Lecture V

Transducers (2)

Realisation by representations

In this lecture, we start again the study of transducers, this time by means of their *realisation by representations*. To that end, one term of the direct product plays a particular role and this yields an almost new computation model which proves to be more natural and apt to many specialisations.

After a new presentation of the Composition Theorem in this other framework, the remaining of the lecture presents such specialisations. We first present the *rational uniformisation* property which is obtained by the construction of the Schützenberger immersions applied to the 'real-time' transducers model. We then sketch the properties of *functional rational relations*.

Contents

1 Real-time transducers and representations

We define a new model of transducers, which leads then naturally to a matrix representation of automata that the realise rational relations.

1.1 Definitions

The definition of real-time transducers we have in mind requires first a slight shift of Definition IV.1 of transducers towards the one of weighted automata, with the transformation of the notion of *initial* and *final states* into the one of *initial* and *final functions*. In a Boolean automaton $\mathcal{A} = \langle A, Q, I, E, T \rangle$, the subsets $I \subseteq Q$ and $T \subseteq Q$ are transformed into the functions $I\colon Q \to \mathfrak{P}(A^*)$ and $T\colon Q \to \mathfrak{P}(A^*)$ definied by:

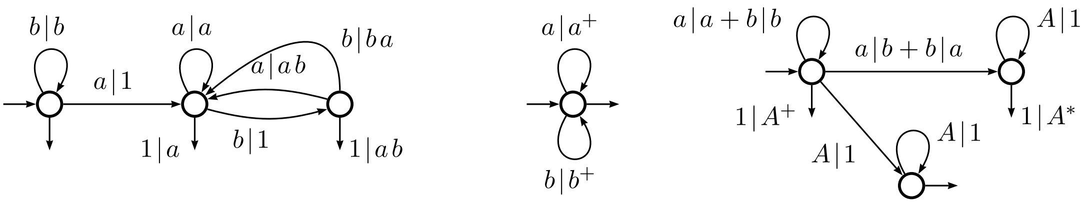
$$I(q) = \begin{cases} 1_{A^*} & \text{if } q \text{ is an initial state} \\ \emptyset & \text{otherwise}, \end{cases} \qquad T(p) = \begin{cases} 1_{A^*} & \text{if } p \text{ is a final state} \\ \emptyset & \text{otherwise}. \end{cases}$$

The definition of the label of a computation is changed accordingly so that the language accepted by the automaton, its *behaviour*, stays unchanged.

Definition 1. A *real-time transducer*[1] on $A^* \times B^*$, $\mathcal{T} = \langle A^* \times B^*, Q, I, E, T \rangle$, is an automaton the transitions of which are labelled by elements of $A \times \mathfrak{P}(B^*)$ and with initial and final functions with values in $\mathfrak{P}(B^*)$, that is, $E \subseteq Q \times A \times \mathfrak{P}(B^*) \times Q$ and $I, T\colon Q \to \mathfrak{P}(B^*)$.

The transducer $\mathcal{T}$ is said to be *finite* if E is finite, if every transition is labelled in $A \times \operatorname{Rat} B^*$ and if I and T are with values in $\operatorname{Rat} B^*$.

Example 2. Figure 1 shows three real-time transducers.



(a) (b) $\mathcal{G}_1$ (c)

Figure 1: Three real-time transducers

More generally, the transitions of $\mathcal{T}$ are thus of the form:

$$p \xrightarrow[\mathcal{T}]{a|K_{a,p,q}} q \qquad \text{with} \qquad a \in A\,,\ K_{a,p,q} \subseteq B^* \quad ,$$

[1] In French: *transducteur "temps-réel"*, a terminology that is not completely satisfactory but that I use for lack of a better translation.

from which we deduce the form of the computations of $\mathcal{T}$:

$$c = \quad \xrightarrow{I(p_0)} p_0 \xrightarrow{a_1|K_{a_1,p_0,p_1}} p_1 \xrightarrow{a_2|K_{a_2,p_1,p_2}} p_2 \cdots p_{n-1} \xrightarrow{a_n|K_{a_n,p_{n-1},p_n}} p_n \xrightarrow{T(p_n)} \quad ,$$

and the one of their label:

$$|c| = (1_{A^*}, I(p_0))\,(a_1, K_{a_1,p_0,p_1})\,(a_2, K_{a_2,p_1,p_2}) \cdots (a_n, K_{a_n,p_{n-1},p_n})\,(1_{A^*}, T(p_n)) \quad .$$

The relation $\theta\colon A^* \to B^*$ realised by $\mathcal{T}$ is thus, for every $w = a_1 a_2 \cdots a_n$ in A^*:

$$\theta(w) = \bigcup_{\substack{c \text{ calcul de } \mathcal{T} \\ \pi_{A^*}(|c|)=w}} I(p_0)\, K_{a_1,p_0,p_1}\, K_{a_2,p_1,p_2} \cdots K_{a_n,p_{n-1},p_n}\, T(p_n) \quad .$$

1.2 Realisation of rational relations

Theorem 3. *A relation $\theta\colon A^* \to B^*$ is* rational *if and only if it is realised by a* finite *real-time transducer.*

Proof. (i) The condition is sufficient. Let $\mathcal{T}$ be a *finite* real-time transducer. If K is a rational language of B^* accepted by $\mathcal{A}$ (Figure 2(a)), a transition $p \xrightarrow{a|K} q$ of $\mathcal{T}$ (Figure 2(b)) is replaced by a set of labelled transitions (Figure 2(c)), the initial and final functions $I(q) = K$ and $T(p) = K$ by two sets of labelled transitions (Figure 2(d) and (e)).

The transducer we obtain in this way is easily seen to be equivalent to the tranducer $\mathcal{T}$ we started from. This construction may yield spontaneous transitions which are then eliminated by the classical algorithm and the result is then a normalised transducer, still equivalent to $\mathcal{T}$. Figure 3 shows this construction applied to the transducer $\mathcal{G}_1$ of Figure 1(b).

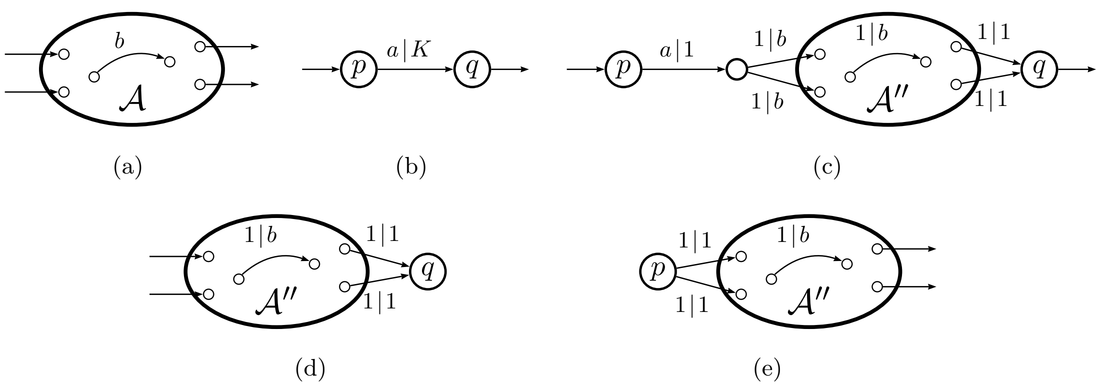


Figure 2: Transforming a real-time transducer into a normalised transducer

(ii) The condition is necessary. Let $\theta\colon A^* \to B^*$ be a rational relation and $\mathcal{T}$ a subnormalised transducer which realises $\widehat{\theta}$. The transducer $\mathcal{T}$ is written under the matrix form: $\mathcal{T} = \langle\, I, E, T \,\rangle$ where I and T are Boolean vectors of dimension Q and where E is a matrix of dimension $Q \times Q$ the entries of which are subsets of $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B)$ — if $\mathcal{T}$ is normalised — or of $(A \times \{1_{B^*}\}) \cup (\{1_{A^*}\} \times B) \cup (A \times B)$ — if $\mathcal{T}$ is subnormalised. In any case,

$$\widehat{\theta} = \bigl|\mathcal{T}\bigr| = I \cdot E^* \cdot T \tag{1.1}$$

holds. In any case also, we can write

$$E = F + G \qquad \text{with} \qquad G \in (\{1_{A^*}\} \times B)^{Q \times Q}$$
$$\text{and} \qquad F \in (A \times \{1_{B^*}\})^{Q \times Q} \qquad \text{or} \qquad F \in \bigl(\,(A \times \{1_{B^*}\}) \cup (A \times B)\,\bigr)^{Q \times Q}$$

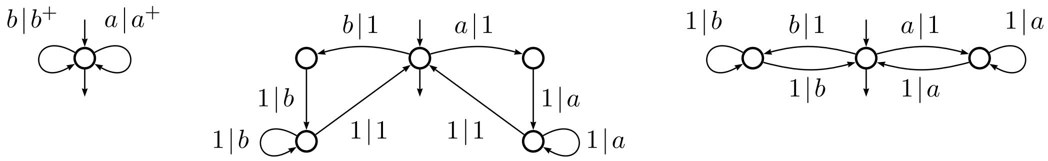
according to whether $\mathcal{T}$ is normalised or subnormalised. Equation (1.1) reads then:

$$\bigl|\mathcal{T}\bigr| = I \cdot E^* \cdot T = I \cdot (F + G)^* \cdot T = I \cdot (G^* \cdot F)^* \cdot G^* \cdot T = I \cdot G^* \cdot (F \cdot G^*)^* \cdot T\,.$$

This shows that $\mathcal{T}$ is equivalent to the two transducers $\mathcal{T}'$ and $\mathcal{T}''$ with:

$$\mathcal{T}' = \langle\, I, G^* \cdot F, G^* \cdot T \,\rangle \qquad \text{and} \qquad \mathcal{T}'' = \langle\, I \cdot G^*, F \cdot G^*, T \,\rangle \ .$$

The entries of G^* belong to $(1_{A^*} \times \operatorname{Rat} B^*)$, as do those as those of $F \cdot G^*$ belong to $(A \times \operatorname{Rat} B^*)$: $\mathcal{T}'$ and $\mathcal{T}''$ are two finite real-time transducers. Example 4 shows these computations for the transducer $\mathcal{G}_2$. ■



(a) $\mathcal{G}_1$ (b) after construction (c) $\mathcal{G}_2$, after closure and quotient

Figure 3: A real-time transducer transformed into a normalised transducer

Example 4. The matrix representation of $\mathcal{G}_2$ is:

$$I_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & (a,1) & (b,1) \\ (1,a) & (1,a) & 0 \\ (1,b) & 0 & (1,b) \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ ,$$

from which we compute:

$$F_2 = \begin{pmatrix} 0 & (a,1) & (b,1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ (1,a) & (1,a) & 0 \\ (1,b) & 0 & (1,b) \end{pmatrix}, \quad G_2^* = \begin{pmatrix} (1,1) & 0 & 0 \\ (1,a^+) & (1,a^*) & 0 \\ (1,b^+) & 0 & (1,b^*) \end{pmatrix} .$$

The transducer $\mathcal{G}_2' = \langle\, I_2, G_2^* \cdot F_2, G_2^* \cdot T_2 \,\rangle$ is shown at Figure 5 and it is easily seen that $\mathcal{G}_2'' = \langle\, I_2 \cdot G_2^*, F_2 \cdot G_2^*, T_2 \,\rangle$ is equal to $\mathcal{G}_1$.

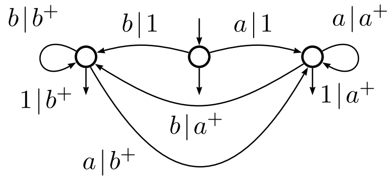


Figure 4: The real-time transducer $\mathcal{G}'_2$

1.3 Representations of rational relations

Definition 5. Let A^* and B^* be two free monoids. A *representation* of A^* into $\operatorname{Rat} B^*$ *of dimension* Q is a triple $\langle I, \mu, T \rangle$ where $\mu \colon A^* \to (\operatorname{Rat} B^*)^{Q \times Q}$ is a *morphism* (hence entirely defined by the matrices $\mu(a)$ for a in A), and where I and T are respectively row and column vectors in $(\operatorname{Rat} B^*)^Q$.

Theorem 6. *A relation* $\theta \colon A^* \to B^*$ *is* rational *if and only if there exists a representation* $\langle I, \mu, T \rangle$ *of* A^* *into* $\operatorname{Rat} B^*$ *which realises* θ*, that is, such that*

$$\forall w \in A^* \qquad \theta(w) = I \cdot \mu(w) \cdot T \ .$$

Proof. If $\mathcal{T} = \langle I, E, T \rangle$ is a real-time transducer, the matrix E defines the morphism $\mu \colon A^* \to (\operatorname{Rat} B^*)^{Q \times Q}$ by

$$E = \sum_{a \in A} (a, 1)\,(1, \mu(a)) \tag{1.2}$$

that is,

$$\forall a \in A\,,\ \forall p, q \in Q \qquad \mu(a)_{p,q} = \begin{cases} K_{a,p,q} & \text{if } p \xrightarrow[\mathcal{T}]{a|K_{a,p,q}} q \\ \emptyset & \text{otherwise}\,, \end{cases}$$

under the hypothesis, necessary for the writing $\mathcal{T} = \langle I, E, T \rangle$, that for every a in A and every pair p, q in Q, there exists at most one transition in $\mathcal{T}$ which goes from p to q and whose first component is a.

Conversely, a morphism $\mu \colon A^* \to (\operatorname{Rat} B^*)^{Q \times Q}$ defines, via the same Equation (1.2), the adjacency matrix of a finite real-time transducer. By induction on the length of the words w, it is then checked that:

$$\forall w \in A^*\,,\ \forall p, q \in Q \qquad \mu(w)_{p,q} = L \quad \Longleftrightarrow \quad L = \bigcup \left\{ H \;\middle|\; p \xrightarrow[\mathcal{T}]{w|H} q \right\}$$

from which it follows that, for every w in A^*, $\theta(w) = I \cdot \mu(w) \cdot T$. ■

Example 7. The representations of the real-time transducers of Example 4 are the following:

$$\text{(a)} \quad I = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mu(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & a & 0 \\ 0 & ab & 0 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & 1 \\ 0 & ba & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 \\ a \\ ab \end{pmatrix};$$

(b) $I = \begin{pmatrix} 1 \end{pmatrix}, \quad \mu(a) = \begin{pmatrix} a^+ \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} b^+ \end{pmatrix}, \quad T = \begin{pmatrix} 1 \end{pmatrix};$

(c) $I = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \mu(a) = \begin{pmatrix} a & b & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} b & a & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} A^+ \\ A^* \\ 1 \end{pmatrix}.$

2 Composition and evaluation theorems

The realisation of rational relations by representations allows to state a *new* composition theorem. The result in itself is not new, since we proved it already in Lecture IV: *the composition of two rational relations is a rational relation* (Theorem IV.25). But, at the same time, what we present here is more than a new proof. This result is indeed now the *consequence* of another result: *the composition of two representations is a representation*, which is new.

The idea is to generalise the *composition of morphisms* between free monoids to representations. In Lecture IV, we had deduce the 'evaluation theorem' from the closure under composition of rational relations. We now proceed the other way around and establish the evaluation theorem first.

2.1 Evaluation Theorem

In the sequel of the section, $\langle I, \mu, T \rangle$ is a representation of A^* into $\operatorname{Rat} B^*$ of dimension Q, that is:

$\mu\colon A^* \to (\operatorname{Rat} B^*)^{Q\times Q}$ is a morphism, $I \in (\operatorname{Rat} B^*)^{1\times Q}$ and $T \in (\operatorname{Rat} B^*)^{Q\times 1}$.

Recall that any application extends additively, that is, if $K \subseteq A^*$, then:

$$\mu(K) = \sum_{w\in K} \mu(w) \quad \text{that is,} \quad \forall p, q \in Q \qquad \mu(K)_{p,q} = \bigcup_{w\in K} \mu(w)_{p,q} \quad .$$

Proposition 8. *If* $\mu\colon A^* \to (\operatorname{Rat} B^*)^{Q\times Q}$ *is a morphism, then*

$$K \in \operatorname{Rat} A^* \quad \Longrightarrow \quad \mu(K) \in (\operatorname{Rat} B^*)^{Q\times Q} \quad ,$$

that is, for every p *and* q *in* Q, $\mu(K)_{p,q}$ *belongs to* $\operatorname{Rat} B^*$.

From which follows:

Corollary 9. *If* $\theta\colon A^* \to B^*$ *is a rational relation, then*

$$K \in \operatorname{Rat} A^* \quad \Longrightarrow \quad \theta(K) \in \operatorname{Rat} B^* \quad .$$

Proof. If θ is a rational relation, then θ is realised by a representation $\langle I, \mu, T \rangle$ and

$$\theta(K) = \bigcup_{w \in K} \theta(w) = \bigcup_{w \in K} I \cdot \mu(w) \cdot T = I \cdot \left(\bigcup_{w \in K} \mu(w) \right) \cdot T = I \cdot \mu(K) \cdot T \quad . \qquad \blacksquare$$

Proof of Proposition 8. (i) Let us recall first the theorem (see proof of Proposition I.17):

Theorem 10. *Let M be a monoid and E a matrix of dimension $Q \times Q$, the entries of which are in $\mathfrak{P}(M)$. Then, the entries of E^* belong to the* rational closure *of the entries of E.* ■

(ii) Preparation. We check successively the 'closure' by union, product, and star:

$$\mu(K), \mu(L) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \quad \Longrightarrow \quad \mu(K \cup L) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \tag{2.1}$$

since, for every p and q in Q, $\mu(K \cup L)_{p,q} = \mu(K)_{p,q} \cup \mu(L)_{p,q}$.

$$\mu(K), \mu(L) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \quad \Longrightarrow \quad \mu(K\, L) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \tag{2.2}$$

since, on one hand-side:

$$\begin{aligned} \mu(K\, L) &= \bigcup \{\mu(w) \mid w \in K\, L\} = \bigcup \{\mu(u\, v) \mid u \in K\, ,\ v \in L\} \\ &= \bigcup \{\mu(u)\, \mu(v) \mid u \in K\, ,\ v \in L\} \\ &= \left(\bigcup \{\mu(u) \mid u \in K\} \right) \left(\bigcup \{\mu(v) \mid v \in L\} \right) = \mu(K)\, \mu(L) \quad , \end{aligned}$$

and, on the other, since $\mathrm{Rat}\, B^*$ is a semiring, $(\mathrm{Rat}\, B^*)^{Q \times Q}$ is closed by product. And, finally:

$$\mu(K) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \quad \Longrightarrow \quad \mu(K^*) \in (\mathrm{Rat}\, B^*)^{Q \times Q} \tag{2.3}$$

since, by (2.1) and (2.2)

$$\mu(K^*) = (\mu(K))^* \quad ,$$

and Theorem 10 applies.

(iii) Proposition 8 is the consequence of the three equations (2.1), (2.2) and (2.3), by induction on the depth of a rational expression which denotes K. ■

2.2 Composition of representations

Definition 11. Let $\mu \colon A^* \to (\mathrm{Rat}\, B^*)^{Q \times Q}$ and $\nu \colon B^* \to (\mathrm{Rat}\, C^*)^{R \times R}$ be two morphisms. The *composition of μ by ν* is the map $\pi = \nu \circ \mu$ from A^* to $(\mathrm{Rat}\, C^*)^{(Q \times R) \times (Q \times R)}$, defined by the block decomposition:

$$\forall w \in A^* \qquad \pi(w)_{p \times R, q \times R} = \nu\Big(\mu(w)_{p,q}\Big) \quad .$$

Remark that this definition is built upon Proposition 8 in the sense that the entries of $\pi(w)$ are *a priori* in $\mathfrak{P}(A^*)$ and it is the proposition that insures that they are in $\mathrm{Rat}\, C^*$.

Examples 12. (i) Definition 11 coincides with the definition of monoid morphisms composition, in the case where μ and ν are such morphisms.

(ii) If $\mu_1(a) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$, $\mu_1(b) = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$ and $\nu_1(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\nu_1(b) = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$, then:

$$\pi_1(a) = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi_1(b) = \begin{pmatrix} 0 & b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & a & 0 \end{pmatrix} \quad .$$

Definition 11 is legitimate thanks to the following proposition:

Proposition 13. *$\pi = \nu \circ \mu \colon A^* \to (\mathrm{Rat}\, C^*)^{(Q\times R)\times(Q\times R)}$ is a morphism.*

Proof. We want to prove that, for every u and v in A^*, it holds:

$$[\nu \circ \mu](uv) = [\nu \circ \mu](u)\, [\nu \circ \mu](v) \quad .$$

For every p and q in Q, it holds:

$$\begin{aligned}
([\nu \circ \mu](uv))_{p\times R, q\times R} &= \nu\Big(\mu(uv)_{p,q}\Big) = \nu\left(\sum_{r\in Q} \Big(\mu(u)_{p,r}\ \mu(v)_{r,q}\Big)\right) \\
&= \sum_{r\in Q} \nu\Big(\mu(u)_{p,r}\ \mu(v)_{r,q}\Big) = \sum_{r\in Q} \Big(\nu\Big(\mu(u)_{p,r}\Big)\ \nu\Big(\mu(v)_{r,q}\Big)\Big) \\
&= \sum_{r\in Q} ([\nu \circ \mu](u)_{p\times R, r\times R}\, [\nu \circ \mu](v)_{r\times R, q\times R}) \\
&= ([\nu \circ \mu](u) \cdot [\nu \circ \mu](v))_{r\times R, q\times R} \quad .
\end{aligned}$$

∎

Theorem 14. *Let $\theta \colon A^* \to B^*$ and $\sigma \colon B^* \to C^*$ be two rational relations, realised by the representations $\langle\, I, \mu, T \,\rangle$ and $\langle\, J, \kappa, U \,\rangle$ respectively. Then, $\sigma \circ \theta \colon A^* \to C^*$ is the rational relation realised by the representation $\langle\, K, \pi, V \,\rangle$, with:*

$$\pi = \nu \circ \mu\,, \qquad K = J \cdot \nu(I) \qquad \text{and} \qquad V = \nu(T) \cdot U \quad .$$

Proof.

$$\begin{aligned}
\forall w \in A^* \qquad [\sigma \circ \theta](w) &= \sigma(\theta(w)) = \sigma(I \cdot \mu(w) \cdot T) \\
&= J \cdot \nu(I \cdot \mu(w) \cdot T) \cdot U \\
&= (J \cdot \nu(I)) \cdot \nu(\mu(w)) \cdot (\nu(T) \cdot U) \quad .
\end{aligned}$$

∎

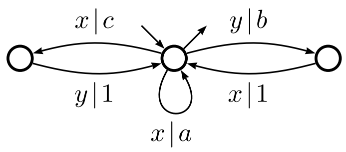
Theorem 14 yields a new method of composition of transducers, by means of the sequence of transformations:

transducer $\rightsquigarrow$ real-time transducer $\rightsquigarrow$ representation $\rightsquigarrow$
composition of representations $\rightsquigarrow$ real-time transducer $\rightsquigarrow$ transducer.

Example 15. The morphism $\varphi_1 \colon \{a,b,c\}^* \to \{x,y\}^*$ defined by:

$$\varphi_1(a) = x\,, \quad \varphi_1(b) = y\,x\,, \quad \varphi_1(c) = x\,y\,.$$

is realised by the representation $\langle\, 1, \varphi_1, 1 \,\rangle$. The relation $\varphi_1{}^{-1} \colon \{x,y\}^* \to \{a,b,c\}^*$ is realised by the real-time transducer below:



and then by the representation $\langle\, J_1, \kappa_1, U_1 \,\rangle$ where:

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad \kappa_1(x) = \begin{pmatrix} a & c & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \kappa_1(y) = \begin{pmatrix} 0 & 0 & b \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The relation $\varphi_1^{-1} \circ \varphi_1 \colon A^* \to A^*$ is realised by the representation
$\langle\, J_1, \kappa_1, U_1 \,\rangle \circ \langle\, 1, \varphi_1, 1 \,\rangle = \langle\, J_1, \pi_1, U_1 \,\rangle$ with $\pi_1 = \kappa_1 \circ \varphi_1$, that is: $\pi_1(a) = \kappa_1(x)$,

$$\pi_1(b) = \kappa_1(y\,x) = \begin{pmatrix} b & 0 & 0 \\ a & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \pi_1(c) = \kappa_1(x\,y) = \begin{pmatrix} c & 0 & a\,b \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix},$$

which corresponds to the real-time transducer below.

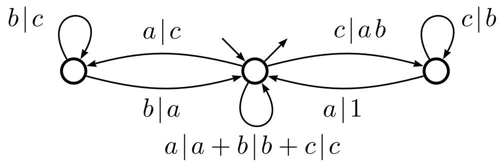


Figure 5: A transducer for $\varphi_1^{-1} \circ \varphi_1$

3 Uniformisation of rational relations

As a first illustration of the realisation of rational relations by representations, we establish a *Rational Uniformisation Theorem.*

3.1 Uniformisation of a relation

The notion of *uniformisation of a relation* (and the terminology) comes from logic (more precisely from descriptive set theory). In these notes, it will be only a definition that allows us to state a result.

If θ is any relation, from a set E into another set F, a function τ from E to F *uniformises* θ (or is *a uniformisation* of θ) if, for every e in the domain of θ, $\tau(e)$ is an element of $\theta(e)$, that is:

$$\mathsf{Dom}\,\tau = \mathsf{Dom}\,\theta \quad \text{and} \quad \forall e \in \mathsf{Dom}\,\theta \qquad \tau(e) \in \theta(e) \quad .$$

A uniformisation τ of a relation θ consists then in a *choice function*, a choice that is repeated for every element of the domain of θ, in the *selection* of an element $\tau(e)$ in each of the subsets $\theta(e)$ (*cf.* Figure 6).

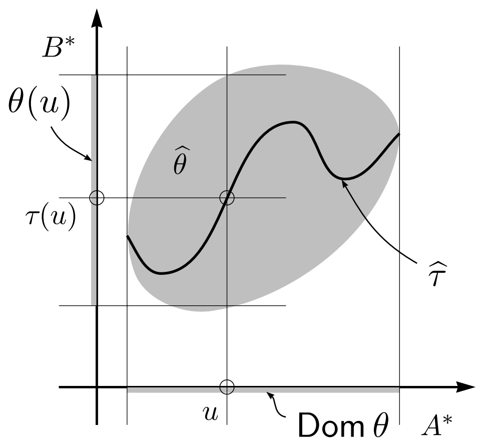


Figure 6: A uniformisation τ of a relation θ

Example 16. If θ is a relation from A^* (From an arbitrary set E indeed) in a free monoid B^*, the *radix uniformisation*[2] is the function θ_{rad} which associate with every element e of the domain of θ the smallest element, in the radix order, of $\theta(e)$.

Along the same lines, we can define the *lexicographic selection* θ_{lex}: it is the function which associates with every element e of the domain of θ the smallest element, in the lexicographic order, of $\theta(e)$, *if it exixts.* As the lexicographic order *is not a well ordering*, this minimum element does not necessarily exist and θ_{lex} is not always a uniformisation, that is, it is a function the domain of which *strictly contained* in the one of θ.

[2] If radix order is called, a tempting option, *military order* (the oldest in the highest rank), we then get a *military uniformisation*...

3.2 The Rational Uniformisation Theorem

After the definition of uniformisation, the problem consists in knowing whether it is possible, when θ belongs to a certain family of relations, to define *by selecting* for every e an element in $\theta(e)$, a uniformisation τ which belongs to that familiy of relations, or to another given family of functions. The problem is solved for the family of rational relations by the following result:

Theorem 17 (Eilenberg 1974)**.** *Every rational relation is uniformised by an unambiguous functional rational relation.*

Proof. Recall that if $\mathcal{A}$ is an automaton over A^* and $\widehat{\mathcal{A}}$ is its determinisation, the accessible part $\mathcal{S}$ of $\widehat{\mathcal{A}}\times\mathcal{A}$ is a covering of $\mathcal{A}$, called the *Schützenberger covering* or *S-covering* of $\mathcal{A}$, and that the projection $\pi_{\widehat{\mathcal{A}}}$ of $\mathcal{S}$ onto $\widehat{\mathcal{A}}$ is an In-morphism. As a result we can, by eliminating some transitions, construct a sub-automaton $\mathcal{T}$ of $\mathcal{S}$, called an *S-immersion* of $\mathcal{A}$, which is unambiguous and equivalent to $\mathcal{A}$.

Let $\mathcal{C}$ be a real-time transducer which realises a relation θ (corresponding to a representation $\langle I,\mu,T\rangle$ of θ), $\mathcal{A}$ its underlying input automaton, and $\mathcal{T}$ an S-immersion in $\mathcal{A}$. Since $\mathcal{T}$ is an immersion in $\mathcal{A}$ each transition (r,a,s) of $\mathcal{T}$ corresponds to a *unique transition* (p,a,q) in $\mathcal{A}$ and hence to a unique transition $(p,(a,\mu(a)_{p,q}),q)$ in $\mathcal{C}$. If we choose, *arbitrarily*, a word w in $\mu(a)_{p,q}$, we construct a transducer $\mathcal{U}$ by replacing each transition (r,a,s) in $\mathcal{T}$ by $(r,(a,w),s)$. Since $\mathcal{T}$ is unambiguous, the relation τ realised by $\mathcal{U}$ is an unambiguous function, and since $\mathcal{T}$ is equivalent to $\mathcal{A}$ it has the same domain as θ, and its graph is contained in that of θ, by the choice of w. ■

Example 18. .— Let θ_2 be the relation from $\{a,b\}^*$ into itself which replace in every word one of its factor ab by the set b^+a (and which is not defined on the words that do not contain such a factor). Figure 7 shows a transducer $\mathcal{E}_2$ which realises θ_2 and whose underlying input automaton is $\mathcal{A}_1$ (on the left, verticaly), its determinisation $\widehat{\mathcal{A}_1}$ (horizontaly, at the top) and the result of the construction decribed in the proof above. □

Since the only possible uniformisation of a function is the function itself, Theorem 17 implies:

Corollary 19.
Every functional rational relation is an unambiguous rational relation.

4 Functional rational relations

The realisation by representation yields handy criteria for defining or characterising classes of relations. The first class that we investigate from this point of view is the

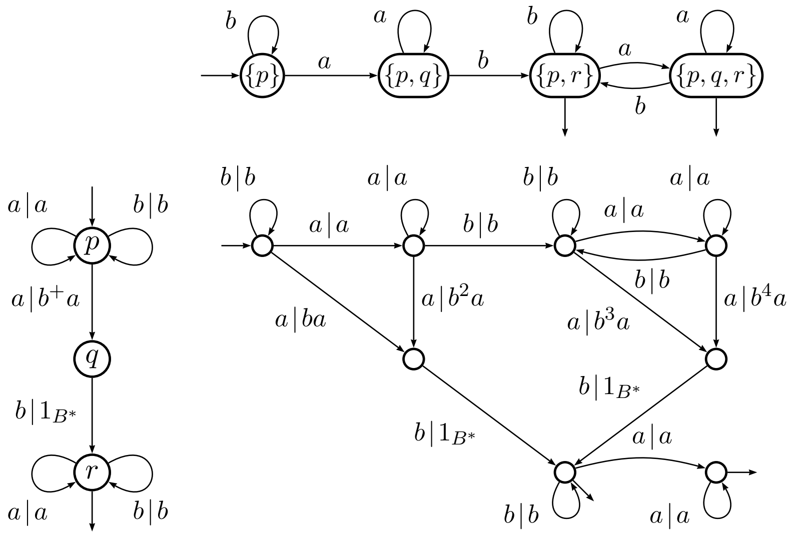


Figure 7: The transducer $\mathcal{E}_2$ and a S-uniformisation of θ_2

one of functional rational relations (which we also call *rational functions*, in spite of the unfortunate collision with a classical terminology in mathematics).

Proposition 20. *Let* $\theta\colon A^* \to B^*$ *be a rational relation realised by a* trim *representation* $\langle\, I, \mu, T\,\rangle$. *If* θ *is a function, then all non zero entries of the matrices* $\mu(a)$ *are words (*monomials*).* ■

Theorem 21. *It is decidable whether a rational relation is functional or not (with a quadratic complexity).* ■

Example 22. Figure 8 shows a transducer which realises a functional relation (which is not so obvious at first sight).

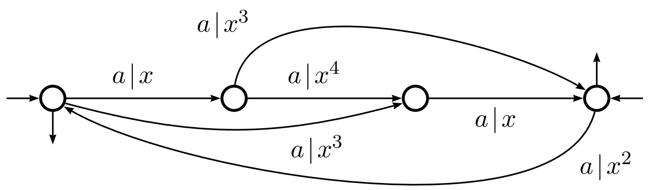


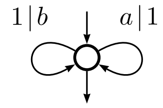
Figure 8: A functional transducer

Definition 23. A rational function is *sequential* (resp. *co-sequential*) if it is realised by a *row-monomial* representation (resp. *column-monomial* representation), that is, if the underlying input automaton of a real-time transducer which realises it is *deterministic* (resp. *co-deterministic*).

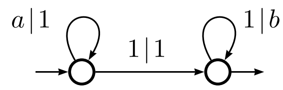
Theorem 24. *Every rational function is the composition of a sequential function by a co-sequential function.* ■

5 Exercises

1. Apply the construction of the proof of Theorem 6 in order to build real-time transducers from the two transducers below which realise the universal relation on $\{a\}^* \times \{b\}^*$.



(a) $\mathcal{U}_1$



(b) $\mathcal{U}_2$

2. Give a realisation by representation of the following relations:

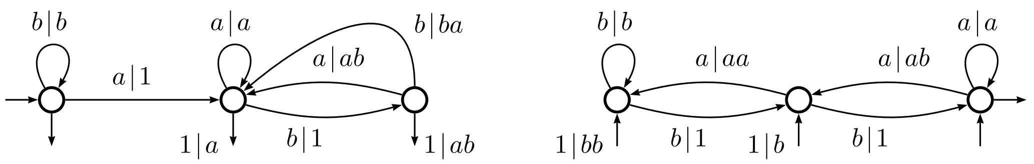
(a) the complement of the identity; (b) the lexicographic order; (c) the radix order.

3. **Finite and infinite components of a rational relation.** Let $\tau\colon A^* \to B^*$ be a relation. The *finite* and *infinite components* τ_{f} and τ_∞ of τ are defined by:

$$\tau_{\mathsf{f}}(w) = \begin{cases} \tau(w) & \text{if } \|\tau(w)\| \text{ is finite} \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{et} \qquad \tau_\infty(w) = \begin{cases} \emptyset & \text{if } \|\tau(w)\| \text{ is finite} \\ \tau(w) & \text{otherwise} \end{cases}$$

Show that if τ is rational, then τ_{f} and τ_∞ are rational and effectively computable from τ.

4. **Fibonacci reduction.** Give a transducer which realises the composition of the relations realised by the transducers below (the transducer on the left by the transducer on the right).



5. **Choosing the uniformisation.** Let $A = \{a, b, c\}$ be a totally ordered alphabet, where $a < b < c$, and let θ be the rational relation from A^* into itself whose graph is:

$$\widehat{\theta} = (a,a)^* \, (b,1)^* \, (1,b) \, \cup \, (a,1)^* \, (b,a)^* \, (1,c) \quad .$$

Show that neither the radix uniformisation θ_{rad} nor the lexicographic selection θ_{lex} are rational functions.

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