MASTER PARISIEN DE RECHERCHE EN INFORMATIQUE

COURS 2-16 "MODÉLISATION PAR AUTOMATES FINIS"

7 December 2012 — Exam (1)

Books and computers forbidden — Lecture and personal notes allowed

The length of the text and the number of exercises should not frighten you nor be understood as part of the difficulty of the exam but as the opportunity given to a student who is 'stuck' on a question to try to solve another one.

I. Exercises on rational relations

1 .— It is known that it is not true in general that the image of a recognisable set is recognisable. In this exercise, it is asked to show that what is false 'in the general case' may hold in some particular cases.

Let $h\colon A_1^* \times A_2^* \to B_1^* \times B_2^*$ be a monoid morphism for which there exist two morphisms

$$h_1 : A_1^* \to B_1^* \quad \text{and} \quad h_2 : A_2^* \to B_2^*$$

such that $h(x, y) = (h_1(x), h_2(y))$.

(i) Check that such an h is indeed a morphism.

(ii) Give an example of a morphism from $A_1^* \times A_2^*$ into $B_1^* \times B_2^*$ which is not of this form.

(iii) Show that if $R \subseteq A_1^* \times A_2^*$ is recognisable, then $h(R)$ is recognisable in $B_1^* \times B_2^*$. [Hint: one could consider an h which is the composition of two morphisms, each one leaving invariant one component.]

2 .— Let us consider the following two predicates

$$S(x, y) \equiv x \geq 2y, \quad I(x, y) \equiv y \leq x \leq 2y \ .$$

Show that there exist non-synchronous ternary rational relations such that every projection on any strict subset of components is synchronous. Hint: consider the relation

$$\{(a^m, a^n, a^p) \mid S(m, n) \text{ and } S(p, n)\} \cup \{(a^m, a^n, a^p) \mid I(m, n) \text{ and } I(p, n)\}$$

and the following four letter alphabet

$$u = (a, a, a), \quad v = (a, \#, a), \quad w = (a, \#, \#) \quad \text{and} \quad z = (\#, \#, a).$$

II. Problem on the growth function of rational languages

In this problem, the function $\mathsf{g}_L(n)$ which associates with every integer n the number of words of length n in a language L is studied:

$$\forall n \in \mathbb{N} \qquad \mathsf{g}_L(n) \ = \ \|L \cap A^n\| \ .$$

If $\mathcal{A} = \langle\, I, E, T \,\rangle$ is a *Boolean* automaton on A^* (of dimension Q), let

$$\mathcal{A}^{\#} = \langle\, I, F, T \,\rangle$$

be the $\mathbb{N}$-automaton on z^* with the same dimension and such that, for every p and q in Q,

$$F_{p,q} = m_{p,q}\, z \ \ ,$$

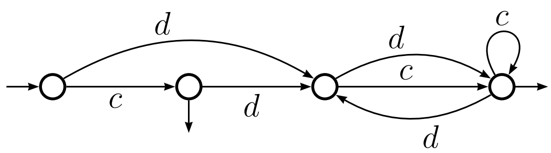
where $m_{p,q}$ is the *number* of transitions from p to q in $\mathcal{A}$ (assuming that every transition in $\mathcal{A}$ is labelled by one letter).

1 .— Let $\mathcal{A}$ be a *deterministic* automaton over A which accepts the language L of A^*. Show that the behaviour of $\mathcal{A}^{\#}$ is the generating series

$$|\mathcal{A}^{\#}| = \mathsf{G}_L(z) = \sum_{n\in\mathbb{N}} \mathsf{g}_L(n)\ z^n \ \ .$$

Is the hypothesis that $\mathcal{A}$ be deterministic necessary?

Let now $\mathcal{B}$ be the automaton of the following figure and K the language it accepts.



2 .— Verify that

$$K = (c + d\,c + d\,d)^* \setminus \{c\,c\,(c+d)^* \cup 1_{B^*}\} \ \ .$$

3 .— Give $\mathcal{B}^{\#}$ and the corresponding representation for $|\mathcal{B}^{\#}|$.

4 .— Compute a reduced representation for $|\mathcal{B}^{\#}|$.
(Every step of the computation will be detailed.)

5 .— Compute $\mathsf{g}_K(n)$.

We come back now to the general case. The deterministic automaton $\mathcal{A}$ of dimension Q and $L = L(\mathcal{A})$ are supposed to be fixed and, for easy writing, we denote by $\mathsf{g}(n)$ the number of words of length n in L. For every p in Q, we write

$$L_p = \{f \in A^* \mid p \xrightarrow[\mathcal{A}]{f} t, t \in T\}$$

and $\mathsf{g}_p(n)$ the number of words of length n in L_p [in particular, $L = L_i$ and $\mathsf{g}(n) = \mathsf{g}_i(n)$ if i is the initial state of $\mathcal{A}$].

6 .— Show that for every $n > 0$ and every p in Q, $\mathsf{g}_p(n)$ is a linear combination, with non negative integer coefficients, of the $\mathsf{g}_q(n-1)$, with q in Q.

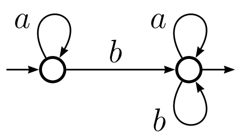
7 .— Deduce that the function $\mathsf{g}(n)$ satisfies a linear recurrence relation of the form:

$$\mathsf{g}(n+k) = z_1\,\mathsf{g}(n+k-1) + z_2\,\mathsf{g}(n+k-2) + \cdots + z_k\,\mathsf{g}(n) \quad ,$$

where k is the number of states of $\mathcal{A}$ and where the z_j are (relative) integers.

8 .— Explain how to compute the initial conditions, that is, : $\mathsf{g}(0)$, $\mathsf{g}(1)$,..., $\mathsf{g}(k-1)$.

9 .— Apply the method and give the linear recurrence relation in the case of the automaton $\mathcal{C}$ below.



III. Finite and infinite components of a relation

If τ is a relation from A^* to M, we define τ_{f} and τ_∞ to be the *finite* and *infinite* 'parts' of τ respectively as follows:

if $\|f\tau\|$ is finite, then $f\tau_{\mathsf{f}} = f\tau$ and $f\tau_\infty = \emptyset$, otherwise $f\tau_{\mathsf{f}} = \emptyset$ and $f\tau_\infty = f\tau$.

1 .— Show that if τ is rational, then τ_{f} and τ_∞ are rational and effectively computable from τ .