Galois descent for vector spaces in half a page

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Let L/K be a finite Galois extension, of degree n and Galois group $G = \{\sigma_1, \dots, \sigma_n\}$. Let V be a finite dimensional L-vector space equipped with a *skew-linear* action of G, which means $\sigma(lv) = \sigma(l)\sigma(v)$ for all $\sigma \in G$, $l \in L$, $v \in V$. Let $V^G \subseteq V$ be the K-subspace of vectors fixed by G.

Theorem 1. *Under these hypotheses, the natural L-linear map*

$$V^G \otimes_K L \longrightarrow V$$

compatible with the action of G, is an isomorphism*.*

Proof. By linear independence of characters and equality of dimension, the map

$$\begin{array}{ccc} L \otimes_K L & \longrightarrow & L^n \\ x \otimes y & \longmapsto & (x\sigma_1(y), \dots, x\sigma_n(y)) \end{array}$$

is an isomorphism. Thus we can find $x_1, \dots, x_n, y_1, \dots, y_n \in L$ such that

$$\sum_i x_i y_i = 1, \qquad \sum_i x_i \sigma(y_i) = 0 \text{ for } \sigma \neq \mathrm{id}\,.$$

For any $v \in V$ we then have

$$v = \sum_{\sigma \in G} \sum_i x_i \sigma(y_i)\sigma(v) = \sum_i x_i \sum_{\sigma \in G} \sigma(y_i v).$$

Since $\sum_{\sigma \in G} \sigma(y_i v) \in V^G$, this shows V is generated by V^G over L.

From this we conclude. Pick $u_1, \dots, u_N \in V^G$ that form a basis of V over L. Let $U \subseteq V^G$ be their K-linear span, so $u_1, \dots, u_N$ also form a basis of U over K, and we have a natural L-linear G-compatible isomorphism

$$U \otimes_K L \xrightarrow{\sim} V.$$

But then we get $U = (U \otimes_K L)^G = V^G$, which finishes the proof. $\square$