrow \quad \begin{cases} c^2(\mu_B) \le 2\eta + c^2(\mu_A), \\ c^2(\mu_A) \le 2\eta + c^2(\mu_B) \end{cases}$$

$$\text{(as } c^2(\mu_B) \le c^2_{\mu_B}(x_A, y_A) \text{ and}$$
$$c^2(\mu_A) \le c^2_{\mu_A}(x_B, y_B))$$
$$\Rightarrow \quad -2\eta + c^2(\mu_B) \le c^2(\mu_A) \le 2\eta + c^2(\mu_B)$$
$$\Rightarrow \quad |c^2(\mu_A) - c^2(\mu_B)| \le 2\eta.$$

The mapping c^2 is therefore 2-Lipschitz. □

Proof of Proposition 9 To prove that $\mathcal{H}^2_\tau$ is a hyperconnection on $\mathcal{F}$ for the overlap mapping $\perp^2_\tau$, we have to show that:

- $0_\mathcal{F} \in \mathcal{H}^2_\tau$,
- $\mathcal{H}^2_\tau$ contains a sup-generating family S of $\mathcal{F}$,
- for any family $\{\mu_i\}$ of elements of $\mathcal{H}^2_\tau$ such that

$$\perp^2_\tau(\{\mu_i\}) = 1, \quad \text{then} \quad \bigvee_i \mu_i \in \mathcal{H}^2_\tau.$$

(1) As $\forall (x, y) \in X^2$, $c^1_{0_\mathcal{F}}(x, y) = 0$, we have $\forall (x, y) \in X^2$, $c^2_{0_\mathcal{F}}(x, y) = 1$ and thus $c^2(0_\mathcal{F}) = 1$. This shows that $\forall \tau \in [0, 1], 0_\mathcal{F} \in \mathcal{H}^2_\tau$.
(2) $\forall x \in X$, $\forall t \in [0, 1]$, $c^2(\delta^t_x) = 1$ and so $\forall x \in X$, $\forall t \in [0, 1]$, $\forall \tau \in [0, 1]$, $\delta^t_x \in \mathcal{H}^2_\tau$. In addition the family $\{\delta^t_x \mid x \in X, t \in [0, 1]\}$ is sup-generating in $\mathcal{F}$.
(3) Let $\{\mu_i\}$ be a family of elements of $\mathcal{H}^2_\tau$ such that $\perp^2_\tau(\{\mu_i\}) = 1$. We denote by μ the supremum of this family: $\mu = \bigvee_i \mu_i$. We have to show that $c^2(\mu) \ge \tau$.

Let x_m be a point for which the global maximum of μ is reached, μ_m be an element of the family $\{\mu_i\}$ such that $\mu(x_m) = \mu_m(x_m)$ and y be a point that belongs to a regional maximum of μ such that $c^2(\mu) = 1 - \mu(y) + c^1_\mu(x_m, y)$ (the existence is guaranteed by Proposition 6). Let k be such that $\mu_k(y) = \mu(y)$, then:

$$c^2(\mu) = 1 - \mu_k(y) + c^1_\mu(x_m, y).$$

As c^1_μ is max-min transitive we have:

$$c^1_\mu(x_m, y) \ge \min(c^1_\mu(x_m, x_k), c^1_\mu(x_k, y)), \tag{11}$$

where x_k is a point for which the global maximum of μ_k is reached.

$$\perp^2_\tau(\{\mu_i\}) = 1$$
$$\Rightarrow \quad \forall \alpha \in [0, 1],$$
$$\bigcap_i \{(\eta^1_{\delta^{h_i}_{x_i}}(\mu_i))_\alpha \mid \alpha \le h_i - 1 + \tau\} \ne \emptyset$$
$$\text{where } h_i = \max_{x \in X} \mu_i(x) = \mu_i(x_i)$$
$$\Rightarrow \quad \forall \alpha \in [0, 1], \quad \alpha \le \min(\mu_m(x_m), \mu_k(x_k)) - 1 + \tau,$$
$$(\eta^1_{\delta^{h_m}_{x_m}}(\mu_m))_\alpha \cap (\eta^1_{\delta^{h_k}_{x_k}}(\mu_k))_\alpha \ne \emptyset$$
$$\text{(considering only indices associated with } \mu_k \text{ and to } \mu_m)$$
$$\Rightarrow \quad \exists l \in L_{x_k, x_m}, \quad l = \{x_0 = x_k, x_1, \ldots, x_n = x_m\}$$
$$\text{such that } \min_{0 \le i \le n} \left(\eta^1_{\delta^{h_k}_{x_k}}(\mu_k) \vee \eta^1_{\delta^{h_m}_{x_m}}(\mu_m)\right)(x_i)$$
$$\ge \min(\mu_m(x_m), \mu_k(x_k)) - 1 + \tau,$$

where L_{x_k, x_m} is the set of discrete paths from x_k to x_m, according to the discrete connectivity c_d defined on X. Indeed according to Proposition 3, $\eta^1_{\delta^{h_k}_{x_k}}(\mu_k) \in \mathcal{H}^1$ and $\eta^1_{\delta^{h_m}_{x_m}}(\mu_m) \in \mathcal{H}^1$ and so according to Proposition 1, all their α-cuts are connected in the sense of the discrete connectivity c_d. If we choose $\alpha_0 = \min(\mu_m(x_m), \mu_k(x_k)) - 1 + \tau$, the α-cuts of $\eta^1_{\delta^{h_k}_{x_k}}(\mu_k)$ and of $\eta^1_{\delta^{h_m}_{x_m}}(\mu_m)(x_i)$ at level α_0 are connected and intersect. Their union is thus connected. This guarantees the existence of the path l. Since

$$c^1_\mu(x, y) = \max_{\substack{l \in L_{x,y} \\ l = \{x_0 = x, x_1, \ldots, x_n = y\}}} \min_{0 \le i \le n} \mu(x_i),$$

we obtain:

$$c^1_{\eta^1_{\delta^{h_k}_{x_k}}(\mu_k) \vee \eta^1_{\delta^{h_m}_{x_m}}(\mu_m)}(x_m, x_k) \ge \mu_k(x_k) - 1 + \tau$$
$$\Rightarrow \quad c^1_\mu(x_m, x_k) \ge \mu_k(x_k) - 1 + \tau$$
$$\text{(as } c^1_\mu \text{ is increasing according to } \mu)$$
$$\Rightarrow \quad c^1_\mu(x_m, x_k) \ge \mu_k(y) - 1 + \tau$$
$$\text{(as } \mu_k(x_k) \ge \mu_k(y)).$$

In addition:

$$\mu_k \in \mathcal{H}^2_\tau$$
$$\Rightarrow \quad c^2(\mu_k) \ge \tau$$
$$\Rightarrow \quad 1 - \min(\mu_k(x_k), \mu_k(y)) + c^1_{\mu_k}(x_k, y) \ge \tau$$
$$\text{(since } \forall (x_1, x_2) \in X^2, c^2_{\mu_k}(x_1, x_2) \ge c^2(\mu_k))$$
$$\Rightarrow \quad 1 - \mu_k(y) + c^1_{\mu_k}(x_k, y) \ge \tau$$
$$\Rightarrow \quad c^1_{\mu_k}(x_k, y) \ge \mu_k(y) - 1 + \tau$$
$$\Rightarrow \quad c^1_\mu(x_k, y) \ge \mu_k(y) - 1 + \tau$$
$$\text{(as } c^1_\mu \text{ is increasing according to } \mu).$$

Therefore we obtain:

$$11 \quad \Rightarrow \quad c^1_\mu(x_m, y) \ge \min(c^1_\mu(x_m, x_k), c^1_\mu(x_k, y))$$
$$\Rightarrow \quad c^1_\mu(x_m, y) \ge \mu_k(y) - 1 + \tau$$

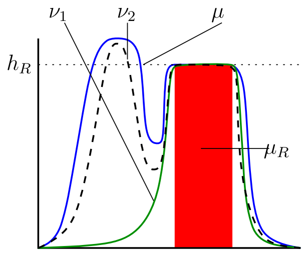


Fig. 23 A 1D fuzzy set μ. μ_R associated with a regional maximum. ν_1 belongs to $\mathcal{H}^1$ and $\mu_R \leq \nu_1$. $\nu_2 \notin \mathcal{H}^1$

$$\begin{aligned}
&\Rightarrow \quad 1-\mu_k(y)+c^1_\mu(x_m,y)\geq\tau\\
&\Rightarrow \quad 1-\mu(y)+c^1_\mu(x_m,y)\geq\tau\\
&\qquad (\text{as } \mu(y)=\mu_k(y))\\
&\Rightarrow \quad 1-\min(\mu(x_m),\mu(y))+c^1_\mu(x_m,y)\geq\tau\\
&\qquad (\text{as } \mu(y)\leq\mu(x_m))\\
&\Rightarrow \quad c^2(\mu)\geq\tau\\
&\Rightarrow \quad \mu\in\mathcal{H}^2_\tau. \qquad \square
\end{aligned}$$

Proof of Proposition 11 According to Proposition 3, if x_i belongs to a regional maximum of μ, then $\eta^1_{\delta^{\mu(x_i)}_{x_i}} \in \mathcal{H}^1$ and Proposition 10 guarantees that $\eta^1_{\delta^{\mu(x_i)}_{x_i}} \in \mathcal{H}^1(\mu)$.

We still have to prove that the set of 1-hyperconnected components $\mathcal{H}^1(\mu)$ of μ according to $\mathcal{H}^1$ is isomorphic to the regional maxima $\{R_i\}$ of μ. To this aim let us show that the elements of $\mathcal{H}^1(\mu)$ are exactly the connected openings $\eta^1_{\mu_R}$ whose origins are associated with the regional maxima of μ. Let $R \subseteq X$ be a regional maximum of μ. Let $\mu_R(x) = \mu(x) = h_R$ if $x \in R$ and 0 otherwise, h_R being the height of this fuzzy set. Let us show that $\eta^1_{\mu_R}(\mu)$ belongs to $\mathcal{H}^1(\mu)$ and that the mapping $\eta^1_{\mu_R}(\mu) : \{R_i\} \to \mathcal{H}^1(\mu)$ is bijective.

The connected opening of origin μ_R can be expressed as: $\eta^1_{\mu_R}(\mu) = \bigvee\{\nu \in \mathcal{H}^1 \mid \mu_R \leq \nu \leq \mu\}$. The set of elements ν that fulfills those conditions intersect (at least) for all levels $\alpha \in [0, h_R]$ since all of them include μ_R. In addition all those elements are such that $\max_{x\in X} \nu(x) = h_R$ (cf. proof of Proposition 3). Indeed otherwise the set ν would have at least two regional maxima and would not belong to $\mathcal{H}^1$ (Fig. 23 illustrates the notations used here. ν_2 is a fuzzy set such that $\max_{x\in X} \nu_2(x) > h_R$ but $\nu_2 \notin \mathcal{H}^1$). We can conclude that $\perp^1(\{\nu \in \mathcal{H}^1 \mid \mu_R \leq \nu \leq \mu\}) = 1$ and therefore that $\eta^1_{\mu_R}(\mu) \in \mathcal{H}^1$. Proposition 10 then guarantees that $\eta^1_{\mu_R}(\mu) \in \mathcal{H}^1(\mu)$.

Let us now show that the mapping $\eta^1_{\mu_R}(\mu) : \{R_i\} \to \mathcal{H}^1(\mu)$ is surjective and thus that each 1-hyperconnected component of μ is associated with a regional maximum of μ. Let $\mu_i \in \mathcal{H}^1(\mu)$ and x_i be a point for which the global maximum of μ_i is reached (note that we have $\mu_i(x_i) = \mu(x_i)$). If x_i belongs to a regional maximum R_i of μ, then $\mu_i = \eta^1_{\mu_{R_i}}(\mu)$ and therefore μ_i is associated with R_i. Let us assume that x_i does not belong to a regional maximum of μ. Let x_m belonging to a regional maximum of μ such that there exists an increasing path from x_i to x_m in μ. We consider the fuzzy set ν such that $\nu(x) = \mu(x)$ for x in that path and $\nu(x) = 0$ otherwise. We can show that $\perp^1(\mu_i, \nu) = 1$ and thus that $\mu_i \vee \nu \in \mathcal{H}^1$. In addition the property $\mu_i < \mu_i \vee \nu \leq \mu$ derives from definition of ν. This contradicts the property $\nexists \nu \in \mathcal{H}^1, \mu_i < \nu \leq \mu$ (since $\mu_i \in \mathcal{H}^1(\mu)$). Therefore this case cannot occur et x_i always belongs to a regional maximum of μ and $\mu_i = \eta^1_{\mu_{R_i}}(\mu)$.

We still have to prove that $\eta^1_{\mu_R}(\mu) : \{R_i\} \to \mathcal{H}^1(\mu)$ is injective. Let R_i and R_j be two regional maxima of μ, $\mu_i = \eta^1_{\mu_{R_i}}(\mu)$ and $\mu_j = \eta^1_{\mu_{R_j}}(\mu)$. Let us assume that $\mu_i = \mu_j$. This leads to $\mu_i \geq \mu_{R_i} \vee \mu_{R_j}$. As $\mu_i \leq \mu$ and as R_i and R_j are distinct regional maxima, μ_i has at least two regional maxima, which contradicts $\mu_i \in \mathcal{H}^1(\mu)$. μ_i and μ_j are thus distinct.

$\mathcal{H}^1(\mu)$ and the set of regional maxima of μ are thus isomorphic. $\square$

Proof of Proposition 12 Let μ be a fuzzy set and $\mathcal{H}^1(\mu) = \{\mu_i\}$. Proposition 11 guarantees that each μ_i is associated with one regional maximum of μ. We denote by x_i a point that belongs to the regional maximum associated with μ_i and x_j a point that belongs to the regional maximum associated with μ_j. Let us prove that:

$$c^1_\mu(x_i, x_j) = \max_{x\in X} \min(\mu_i(x), \mu_j(x)).$$

First we will show that:

$$c^1_\mu(x_i, x_j) \geq \max_{x\in X} \min(\mu_i(x), \mu_j(x)).$$

The measure c^1_μ is max-min transitive and so:

$$c^1_{\mu_i\vee\mu_j}(x_i, x_j) \geq \max_{x\in X} \min(c^1_{\mu_i\vee\mu_j}(x_i, x), c^1_{\mu_i\vee\mu_j}(x, x_j)).$$

In addition $c^1_{\mu_i\vee\mu_j}(x_i, x) \geq c^1_{\mu_i}(x_i, x)$, since c^1_μ is increasing according to μ. As by definition $\mu_i \in \mathcal{H}^1$ and as x_i is a point for which the global maximum of μ_i is reached, we have the equality: $c^1_{\mu_i}(x_i, x) = \min(\mu_i(x_i), \mu_i(x)) = \mu_i(x)$. We thus obtain $c^1_{\mu_i\vee\mu_j}(x_i, x) \geq \mu_i(x)$ and in the same manner $c^1_{\mu_i\vee\mu_j}(x, x_j) \geq \mu_j(x)$.

So:

$$\begin{aligned}
&c^1_{\mu_i\vee\mu_j}(x_i, x_j) \geq \max_{x\in X} \min(c^1_{\mu_i\vee\mu_j}(x_i, x), c^1_{\mu_i\vee\mu_j}(x, x_j))\\
&\quad\Rightarrow \quad c^1_{\mu_i\vee\mu_j}(x_i, x_j) \geq \max_{x\in X} \min(\mu_i(x), \mu_j(x))\\
&\quad\Rightarrow \quad c^1_\mu(x_i, x_j) \geq \max_{x\in X} \min(\mu_i(x), \mu_j(x))\\
&\qquad\quad (\text{as } c^1_\mu \text{ is increasing according to } \mu).
\end{aligned}$$

We will now prove by contradiction that: $c^1_\mu(x_i, x_j) \leq \max_{x\in X} \min(\mu_i(x), \mu_j(x))$. Let us assume that $c^1_\mu(x_i, x_j) > \max_{x\in X} \min(\mu_i(x), \mu_j(x))$ and let x_a be the point for which the max-min criterion in $c^1_\mu(x_i, x_j)$ is reached. We thus have $\mu(x_a) > \max_{x\in X} \min(\mu_i(x), \mu_j(x))$ and so $\mu(x_a) > \mu_i(x_a)$ or $\mu(x_a) > \mu_j(x_a)$. In addition the 1-hyperconnected components μ_i $(\in \mathcal{H}^1)$ are such that $\forall x \in X$, $\mu_i(x) = c^1_\mu(x, x_i)$ (cf. Propositions 3 and 11). We obtain $\mu_i(x_a) = c^1_\mu(x_a, x_i)$, and since $c^1_\mu(x_a, x_i) = \mu(x_a)$ (considering the path used to compute $c^1_\mu(x_i, x_j)$), we have $\mu_i(x_a) = \mu(x_a)$. In the same way we have $\mu_j(x_a) = c^1_\mu(x_a, x_j) = \mu(x_a)$, which contradicts $\mu(x_a) > \mu_i(x_a)$ or $\mu(x_a) > \mu_j(x_a)$. □

Proof of Proposition 13 As the set of leaves $\mathcal{L}$ of the tree $T = T(\mu)$ is isomorphic to the regional maxima $\{R_i\}$ of μ, Proposition 11 guarantees that $\mathcal{L}$ is isomorphic to $\mathcal{H}^1(\mu)$.

Let us now show that each connected component μ_i is exactly represented by the branch $P^{l_i}(h(l_i))$ of the tree. We denote by $E(G)(x) = \max_{v\in G \mid x\in Pt(v)} h(v)$ the function that returns the fuzzy set associated with a sub-tree G of $T(\mu)$. We recall that the quantification of membership degrees and tree's levels are similar and thus $\forall \mu \in \mathcal{F}, \mu = E(\mathcal{V})$, where $\mathcal{V}$ is the set of vertices of the tree $T(\mu)$ associated with μ. We will first show that $E(P^{l_i}) \in \mathcal{H}^1$. By definition a vertex of the tree $T(\mu)$ of height h is associated with a connected component of the α-cut of μ of height h and edges are induced by the inclusion relation between connected components for successive values of h. In addition, since P^{l_i} is a branch of the tree, all vertices of P^{l_i} have at most one child vertex in P^{l_i}. Therefore if $v \in P^{l_i}$, all vertices $v' \in P^{l_i}$ whose height is higher than $h(v)$ are descendants of v and $Pt(v') \subseteq Pt(v)$. Let α be a fixed level and v_m be the vertex of P^{l_i} such that $h(v_m) = \alpha$. We have $\forall x \in (E(P^{l_i}))_\alpha$, $\exists v \in P^{l_i}$ such that $x \in Pt(v)$ and $h(v) \geq \alpha$. Since for any vertex v of P^{l_i} such that $h(v) \geq \alpha$ we have $Pt(v) \subseteq Pt(v_m)$, we obtain $\forall x \in (E(P^{l_i}))_\alpha, x \in Pt(v_m)$. In addition $\forall x \in Pt(v_m), x \in (E(P^{l_i}))_\alpha$. Therefore $(E(P^{l_i}))_\alpha$ is the connected component $Pt(v_m)$ represented by v_m and all α-cuts of $E(P^{l_i})$ are connected. According to Proposition 1, $E(P^{l_i})$ belongs to $\mathcal{H}^1$.

We still have to show that there does not exist a fuzzy set $\nu \in \mathcal{H}^1$ such that $E(P^{l_i}) < \nu \leq \mu$. Suppose that such a fuzzy set ν exists. ν is such that $\exists \alpha \leq h(l_i), (E(P^{l_i}))_\alpha \subset (\nu)_\alpha$. By definition $(E(P^{l_i}))_\alpha = Pt(v)$, where $h(v) = \alpha$ and $Pt(v)$ is a connected component of $(\mu)_\alpha$. Therefore $(\nu)_\alpha$ presents two connected components and $\nu \notin \mathcal{H}^1$. This contradicts the hypothesis and we obtain $\forall l_i \in \mathcal{L}, E(P^{l_i}) \in \mathcal{H}^1(\mu)$. In addition since $\mathcal{H}^1(\mu)$ is isomorphic to the leaves of the tree, we obtain that all 1-hyperconnected components of μ are represented by a branch of the tree. □

Proof of Proposition 14 Let us show that:

$$c^2(\mu) = 1 - \max_{(l_1,l_2)\in\mathcal{L}^2} (\min(h(l_1), h(l_2)) - i^{l_1,l_2}).$$

We recall that $i^{l_1,l_2} = h(P^{l_1}(h(l_1)) \wedge P^{l_2}(h(l_2)))$.

Proposition 6 guarantees that the degree of connectivity $c^2(\mu) = \min_{(x,y)\in X^2} c^2_\mu(x, y)$ of a fuzzy set μ is reached when x and y belong to regional maxima of μ. If x_1 and x_2 belong to two regional maxima, they are associated with two 1-hyperconnected components of μ according to Proposition 11. We denote by μ_1 and μ_2 those components and by l_1 and l_2 the leaves of the tree associated with x_1 and x_2. We can reformulate $c^2_\mu(x_1, x_2)$ as $c^2_\mu(x_1, x_2) = 1 - \min(h(l_1), h(l_2)) + c^1_\mu(x_1, x_2)$.

Let us show that $c^1_\mu(x_1, x_2) = i^{l_1,l_2}$. According to Proposition 12, we have $c^1_\mu(x_1, x_2) = \max_{x\in X} \min(\mu_1(x), \mu_2(x))$ and according to Proposition 13, we have $\mu_1 = E(P^{l_1})$ and $\mu_2 = E(P^{l_2})$. Moreover we have:

$$E(P^{l_1} \wedge P^{l_2})(x) = \max_{v\in P^{l_1}\wedge P^{l_2} \mid x\in Pt(v)} h(v).$$

Since x cannot belong to two nodes that are not either linked by an edge (or a connected series of edges) or equal, we have:

$$\max_{v\in P^{l_1} \mid x\in Pt(v)} h(v) \leq \max_{v\in P^{l_1}\wedge P^{l_2} \mid x\in Pt(v)} h(v) \quad \text{or}$$
$$\max_{v\in P^{l_2} \mid x\in Pt(v)} h(v) \leq \max_{v\in P^{l_1}\wedge P^{l_2} \mid x\in Pt(v)} h(v).$$

Therefore we obtain:

$$\begin{aligned} &E(P^{l_1} \wedge P^{l_2})(x) \\ &\quad = \min\Bigl(\max_{v\in P^{l_1} \mid x\in Pt(v)} h(v), \max_{v\in P^{l_2} \mid x\in Pt(v)} h(v)\Bigr) \\ &\quad = (\mu_1 \wedge \mu_2)(x) \end{aligned}$$

and thus

$$\begin{aligned} i^{l_1,l_2} &= h(P^{l_1}(h(l_1)) \wedge P^{l_2}(h(l_2))) \\ &= \max_{x\in X} \min(\mu_1(x), \mu_2(x)) = c^1_\mu(x_1, x_2). \end{aligned}$$

We obtain $c^2_\mu(x_1, x_2) = 1 - \min(h(l_1), h(l_2)) + i^{l_1,l_2}$. The connectivity degree $c^2(\mu)$ is then obtained considering all leaves of the tree:

$$\begin{aligned} c^2(\mu) &= \min_{(l_1,l_2)\in\mathcal{L}^2} (1 - \min(h(l_1), h(l_2)) + i^{l_1,l_2}) \\ &= 1 - \max_{(l_1,l_2)\in\mathcal{L}^2} (\min(h(l_1), h(l_2)) - i^{l_1,l_2}). \end{aligned}$$ □

Proof of Proposition 15 Let μ be a fuzzy set and G a subtree belonging to $S_{T(\mu)}$ (we recall that a subtree is assimilated to

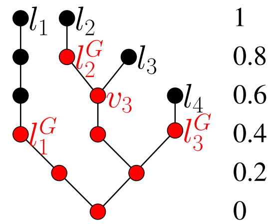


Fig. 24 A tree $T(\mu)$ that presents four leaves l_1, l_2, l_3 and l_4. A sub-tree G of $T(\mu)$ is shown in red. Its tree leaves are denoted by l_1^G, l_2^G and l_3^G

its set of vertices). We denote by $E(G)$ the fuzzy set associated with G. We will show that:

$$c^2(E(G)) = \min\Big(1, \min_{(l_1,l_2)\in\mathcal{L}^2} (1 - \min(h_G^{l_1}, h_G^{l_2}) + i_{T(\mu)}^{l_1,l_2})\Big),$$

where $h_G^{l_i}$ is the maximal height of the vertices of G on the branch associated with the leaf l_i. In addition we specify here the tree in which i^{l_1,l_2} is computed, and we note $i_{T(\mu)}^{l_1,l_2}$. Considering the illustration in Fig. 24, we have for instance $h_G^{l_1} = 0.4$, for the subtree G in red.

Proposition 14 guarantees that:

$$c^2(E(G)) = \min_{(l_1^G,l_2^G)\in{\mathcal{L}_G}^2} (1 - \min(h(l_1^G), h(l_2^G)) + i_G^{l_1^G,l_2^G})$$

(we specify in that case that we consider the leaves of G and not the leaves of $T(\mu)$).

Let l_a^G and l_b^G be leaves of G for which this minimum is reached: $c^2(E(G)) = 1 - \min(h(l_a^G), h(l_b^G)) + i_G^{l_a^G,l_b^G}$ (those leaves are l_1^G and l_2^G in Fig. 24). Whatever the leaf l^G of G, there exists at least one leaf l of $T(\mu)$ such that $l^G \in P^l(h(l))$ and $h_G^l = h(l^G)$ (the leaf l_4 satisfies those conditions for l_3^G in Fig. 24). l is the leaf of a branch of $T(\mu)$ that contains l^G. If we denote by l_a and l_b the two leaves associated with l_a^G and l_b^G, we obtain: $c^2(E(G)) = 1 - \min(h_G^{l_a}, h_G^{l_b}) + i_G^{l_a^G,l_b^G}$.

If the leaves l_a^G and l_b^G are distinct, we have $i_G^{l_a^G,l_b^G} = i_{T(\mu)}^{l_a,l_b}$. In fact we have $i_G^{l_a^G,l_b^G} = h(P^{l_a}(h_G^{l_a}) \wedge P^{l_b}(h_G^{l_b}))$ and $i_{T(\mu)}^{l_a,l_b} = h(P^{l_a}(h(l_a)) \wedge P^{l_b}(h(l_b)))$. Since the leaves l_a^G and l_b^G are distinct, we have $h(P^{l_a}(h_G^{l_a}) \wedge P^{l_b}(h_G^{l_b})) < \min(h_G^{l_a}, h_G^{l_b})$ and therefore $P^{l_a}(h_G^{l_a}) \wedge P^{l_b}(h_G^{l_b}) = P^{l_a}(h(l_a)) \wedge P^{l_b}(h(l_b))$. For instance in Fig. 24 we have $i_G^{l_2^G,l_3^G} = 0.2 = i_{T(\mu)}^{l_2,l_4}$.

If l_a^G and l_b^G are actually the same leaf, we have $i_G^{l_a^G,l_b^G} = h(l_a^G) = h(l_b^G)$ and $i_{T(\mu)}^{l_a,l_b} \geq \min(h_G^{l_a}, h_G^{l_b}) = h_G^{l_a} = h_G^{l_b}$. Therefore we obtain $1 - \min(h(l_a^G), h(l_b^G)) + i_G^{l_a^G,l_b^G} = 1 = \min(1, 1 - \min(h_G^{l_a}, h_G^{l_b}) + i_{T(\mu)}^{l_a,l_b})$.

So in all cases, $c^2(E(G)) = \min(1, 1 - \min(h_G^{l_a}, h_G^{l_b}) + i_{T(\mu)}^{l_a,l_b})$.

Let l_1 and l_2 be any two leaves of $T(\mu)$. We denote by v_1 the vertex of G associated with l_1 ($v_1 \in P^{l_1}(h(l_1))$ and $h(v_1) = h_G^{l_1}$) and by v_2 the vertex of G associated with l_2 (in Fig. 24, we show the vertex v_3 associated with the leaf l_3. The other leaves of the tree are associated with leaves of G: l_1^G is associated with l_1, l_2^G to l_2 and l_3^G to l_4). We denote by l_1^G and l_2^G two leaves of G such that v_1 and v_2 belong to associated branches (in Fig. 24, v_3 is associated with l_2^G).

$$\begin{aligned}
&1 - \min(h_G^{l_1}, h_G^{l_2}) + i_{T(\mu)}^{l_1,l_2}\\
&\quad = 1 - \min(h(v_1), h(v_2)) + i_{T(\mu)}^{l_1,l_2}\\
&\quad \geq 1 - \min(h(v_1), h(v_2)) + i_G^{v_1,v_2} \quad \text{(see below $*1$)}\\
&\quad \geq 1 - \min(h(l_1^G), h(l_2^G)) + i_G^{l_1^G,l_2^G} \quad \text{(see below $*2$)}\\
&\quad \geq 1 - \min(h(l_a^G), h(l_b^G)) + i_G^{l_a^G,l_b^G}\\
&\qquad \text{(since the minimum of } 1 - \min(h(l_1^G), h(l_2^G)) + i_G^{l_1^G,l_2^G}\\
&\qquad \text{is reached for the leaves } l_a^G \text{ and } l_b^G\text{)}\\
&\quad \geq 1 - \min(h_G^{l_a}, h_G^{l_b}) + i_{T(\mu)}^{l_a,l_b}.
\end{aligned}$$

The minimum is thus reached by the leaves l_a and l_b. Therefore:

$$c^2(E(G)) = \min_{(l_1,l_2)\in\mathcal{L}^2} (\min(1, 1 - \min(h_G^{l_1}, h_G^{l_2}) + i_{T(\mu)}^{l_1,l_2}))$$

can be rewritten as

$$c^2(E(G)) = \min\Big(1, \min_{(l_1,l_2)\in\mathcal{L}^2} (1 - \min(h_G^{l_1}, h_G^{l_2}) + i_{T(\mu)}^{l_1,l_2})\Big).$$

$*1$:

$$\begin{aligned}
i_{T(\mu)}^{l_1,l_2} &= h(P^{l_1}(h(l_1)) \wedge P^{l_2}(h(l_2)))\\
&= h(P^{l_1}(h(l_1)) \wedge P^{l_2}(h(l_2)))\\
&\geq h(P^{l_1}(h(v_1)) \wedge P^{l_2}(h(v_2)))
\end{aligned}$$

since $h(v_1) \leq h(l_1)$ and $h(v_2) \leq h(l_2)$. Therefore $P^{l_1}(h(v_1)) \wedge P^{l_2}(h(v_2)) \leq P^{l_1}(h(l_1)) \wedge P^{l_2}(h(l_2))$. In addition $P^{l_1}(h(v_1)) = P^{l_1^G}(h(v_1))$ and $P^{l_2}(h(v_2)) = P^{l_2^G}(h(v_2))$. Thus we derive:

$$i_{T(\mu)}^{l_1,l_2} = h(P^{l_1^G}(h(v_1)) \wedge P^{l_2^G}(h(v_2))) = i_G^{v_1,v_2}.$$

$*2$: Let us show that if v_1 and v_2 are two vertices that belong to the same branch such that $h(v_1) \leq h(v_2)$, we have $\forall v \in T(\mu)$:

$$\min(h(v_1), h(v)) - i_{T(\mu)}^{v_1,v} \leq \min(h(v_2), h(v)) - i_{T(\mu)}^{v_2,v}.$$

If $i_{T(\mu)}^{v_1,v} < h(v_1)$, we have $i_{T(\mu)}^{v_1,v} = i_{T(\mu)}^{v_2,v}$ as v_1 and v_2 belong to the same branch of the tree. Since $h(v_1) \leq h(v_2)$, the inequality is thus satisfied. If $i_{T(\mu)}^{v_1,v} = h(v_1)$, we have

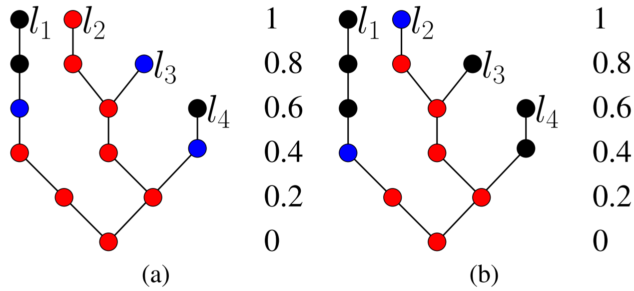


Fig. 25 (Color online) (**a**) A tree $T(\mu)$ and a sub-tree G in *red*. In *blue* and *red* $\delta_{T(\mu)}(G, 0.2)$. (**b**) In *blue* and *red*, a sub-tree G. In red $\varepsilon_{T(\mu)}(G, 0.2)$

$\min(h(v_1), h(v)) - i^{v_1,v}_{T(\mu)} = 0$ and since $\min(h(v_2), h(v)) - i^{v_2,v}_{T(\mu)} \geq 0$, the inequality is also satisfied in that case. □

Proof of Proposition 16 We denote by $E(G)$ the fuzzy set associated with a sub-tree $G \in S_{T(\mu)}$ of $T(\mu)$.

Let us prove that for any sub-tree $G \in S_{T(\mu)}$:

$$c^2(E(G)) \geq \tau \;\Rightarrow\; c^2(E(\delta_{T(\mu)}(G, r))) \geq \max(0, \tau - r),$$

$$c^2(E(G)) \geq \tau \;\Rightarrow\; c^2(E(\varepsilon_{T(\mu)}(G, r))) \geq \min(1, \tau + r).$$

An example is presented in Fig. 25. The dilation of a subtree G (in red), $\delta_{T(\mu)}(G, 0.2)$ is displayed in blue and red (a). The contraction $\varepsilon_{T(\mu)}(G, 0.2)$ (b) of this sub-tree is displayed in red. According to Proposition 15, the connectivity degree of G is: $c^2(E(G)) = 1 - \min(h^{l_1}_G, h^{l_2}_G) + i^{l_1,l_2} = 1 - 0.4 + 0 = 0.6$. The connectivity degree of $\delta_{T(\mu)}(G, 0.2)$ (presented in blue and red) is $c^2(E(\delta_{T(\mu)}(G, 0.2))) = 1 - \min(h^{l_1}_\delta, h^{l_2}_\delta) + i^{l_1,l_2} = 1 - 0.6 + 0 = 0.4$. Therefore the inequality $c^2(E(\delta_{T(\mu)}(G, 0.2))) \geq \max(0, c^2(E(G)) - 0.2)$ is satisfied.

Let us first prove the property for the dilation. We denote by δ the dilation:

$$\delta_{T(\mu)}(G, r) = \bigvee_{l \in \mathcal{L}} P^l(\min(h(l), h^l_G + r)),$$

$$\begin{aligned}
& c^2(E(G)) \geq \tau \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \quad 1 - \min(h^{l_1}_G, h^{l_2}_G) + i^{l_1,l_2} \geq \tau \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 - \min(h^{l_1}_G + r, h^{l_2}_G + r) + i^{l_1,l_2} \geq \tau - r \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \quad 1 - \min(\min(h(l_1), h^{l_1}_G + r), \\
& \qquad\quad \min(h(l_2), h^{l_2}_G + r)) + i^{l_1,l_2} \geq \tau - r \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 - \min(h^{l_1}_\delta, h^{l_2}_\delta) + i^{l_1,l_2} \geq \tau - r \\
& \qquad\quad (\text{as } h^l_\delta = \min(h(l), h^l_G + r) \text{ (see below } *1)) \\
& \quad \Rightarrow \quad c^2(E(\delta)) \geq \max(0, \tau - r).
\end{aligned}$$

Let us now prove the property for the contraction. An example is presented in Fig. 25 (b). The sub-tree G is displayed is blue and red and its connectivity degree according to c^2 is 0.6. The connectivity degree c^2 of $\varepsilon_{T(\mu)}(G, 0.2)$ presented in red is $c^2(E(\varepsilon_{T(\mu)}(G, 0.2))) = 1 - \min(h^{l_1}_\varepsilon, h^{l_2}_\varepsilon) + i^{l_1,l_2} = 1 - 0.2 + 0 = 0.8$. We obtain $c^2(E(\varepsilon_{T(\mu)}(G, 0.2))) \geq \min(1, c^2(E(G)) + 0.2)$.

We denote by ε the contraction:

$$\varepsilon_{T(\mu)}(G) = \vee_{l \in \mathcal{L}} P^l(\max(0, h^l_G - r))$$

$$\begin{aligned}
& c^2(E(G)) \geq \tau \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \quad 1 - \min(h^{l_1}_G, h^{l_2}_G) + i^{l_1,l_2} \geq \tau \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 - \min(h^{l_1}_G - r, h^{l_2}_G - r) + i^{l_1,l_2} \geq \tau + r \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad \min(1, 1 - \min(h^{l_1}_G - r, h^{l_2}_G - r) + i^{l_1,l_2}) \\
& \qquad\qquad \geq \min(1, \tau + r) \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 + i^{l_1,l_2} + \min(-i^{l_1,l_2}, -\min(h^{l_1}_G - r, h^{l_2}_G - r)) \\
& \qquad\qquad \geq \min(1, \tau + r) \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 + i^{l_1,l_2} + \min(0, -\min(h^{l_1}_G - r, h^{l_2}_G - r)) \\
& \qquad\qquad \geq \min(1, \tau + r) \\
& \qquad\quad (\text{as } i^{l_1,l_2} \geq 0) \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 + i^{l_1,l_2} - \max(0, \min(h^{l_1}_G - r, h^{l_2}_G - r)) \\
& \qquad\qquad \geq \min(1, \tau + r) \\
& \quad \Rightarrow \quad \forall (l_1, l_2) \in \mathcal{L}^2, \\
& \qquad\quad 1 + i^{l_1,l_2} - \min(\max(0, h^{l_1}_G - r), \max(0, h^{l_2}_G - r)) \\
& \qquad\qquad \geq \min(1, \tau + r).
\end{aligned}$$

In general $h^l_\varepsilon \geq \max(0, h^l_G - r)$ (in the example of Fig. 25(b), we have for instance $h^{l_3}_\varepsilon = 0.6$ and $\max(0, h^{l_3}_G - 0.2) = 0.4$). However if $h^l_\varepsilon > \max(0, h^l_G - r)$, the branch of ε associated with the leaf l does not contain a leaf of ε (since there exists another leaf l' such that $P^l(\max(0, h^l_G - r)) < P^{l'}(\max(0, h^{l'}_G - r))$), and this branch is not associated with a regional maximum of $E(\varepsilon)$. The minimum in $c^2(E(\varepsilon)) = \min_{(l_1,l_2) \in \mathcal{L}^2}(1 - \min(h^{l_1}_\varepsilon, h^{l_2}_\varepsilon) + i^{l_1,l_2})$ is reached for l_1 and l_2 such that $h^l_\varepsilon = \max(0, h^l_G - r)$.

$\forall(l_1,l_2)\in\mathcal{L}^2,$

$$1+i^{l_1,l_2}-\min(\max(0,h_G^{l_1}-r),\max(0,h_G^{l_2}-r))$$
$$\geq \min(1,\tau+r)$$
$$\Rightarrow \min_{(l_1,l_2)\in\mathcal{L}^2}(1+i^{l_1,l_2}-\min(h_\varepsilon^{l_1},h_\varepsilon^{l_2}))\geq\min(1,\tau+r)$$
$$\Rightarrow c^2(E(\varepsilon))\geq\min(1,\tau+r).$$

$*1$: Let us show that $h_\delta^l=\min(h(l),h_S^l+r)$, where S is a subtree of $T(\mu)$, l is a leaf of $T(\mu)$ and $\delta=\delta_{T(\mu)}(S,r)$. Since by definition $\delta=\bigvee_{l'\in\mathcal{L}}P^{l'}(\min(h(l'),h_S^{l'}+r))$, we have $h_\delta^l\geq\min(h(l),h_S^l+r)$. Let us show that for any leaf $l'\in\mathcal{L}$, the height of $P^{l'}(\min(h(l'),h_S^{l'}+r))$ on the branch associated with l is smaller than or equal to $\min(h(l),h_S^l+r)$. This height can be written as $h_{l'}=h(P^{l'}(\min(h(l'),h_S^{l'}+r))\wedge P^l(h(l)))$ and we have $h_{l'}\leq h(P^{l'}(h(l'))\wedge P^l(h(l)))=i^{l,l'}$ and $h_{l'}\leq\min(h(l'),h_S^{l'}+r)$.

If $h_S^{l'}+r<i^{l,l'}$, then branches of S associated with l and l' are equal and $h_S^{l'}=h_S^l$. Therefore $h_{l'}\leq\min(h(l'),h_S^{l'}+r)=h_S^{l'}+r=\min(h(l),h_S^l+r)$ (since $h_S^{l'}+r<i^{l,l'}\leq\min(h(l'),h(l))$).

If $h_S^{l'}+r\geq i^{l,l'}$, then we have also $h_S^l+r\geq i^{l,l'}$ (since the branches of S associated to l and l' are equal below the level $i^{l,l'}$) and $h_{l'}\leq i^{l,l'}\leq\min(h(l),h_S^l+r)$.

Therefore we obtain $h_\delta^l=\min(h(l),h_S^l+r)$. □

Proof of Proposition 17 To simplify the notations, a fuzzy set and its associated tree or sub-tree will be denoted by the same variable in this proof.

We will prove that the set of τ-hyperconnected components of μ ($\mathcal{H}_\tau^2(\mu)$) is isomorphic to the set of 1-hyperconnected components of $\varepsilon_\mu^{1-\tau}(\mu)$ ($\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$), where $\varepsilon_\mu^{1-\tau}(\mu)=\varepsilon_{T(\mu)}(T(\mu),1-\tau)$. Let us first prove that the mappings $\delta_\mu^{1-\tau}$ and $\varepsilon_\mu^{1-\tau}$ can respectively be defined as mappings from $\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$ to $\mathcal{H}_\tau^2(\mu)$ and from $\mathcal{H}_\tau^2(\mu)$ to $\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$. We will then prove that those mappings are bijective.

Let us first show that the contraction of size $1-\tau$ of a fuzzy set in $\mathcal{H}_\tau^2(\mu)$ belongs to $\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$. Let μ_i be a τ-hyperconnected component of μ. According to Proposition 16, $\varepsilon_\mu^{1-\tau}(\mu_i)$ is $\min(1,\tau+(1-\tau))$-hyperconnected. Therefore it belongs to $\mathcal{H}^1$ and we will prove that it is a 1-hyperconnected component of $\varepsilon_\mu^{1-\tau}(\mu)$ (thus maximal). $\varepsilon_\mu^{1-\tau}(S)$ and $\delta_\mu^{1-\tau}(S)$ are increasing with respect to the subtree S (since they are expressed as the supremum over increasing operators of the height of the branches of S) and therefore $\varepsilon_\mu^{1-\tau}(\mu_i)\leq\varepsilon_\mu^{1-\tau}(\mu)$.

Suppose that $\nu\in\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$ be such that:

$$\varepsilon_\mu^{1-\tau}(\mu_i)\leq\nu\leq\varepsilon_\mu^{1-\tau}(\mu)$$
$$\Rightarrow \delta_\mu^{1-\tau}\varepsilon_\mu^{1-\tau}(\mu_i)\leq\delta_\mu^{1-\tau}(\nu)\leq\delta_\mu^{1-\tau}\varepsilon_\mu^{1-\tau}(\mu)=\mu$$
$$\text{(justification 1)}$$
$$\Rightarrow \mu_i\leq\delta_\mu^{1-\tau}(\nu)\leq\mu\quad\text{(justification 2)}$$
$$\Rightarrow \mu_i=\delta_\mu^{1-\tau}(\nu)\quad\text{(justification 3)}$$
$$\Rightarrow \varepsilon_\mu^{1-\tau}(\mu_i)=\nu\quad\text{(justification 4).}$$

The contraction of size $1-\tau$ of a τ-hyperconnected component of μ is thus a 1-hyperconnected component of the contraction of μ.

Let us now show that the dilation of size $1-\tau$ of a 1-hyperconnected component of the contraction of size $1-\tau$ of μ is a τ-hyperconnected component of μ. Let ε_i be a 1-hyperconnected component of $\varepsilon_\mu^{1-\tau}(\mu)$. We want to prove that $\delta_\mu^{1-\tau}(\varepsilon_i)$ is a τ-hyperconnected component of μ. According to Proposition 16, $\delta_\mu^{1-\tau}(\varepsilon_i)$ is $\max(0,1-(1-\tau))$-hyperconnected and therefore it belongs to $\mathcal{H}_\tau^2$. We still have to prove that it is a τ-hyperconnected component (i.e. that it is maximal). Suppose that $\nu\in\mathcal{H}_\tau^2(\mu)$ is such that $\delta_\mu^{1-\tau}(\varepsilon_i)\leq\nu\leq\mu$. As the contraction is increasing we have:

$$\varepsilon_\mu^{1-\tau}\delta_\mu^{1-\tau}(\varepsilon_i)\leq\varepsilon_\mu^{1-\tau}(\nu)\leq\varepsilon_\mu^{1-\tau}(\mu)$$
$$\Rightarrow \varepsilon_i\leq\varepsilon_\mu^{1-\tau}(\nu)\leq\varepsilon_\mu^{1-\tau}(\mu)\quad\text{(justification 4)}$$
$$\Rightarrow \varepsilon_i=\varepsilon_\mu^{1-\tau}(\nu)\quad\text{(as } \varepsilon_i \text{ is a 1-hyperconnected component of } \varepsilon_\mu^{1-\tau}(\mu))$$
$$\Rightarrow \delta_\mu^{1-\tau}(\varepsilon_i)=\delta_\mu^{1-\tau}\varepsilon_\mu^{1-\tau}(\nu)$$
$$\Rightarrow \delta_\mu^{1-\tau}(\varepsilon_i)=\nu\quad\text{(justification 2).}$$

Therefore we obtain $\delta_\mu^{1-\tau}(\varepsilon_i)\in\mathcal{H}_\tau^2(\mu)$.

We consider now the mappings $\delta_\mu^{1-\tau}:\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))\to\mathcal{H}_\tau^2(\mu)$ and $\varepsilon_\mu^{1-\tau}:\mathcal{H}_\tau^2(\mu)\to\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$. Those are bijective and inverse of each other. Indeed if $\mu_i\in\mathcal{H}_\tau^2(\mu)$, we can show that $\delta_\mu^{1-\tau}\varepsilon_\mu^{1-\tau}(\mu_i)=\mu_i$ (justification 2). Conversely if $\varepsilon_i\in\mathcal{H}^1(\varepsilon_\mu^{1-\tau}(\mu))$, we have $\varepsilon_\mu^{1-\tau}\delta_\mu^{1-\tau}(\varepsilon_i)=\varepsilon_i$ (justification 4).

Justification 1: The mapping $\delta_\mu^\tau\varepsilon_\mu^\tau$ is extensive. In fact if we rewrite the fuzzy set a, its contraction and the dilation of it as:

$$a=\bigvee_{l\in\mathcal{L}}P_l(h_a^l),$$
$$\varepsilon_\mu^\tau(a)=\bigvee_{l\in\mathcal{L}}P_l(h_\varepsilon^l),$$
$$\delta_\mu^\tau\varepsilon_\mu^\tau(a)=\bigvee_{l\in\mathcal{L}}P_l(\min(h_l,h_\varepsilon^l+\tau)),$$

we have the following inequalities:

$$h_\varepsilon^l\geq\max(0,h_a^l-\tau),$$

$h^l_\delta = \min(h_l, h^l_\varepsilon + \tau)$.

Therefore we obtain:

$$h^l_\delta \geq \min(h_l, \max(0, h^l_a - \tau) + \tau)$$
$$\Rightarrow \quad h^l_\delta \geq \min(h_l, \max(\tau, h^l_a)) \geq h^l_a,$$

since $h_l \geq h^l_a$. The mapping $\delta^\tau_\mu \varepsilon^\tau_\mu$ is therefore extensive. Note that this property is not always satisfied by the mapping $\varepsilon^\tau_\mu \delta^\tau_\mu$.

Justification 2: According to justification 1, the mapping $\delta^{1-\tau}_\mu \varepsilon^{1-\tau}_\mu$ is extensive. We have thus $\mu_i \leq \delta^{1-\tau}_\mu \cdot \varepsilon^{1-\tau}_\mu(\mu_i) \leq \mu$. Since $\delta^{1-\tau}_\mu \varepsilon^{1-\tau}_\mu(\mu_i)$ is τ-hyperconnected and μ_i is a τ-hyperconnected component of μ, we have $\delta^{1-\tau}_\mu \varepsilon^{1-\tau}_\mu(\mu_i) = \mu_i$.

Justification 3: μ_i is a τ-hyperconnected component of μ and by definition $\nexists \upsilon \in \mathcal{H}^2_\tau, \mu_i < \upsilon \leq \mu$. Since $\delta^{1-\tau}_\mu(\nu) \in \mathcal{H}^2_\tau$, we have $\mu_i = \delta^{1-\tau}_\mu(\nu)$.

Justification 4: if a fuzzy set ν is such that $\nu \leq \varepsilon^{1-\tau}_\mu(\mu)$, we have $\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu) = \nu$. In fact if we rewrite ν, $\delta^{1-\tau}_\mu(\nu)$ and $\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu)$ as:

$$\nu = \bigvee_{l \in \mathcal{L}} P^l(h^l_\nu),$$
$$\delta^{1-\tau}_\mu(\nu) = \bigvee_{l \in \mathcal{L}} P^l(h^l_\delta),$$
$$\varepsilon^{1-\tau}_\mu \delta^\tau_\mu(\nu) = \bigvee_{l \in \mathcal{L}} P^l(h^l_\varepsilon),$$

we have the following inequalities between the heights of the branches:

$$h^l_\delta = \min(h_l, h^l_\nu + 1 - \tau),$$
$$h^l_\varepsilon \geq \max(0, h^l_\delta - 1 + \tau).$$[8]

We can conclude that:

$$h^l_\varepsilon \geq \min(\max(0, h_l - 1 + \tau), h^l_\nu).$$

As $\nu \leq \varepsilon^{1-\tau}_\mu(\nu)$, we have $h^l_\nu \leq \max(0, h_l - 1 + \tau)$ and we obtain: $h^l_\varepsilon \geq h^l_\nu$. The mapping $\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu$ is thus in that case extensive: $\nu \leq \varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu)$. In addition since the mapping $\delta^{1-\tau}_\mu$ is increasing, we have $\delta^{1-\tau}_\mu(\nu) \leq \delta^{1-\tau}_\mu \varepsilon^{1-\tau}_\mu(\mu) \leq \mu$ and since $\varepsilon^{1-\tau}_\mu$ is increasing: $\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu) \leq \varepsilon^{1-\tau}_\mu(\mu)$.

As ν is a 1-hyperconnected component of $\varepsilon^{1-\tau}_\mu(\mu)$ and as $\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu)$ is 1-hyperconnected, we obtain:

$$\varepsilon^{1-\tau}_\mu \delta^{1-\tau}_\mu(\nu) = \nu. \qquad \square$$

Proof of Proposition 18 Let $\mu_C : \mathcal{F} \to [0, 1]$ be a Lipschitz mapping. Let us show that the mapping associating the fuzzy set μ with $\xi_{\mu_C}(\mu) = \bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \mu \text{ and } \max_{x\in X} \nu(x) \leq \mu_C(\nu)\}$ is Lipschitz.

Let μ_1 and μ_2 be two fuzzy sets and $\eta = \sup_{x\in X} |\mu_1(x) - \mu_2(x)|$. $\forall x \in X, \exists \nu_1 \in \mathcal{H}^2_\tau$ such that: $\xi_{\mu_C}(\mu_1)(x) = \nu_1(x)$, $\nu_1 \leq \mu_1$ and $\max_{x\in X} \nu_1(x) \leq \mu_C(\nu_1)$. Let ν_2 be a fuzzy set defined as $\nu_2 = \max(0, \nu_1 - \eta)$. We have $\nu_2 \in \mathcal{H}^2_\tau$, $\nu_2 \leq \mu_2$ and $\sup_{x\in X} \nu_2(x) \leq \max(0\,,\ \mu_C(\nu_1) - \eta)$. Since μ_C is Lipschitz, $|\mu_C(\nu_2) - \mu_C(\nu_1)| \leq \sup_{x\in X} |\nu_2(x) - \nu_1(x)|$. In addition we have $\sup_{x\in X} |\nu_2(x) - \nu_1(x)| \leq \eta$, thus $\mu_C(\nu_1) - \eta \leq \mu_C(\nu_2)$. We derive $\sup_{x\in X} \nu_2(x) \leq \mu_C(\nu_2)$ and so:

$$\xi_{\mu_C}(\mu_2)(x) \geq \xi_{\mu_C}(\mu_1)(x) - \eta.$$

In the same way we derive:

$$\xi_{\mu_C}(\mu_1)(x) \geq \xi_{\mu_C}(\mu_2)(x) - \eta.$$

Therefore we have:

$$\forall x \in X, \quad |\xi_{\mu_C}(\mu_2)(x) - \xi_{\mu_C}(\mu_1)(x)| \leq \eta. \qquad \square$$

Proof of Proposition 19 We want to prove that the filter defined by (7) can be computed over the τ-hyperconnected components of μ if the criterion μ_C is increasing. According to (7), $\xi_{\mu_C}(\mu)$ is defined as:

$$\xi_{\mu_C}(\mu) = \bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \mu \text{ and } \max_{x\in X} \nu(x) \leq \mu_C(\nu)\}.$$

Since $\forall \nu \in \mathcal{H}^2_\tau$ such that $\nu \leq \mu$, the property $\nu \leq \mu_i$ with $\mu_i \in \mathcal{H}^2_\tau(\mu)$ always holds, we can rewrite $\xi_{\mu_C}(\mu)$ as:

$$\xi_{\mu_C}(\mu) = \bigvee_{\mu_i \in \mathcal{H}^2_\tau(\mu)} \bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \mu_i \text{ and } \max_{x\in X} \nu(x) \leq \mu_C(\nu)\}.$$

Sorting the fuzzy sets according to their height m, we obtain: $\xi_{\mu_C}(\mu) = \bigvee_{\mu_i \in \mathcal{H}^2_\tau(\mu)} \bigvee_{m\in[0,1]} \bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \mu_i \text{ and } \max_{x\in X} \nu(x) = m \text{ and } m \leq \mu_C(\nu)\}$.

For $m \in [0, \max_{x\in X} \mu_i(x)]$,[9] the following equality holds:

$$\bigvee\{\nu \in \mathcal{H}^2_\tau \mid \nu \leq \mu_i \text{ and } \max_{x\in X} \nu(x) = m\} = \min(\mu_i, m).$$

Indeed $\min(\mu_i, m) \in \mathcal{H}^2_\tau$ (since $\mu_i \in \mathcal{H}^2_\tau$),[10] $\min(\mu_i, m) \leq \mu_i$ and $\max_{x\in X} \min(\mu_i, m)(x) = m$.

[8] In Sect. 3.4, we have shown that this inequality between the heights of the branches is satisfied by the contraction and dilation. Moreover we have an equality of the heights of the branches in the dilation case.

[9] For $m > \max_{x\in X} \mu_i(x)$, there is no ν such that $\nu \leq \mu_i$ and $\max_{x\in X} \nu(x) = m$.

[10] Let $\mu \in \mathcal{H}^2_\tau$, we have $\forall (x, y) \in X^2, 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, y) \geq \tau$. Let $\mu' = \min(\mu, m)$ and $(x, y) \in X^2$, we have

As μ_C is increasing, we obtain:

$$\xi_{\mu_C}(\mu) = \bigvee_{\substack{\mu_i \in \\ \mathcal{H}_\tau^2(\mu)}} \bigvee_{m\in[0,h_i]} \{\min(\mu_i, m) \mid m \le \mu_C(\min(\mu_i, m))\},$$

where $h_i = \max_{x\in X} \mu_i(x)$. □

Proof of Proposition 20 Let $\underline{A}$ be a fuzzy set and α be its height: $\alpha = \max_{x\in X} \underline{A}(x)$. We will show that $\xi^1_{\underline{A}}(\overline{A}) = \bigvee\{\nu \in \mathcal{H}_\tau^2(\overline{A}) \mid \underline{A} \le \nu\}$ is $\alpha - 1 + \tau$-hyperconnected.

All elements ν in $\{\nu \in \mathcal{H}_\tau^2(\overline{A}) \mid \underline{A} \le \nu\}$ also belong to $\mathcal{H}^2_{\alpha-1+\tau}$ (as they belong to $\mathcal{H}_\tau^2$ and as $\alpha - 1 \le 0$). We will show that all those elements overlap according to the overlap mapping $\perp^2_{\alpha-1+\tau}$.

Let ν be a fuzzy set such that $\nu \in \mathcal{H}_\tau^2(\overline{A})$ and $\underline{A} \le \nu$. Let x_m be a point for which ν reaches its global maximum, $h = \nu(x_m)$ and x_a be a point for which $\underline{A}$ reaches its global maximum (we have $\alpha \le h$ and $\nu(x_a) \le \nu(x_m)$).

$$\begin{aligned}
&\nu \in \mathcal{H}_\tau^2\\
&\Rightarrow \quad 1 - \min(\nu(x_m), \nu(x_a)) + c^1_\nu(x_m, x_a) \ge \tau\\
&\Rightarrow \quad 1 - \tau + c^1_\nu(x_m, x_a) \ge \nu(x_a) \ge \underline{A}(x_a) = \alpha\\
&\Rightarrow \quad c^1_\nu(x_m, x_a) \ge \alpha - 1 + \tau \ge h - 1 + (\alpha - 1 + \tau),
\end{aligned}$$

as $h - 1 \le 0$. According to Proposition 3, for all ν satisfying the conditions we have $\eta^1_{\delta^h_{x_m}}(\nu)(x_a) \ge h - 1 + (\alpha - 1 + \tau)$. The elements of $\{\nu \in \mathcal{H}_\tau^2(\overline{A}) \mid \underline{A} \le \nu\}$ overlap according to $\perp^2_{\alpha-1+\tau}$. The supremum over this set is therefore $\alpha - 1 + \tau$-hyperconnected. □

Proof of Proposition 21 Let us show that the mapping associating $\underline{A}$ with $\xi^2_{\underline{A}}(\overline{A}) = \bigvee\{\nu \in \mathcal{H}_\tau^2 \mid \nu \le \overline{A}$ and $\max_{x\in X} \nu(x) \le \mu_\le(\underline{A}, \nu)\}$ is Lipschitz, as well as the mapping associating $\overline{A}$ with $\xi^2_{\underline{A}}(\overline{A})$. For the second one, according to Proposition 18, if the mapping associating ν with $\mu_\le(\underline{A}, \nu)$ is Lipschitz, then the mapping associating $\overline{A}$ with $\xi^2_{\underline{A}}(\overline{A})$ is Lipschitz. As $\mu_\le(\underline{A}, \nu) = \min_{x\in X} \min(1, 1 - \underline{A}(x), \nu(x))$ is Lipschitz, the mapping associating $\overline{A}$ with $\xi^2_{\underline{A}}(\overline{A})$ is Lipschitz.

Let $\underline{A_1}$ and $\underline{A_2}$ be two fuzzy sets and η be such that $\eta = \sup_{x\in X} |\underline{A_1}(x) - \underline{A_2}(x)|$. Let $\nu_1 \in \mathcal{H}_\tau^2$ such that $\xi^2_{\underline{A_1}}(\overline{A})(x) = \nu_1(x)$, $\nu_1 \le \overline{A}$ and $\max_{x\in X} \nu_1(x) \le \mu_\le(\underline{A_1}, \nu_1)$. We denote by h_1 the height of ν_1: $\max_{x\in X} \nu_1(x)$. Let ν_2 be a fuzzy set defined as $\nu_2 = \min(\nu_1, \max(0, h_1 - \eta))$. We have $\nu_2 \in \mathcal{H}_\tau^2$ and $\nu_2 \le \overline{A}$. In addition:

$$\begin{aligned}
&\max_{x\in X} \nu_1(x) \le \mu_\le(\underline{A_1}, \nu_1)\\
&\Rightarrow \quad \max_{x\in X} \nu_1(x) \le \min_{x\in X} \min(1, 1 - \underline{A_1}(x) + \nu_1(x))\\
&\Rightarrow \quad \max(0, \max_{x\in X} \nu_1(x) - \eta)\\
&\qquad \le \max\Big(0, \min_{x\in X} \min(1, 1 - \underline{A_1}(x) + \nu_1(x)) - \eta\Big)\\
&\Rightarrow \quad \max_{x\in X} \nu_2(x)\\
&\qquad \le \max\Big(0, \min_{x\in X} \min(1, 1 - \underline{A_1}(x) + \nu_1(x) - \eta)\Big)\\
&\Rightarrow \quad \max_{x\in X} \nu_2(x)\\
&\qquad \le \max\Big(0, \min_{x\in X} \min(1, 1 - \underline{A_2}(x) + \nu_1(x))\Big).
\end{aligned}$$

If $x \in X$ is such that $\nu_1(x) > h_1 - \eta$, we have $\nu_2(x) = h_2$. Since the inequality $h_2 \le 1 - \underline{A_2}(x) + h_2$ is always satisfied, we obtain $\nu_2(x) \le 1 - \underline{A_2}(x) + h_2$. Otherwise we have $\nu_2(x) = \nu_1(x)$ and the inequality $h_2 \le \min(1, 1 - \underline{A_2}(x) + \nu_2(x))$ is fulfilled. We obtain $\max_{x\in X} \nu_2(x) \le \min_{x\in X} \min(1, 1 - \underline{A_2}(x) + \nu_2(x))$ and therefore $\xi^2_{\underline{A_2}}(\overline{A})(x) \ge \xi^2_{\underline{A_1}}(\overline{A})(x) - \eta$. In the same way we can obtain $\xi^2_{\underline{A_1}}(\overline{A})(x) \ge \xi^2_{\underline{A_2}}(\overline{A})(x) - \eta$. The mapping associating $\underline{A}$ with $\xi^2_{\underline{A}}(\overline{A})$ is therefore Lipschitz. □

Proof of Proposition 22 Let $\underline{A}$ be a fuzzy set. We denote by α its height: $\alpha = \max_{x\in X} \underline{A}(x)$. We will prove that $\xi^2_{\underline{A}}(\overline{A}) = \bigvee\{\nu \in \mathcal{H}_\tau^2 \mid \nu \le \overline{A}$ and $\max_{x\in X} \nu(x) \le \mu_\le(\underline{A}, \nu)\}$ is $\max(0, \alpha - 1 + \tau)$-hyperconnected.

To this aim let us show that all elements $\nu \in \mathcal{H}_\tau^2$ that fulfill the criteria $\nu \le \overline{A}$ and $\max_{x\in X} \nu(x) \le \mu_\le(\underline{A}, \nu)$ overlap according to $\perp^2_{\alpha-1+\tau}$.

Let ν be a fuzzy set in $\mathcal{H}_\tau^2$ such that $\max_{x\in X} \nu(x) \le \mu_\le(\underline{A}, \nu)$ and $\nu \le \overline{A}$. Let x_a be a point such that $\underline{A}(x_a) = \alpha$, x_m a point such that $\nu(x_m) = h$ where $h = \max_{x\in X} \nu(x)$. We have:

$$\begin{aligned}
&\max_{x\in X} \nu(x) \le \mu_\le(\underline{A}, \nu)\\
&\Rightarrow \quad h \le \min_{x\in X} \min(1, 1 - \underline{A}(x) + \nu(x))\\
&\Rightarrow \quad h \le 1 - \alpha + \nu(x_a).
\end{aligned}$$

$c^1_{\mu'}(x, y) = \min(m, c^1_\mu(x, y))$ and thus $1 - \min(\mu'(x), \mu'(y)) + c^1_{\mu'}(x, y) = 1 - \min(m, \min(\mu(x), \mu(y))) + \min(m, c^1_\mu(x, y))$. If $c^1_\mu(x, y) \ge m$, we have $1 - \min(\mu'(x), \mu'(y)) + c^1_{\mu'}(x, y) = 1 - m + m \ge \tau$ (as $\min(\mu(x), \mu(y)) \ge c^1_\mu(x, y)$). If $c^1_\mu(x, y) \le m$ and $\min(\mu(x), \mu(y)) \ge m$, we have $1 - \min(\mu'(x), \mu'(y)) + c^1_{\mu'}(x, y) = 1 - m + c^1_{\mu'}(x, y) \ge 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, y) \ge \tau$. Finally if $c^1_\mu(x, y) \le m$ and $\min(\mu(x), \mu(y)) \le m$, we obtain $1 - \min(\mu'(x), \mu'(y)) + c^1_{\mu'}(x, y) = 1 - \min(\mu(x), \mu(y)) + c^1_\mu(x, y) \ge \tau$. Therefore we obtain $\forall (x, y) \in X^2$, $1 - \min(\mu'(x), \mu'(y)) + c^1_{\mu'}(x, y) \ge \tau$ and thus $\min(\mu, m) \in \mathcal{H}_\tau^2$.

In addition ν is τ-hyperconnected so:

$$1-\min(\nu(x_a),\nu(x_m))+c^1_\nu(x_a,x_m)\geq\tau$$
$$\Rightarrow\quad 1-\tau+c^1_\nu(x_a,x_m)\geq\nu(x_a).$$

We obtain:

$$h\leq 1-\alpha+1-\tau+c^1_\nu(x_a,x_m)$$
$$\Rightarrow\quad h-1+\alpha-1+\tau\leq c^1_\nu(x_a,x_m).$$

Since according to Proposition 3, $c^1_\nu(x_a,x_m)=\eta^1_{\delta^h_{x_m}}(\nu)(x_a)$, all the fuzzy sets $\eta^1_{\delta^h_{x_m}}(\nu)$ intersect in x_a at least for all levels below $h-1+\alpha-1+\tau$. Therefore the set of elements ν (that satisfy the criteria of the filter) overlap according to $\perp^2_{\alpha-1+\tau}$. Since all those fuzzy sets are $(\alpha-1+\tau)$-hyperconnected, $\xi^2_{\underline{A}}(\overline{A})$ is $\max(0,\alpha-1+\tau)$-hyperconnected. □

Proof of Proposition 23 Let us finally prove that the mapping that associates $\overline{A}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$ is Lipschitz, as well as the mapping that associates $\mu_{S_{\min}}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$.

According to Proposition 18, if $\mu_C(\nu)$ is Lipschitz, then the mapping associating $\overline{A}$ with $\xi_{\mu_C}(\overline{A})$ is Lipschitz. We will prove that the criterion $\mu_C(\nu)=\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu)(s),\ \mu_{S_{\min}}(s))$ (with $\mu_S(\nu)(s)=\sup_{S(\nu_\alpha)\geq s}\alpha$) is Lipschitz. Let ν_1 and ν_2 be two fuzzy sets such that $\max_{x\in X}|\nu_1(x)-\nu_2(x)|=\eta$. Since $\forall\alpha\in[0,1]$, $\nu_1(x)\geq\alpha\ \Rightarrow\ \nu_2(x)\geq\max(0,\alpha-\eta)$, we can derive the following inclusions: $(\nu_1)_\alpha\subseteq(\nu_2)_{\max(0,\alpha-\eta)}$ and $(\nu_2)_\alpha\subseteq(\nu_1)_{\max(0,\alpha-\eta)}$. We thus have $S(\nu_{1\alpha_1(s)})\geq s$ and $S(\nu_{2\alpha_2(s)})\geq s$, where $\alpha_1(s)=\mu_S(\nu_1)(s)$ and $\alpha_2(s)=\mu_S(\nu_2)(s)$. According to the inclusion property given above, we can deduce that $S(\nu_{2\max(0,\alpha_1(s)-\eta)})\geq s$ and that $S(\nu_{1\max(0,\alpha_2(s)-\eta)})\geq s$. Therefore we obtain $\alpha_2(s)\geq\max(0,\alpha_1(s)-\eta)$ and $\alpha_1(s)\geq\max(0,\alpha_2(s)-\eta)$.

Thus:

$$\forall s\in\mathbb{R}^+,\quad \alpha_1(s)-\eta\leq\alpha_2(s)\leq\alpha_1(s)+\eta$$
$$\Rightarrow\quad\forall s\in\mathbb{R}^+,\quad \min(\mu_S(\nu_1)(s),\mu_{S_{\min}}(s))-\eta$$
$$\leq\min(\mu_S(\nu_2)(s),\mu_{S_{\min}}(s))$$
$$\leq\min(\mu_S(\nu_1)(s),\mu_{S_{\min}}(s))+\eta,$$
$$\Rightarrow\quad\mu_C(\nu_1)-\eta\leq\mu_C(\nu_2)\leq\mu_C(\nu_1)+\eta.$$

We can conclude that the mapping that associates $\overline{A}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$ is Lipschitz.

Let us now prove that the mapping that associates $\mu_{S_{\min}}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$ is Lipschitz. Let $\mu_{S^1_{\min}}$ and $\mu_{S^2_{\min}}$ such that $\sup_{s\in\mathbb{R}^{*+}}|\mu_{S^1_{\min}}(s)-\mu_{S^2_{\min}}(s)|=\eta$. Let $x\in X$ and ν_1 such that $\xi^2_{\mu_{S^1_{\min}}}(\overline{A})(x)=\nu_1(x)$, $\nu_1\in\mathcal{H}^2_\tau$, $\nu_1\leq\overline{A}$ and $\max_{y\in X}\nu_1(y)\leq\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_1)(s),\mu_{S^1_{\min}}(s))$. Let ν_2 a fuzzy set such that $\forall y\in X,\ \nu_2(y)=\max(0,\nu_1(y)-\eta)$. We have $\nu_2\leq\overline{A}$ and $\nu_2\in\mathcal{H}^2_\tau$. Moreover let $\alpha_1(s)=\mu_S(\nu_1)(s)$ and $\alpha_2(s)=\mu_S(\nu_2)(s)$. Thus we have $S(\nu_{1\alpha_1(s)})\geq s$ and $S(\nu_{2\alpha_2(s)})\geq s$. Since $\forall\alpha\in[0,1]$, $\nu_{1\alpha}\subseteq\nu_{2\max(0,\alpha-\eta)}$, we obtain $S(\nu_{2\max(0,\alpha_1(s)-\eta)})\geq s$ and therefore:

$$\forall s\in\mathbb{R}^+,\quad\mu_S(\nu_1)(s)\leq\mu_S(\nu_2)(s)+\eta$$
$$\Rightarrow\quad\forall s\in\mathbb{R}^+,$$
$$\min(\mu_S(\nu_1)(s),\mu_{S^1_{\min}}(s))$$
$$\leq\min(\mu_S(\nu_2)(s)+\eta,\mu_{S^2_{\min}}(s)+\eta)$$
$$(\text{since }\mu_{S^1_{\min}}(s)\leq\mu_{S^2_{\min}}(s)+\eta),$$
$$\Rightarrow\quad\forall s\in\mathbb{R}^+,$$
$$\min(\mu_S(\nu_1)(s),\mu_{S^1_{\min}}(s))$$
$$\leq\min(\mu_S(\nu_2)(s),\mu_{S^2_{\min}}(s))+\eta,$$
$$\Rightarrow\quad\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_1)(s),\mu_{S^1_{\min}}(s))$$
$$\leq\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_2)(s),\mu_{S^2_{\min}}(s))+\eta,$$
$$\Rightarrow\quad\max_{y\in X}\nu_1(y)\leq\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_2)(s),\mu_{S^2_{\min}}(s))+\eta,$$
$$\Rightarrow\quad\max\Big(0,\max_{y\in X}\nu_1(y)-\eta\Big)$$
$$\leq\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_2)(s),\mu_{S^2_{\min}}(s)),$$
$$\Rightarrow\quad\max_{y\in X}\nu_2(y)\leq\max_{s\in\mathbb{R}^+}\min(\mu_S(\nu_2)(s),\mu_{S^2_{\min}}(s)).$$

Therefore we obtain $\xi^2_{\mu_{S^2_{\min}}}(\overline{A})(x)\geq\xi^2_{\mu_{S^1_{\min}}}(\overline{A})(x)-\eta$. In the same way we derive $\xi^2_{\mu_{S^2_{\min}}}(\overline{A})(x)\leq\xi^2_{\mu_{S^1_{\min}}}(\overline{A})(x)+\eta$. The mapping that associates $\mu_{S_{\min}}$ to $\xi^2_{\mu_{S_{\min}}}(\overline{A})$ is thus Lipschitz. □

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