Fuzzy connected filters for fuzzy gray scale images

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Abstract

In this paper, we propose an extension of connected filters into the fuzzy set framework. We introduce a modeling of imprecision via the introduction of the fuzzy umbra image concept which allows us to express fuzzy connected operators for fuzzy gray scale images. Using this new tool, a set of filters families is proposed and their properties are discussed. The expression of classical filters in this new framework and the introduction of a new filter dedicated to the processing of Digital Breast Tomosynthesis (DBT) volumes are also described.

Keywords: Fuzzy connected filters, fuzzy umbra images.

1 Introduction

Connected filters [9] are powerful filters widely used to simplify images. Since they rely on the definition of connected components (CC), they may in some cases not handle perfectly the imprecision contained in images (e.g. with the presence of noise, two non connected components can become connected). However, expressing images using fuzzy sets allows modeling gradual connectivity, and thus can overcome this limitation. For this reason it makes sense to extend this kind of operator to fuzzy sets. Some other approaches exist to extend the notion of connectivity as proposed in [12, 2]. However, fuzziness on gray levels has not been considered so far.

First the concept of fuzzy umbra image, which is suitable to model imprecision present in gray scale images is proposed (Section 2), then a definition of connected operators for fuzzy subsets is introduced in order to extend them to grayscale images (Section 3). Finally, the expression of standard operators within this framework is discussed (Sections 4-6), and a new operator used to mark circumscribed masses in DBT volumes is proposed as an illustrative example (Section 7).

Note that due to the lack of space, most of the time, only a sketch of the proofs will be given.

2 Fuzzy umbra images

In the following developments, the notations presented in Table 1 will be used.

Definition 2.1. $F \in \mathcal{F}$ is a fuzzy umbra image (FUI) iff $\forall p \in \Omega, \forall g_1 \in E, \forall g_2 \in E \quad g_1 \leq g_2 \Rightarrow F(p, g_1) \geq F(p, g_2)$.

A fuzzy umbra image is a direct extension of the crisp umbra image concept: the dimension of the nD gray scale image is increased to get a $(n+1)$D binary image. In our case, imprecision in gray scale images is expressed using fuzzy sets in the umbra images. Thus, for a fuzzy umbra image (F), $\forall p \in \Omega, \forall g \in E$: $F(p, g)$ represents the satisfaction degree of the property: *the image intensity is greater or equal to g at point p.*

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Table 1: Notations.

Ω	image bounded domain with a discrete connectivity
E	gray scale set
$\mathcal{I}$	set of natural images $I : \Omega \rightarrow E$
$\mathcal{S}$	set of fuzzy sets defined on Ω ($\Omega \rightarrow [0,1]$)
$\mathcal{F}$	set of fuzzy sets defined on $\Omega \times E$ ($\Omega \times E \rightarrow [0,1]$)
2^{Ω}	set of sets included in Ω ($\Omega \rightarrow \{0,1\}$). Obviously, we have: $2^{\Omega} \subset \mathcal{S}$.
$2^{\Omega \times E}$	set of sets included in $\Omega \times E$ ($\Omega \times E \rightarrow \{0,1\}$). Obviously, we have: $2^{\Omega \times E} \subset \mathcal{F}$.
$\mathcal{K}$	set of fuzzy subsets defined on $\mathbb{R}^+$ ($\mathbb{R}^+ \rightarrow [0,1]$)
$[0,1]^{\mathcal{S}}$	set of fuzzy subsets included in $\mathcal{S}$.

Definition 2.2. The image-to-FUI conversion operator $um : \mathcal{I} \rightarrow 2^{\Omega \times E}$, which describes a conversion process of a gray scale image into an umbra image is defined as:

$$\forall I \in \mathcal{I}, \forall p \in \Omega, \forall g \in E$$
$$um(I)(p,g) = \begin{cases} 1 \ \textit{if } I(p) \geq g \\ 0 \ \textit{otherwise} \end{cases}$$

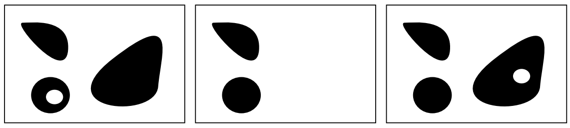
Since images in $\mathcal{I}$ do not explicitly hold imprecision, the resulting umbra image is crisp, but can still be interpreted as a fuzzy subset.

We denote by f_α the α-cut of the set f ($f \in \mathcal{S}$), for $\alpha \in [0,1]$. The same notation will be used for α-cuts of fuzzy sets in $\mathcal{F}$.

Definition 2.3. The FUI-to-image conversion operator $im : [0,1] \times \mathcal{F} \rightarrow \mathcal{I}$ is defined as:

$$\forall \alpha \in [0,1], \forall F \in \mathcal{F}, \forall p \in \Omega$$
$$im_\alpha(F)(p) = sup\{g \in E / F(p,g) \geq \alpha\}$$

This operator provides a way to come back into the gray scale image domain from a fuzzy umbra image. Since this last one is fuzzy, it is necessary to have a way to deal with imprecision (gray scale images cannot represent it). Here this is done using an α-cut, which allows to come back directly to the gray scale domain by removing imprecision.



a b c

Figure 1: Filtering of a crisp set (a) with a connected operator (b) and with a non-connected operator (c).

3 Families of fuzzy connected operators

Definition 3.1. An operator $\phi : \mathcal{S} \rightarrow \mathcal{S}$ is fuzzy connected (FC) iff $\forall f \in \mathcal{S}, \forall \alpha \in [0,1]$ $(C^b(f_\alpha \cap \overline{\phi(f)_\alpha}) \subseteq C^b(f_\alpha)) \wedge (C^b(\overline{f_\alpha} \cap \phi(f)_\alpha) \subseteq C^b(\overline{f_\alpha}))$

with: $C^b(N) = \{\textit{set of CC of } N\}$ and $\overline{N}$ denotes the complementation.

In other words, each α-cut of the filtered image is handled as in the crisp case. The idea is that each CC of the background (resp. the object) can become entirely object (resp. background) or stay as it. That is, a CC cannot be modified and a connected operator cannot create new contours: it can only keep or suppress them. Figure 1 illustrates this process.

Let $\Psi = \{\psi^{l,F} : \mathcal{S} \rightarrow \mathcal{S} / l \in E, F \in \mathcal{F}\}$ be a set of operators.

Definition 3.2. Ψ is a set of weak fuzzy connected operators (SWFCO) iff:

$$\forall l \in E, \forall F \in \mathcal{F} \quad \psi^{l,F} \textit{ is FC} \tag{1}$$
$$\forall l \in E, \forall F \in \mathcal{F}, \forall f \in \mathcal{S} \quad \psi^{l,F}(f) \subseteq f \tag{2}$$

where $\subseteq$ denotes the classical inclusion on $\mathcal{S}$.

A weak connected operator ($\psi^{l,F}$) is the extension of an anti-extensive connected operator in the crisp case. The three indices have different meaning, and may in some cases be suppressed (examples will be shown in this paper). Parameters l and F correspond to external data that will be used in the definition of connected operators dedicated to fuzzy umbra images. They will allow defining operators that rely not only on components embeding but also on an image content (for instance the filtered one). This corresponds to the extension of filters such as the *volume levelling* [10].

Definition 3.3. Ψ is a set of basic fuzzy connected operators (SBFCO) iff:

$$\Psi \; is \; a \; SWFCO \quad (3)$$

$$\begin{array}{l} \forall l \in E, \forall (F,F') \in \mathcal{F}^2, \forall (f,h) \in \mathcal{S}^2 \\ (f \subseteq h) \wedge (F \subseteq F') \Rightarrow \psi^{l,F}(f) \subseteq \psi^{l,F'}(h) \end{array} \quad (4)$$

Because these operators are increasing (Eq. 4), a fuzzy umbra image operator constructed using such a Ψ will be increasing as it will be shown later.

Definition 3.4. Ψ is a set of ordered fuzzy connected operators (SOFCO) iff:

$$\Psi \; is \; a \; SBFCO \quad (5)$$

$$\begin{array}{l} \forall (l_1, l_2) \in E^2, \forall F \in \mathcal{F}, \forall f \in \mathcal{S} \\ l_1 \geq l_2 \Rightarrow \psi^{l_1,F}(f) \subseteq \psi^{l_2,F}(f) \end{array} \quad (6)$$

Here, the definition of Ψ is refined by the introduction of the increasingness with respect to l. It will be shown later that this is a sufficient condition to ensure that filtered FUI are still FUI.

Let Ψ_λ be a set of functions parametrized by $\lambda \in \mathbb{R}^+$: $\Psi_\lambda = \{\psi_\lambda^{l,F} : \mathcal{S} \rightarrow \mathcal{S} / l \in E, F \in \mathcal{F}\}$

Definition 3.5. Ψ_λ is a set of extended fuzzy connected operators (SEFCO) iff:

$$\forall \lambda \quad \Psi_\lambda \; is \; a \; SOFCO \quad (7)$$

$$\begin{array}{l} \forall (l_1, l_2) \in E^2, \forall (\lambda, \mu) \in K^2, \\ \forall F \in \mathcal{F}, \forall f \in \mathcal{S} \\ (\lambda \geq \mu) \wedge (l_1 \geq l_2) \Rightarrow \psi_\lambda^{l_1,F}(f) \subseteq \psi_\mu^{l_2,F}(f) \end{array} \quad (8)$$

$$\forall l \in E, \forall F \in \mathcal{F}, \forall f \in \mathcal{S} \quad \psi_0^{l,F}(f) = f \quad (9)$$

Here the definition of Ψ is refined by adding a parameter λ with respect to which the operator is decreasing. The idea is to provide a framework to express fuzzy extinction filters. Typically λ corresponds to a threshold on an attribute (area, volume, etc.) computed on fuzzy sets. Using this threshold, we can evaluate how a fuzzy connected component vanishes.

4 Extinction operators

Definition 4.1. $S : \mathcal{S} \times \mathcal{S} \rightarrow [0,1]$ is a similarity measure iff [4]:

$$\forall f \in \mathcal{S} \quad S(f,f) = 1 \quad (10)$$

$$\forall (f,h) \in \mathcal{S}^2 \quad S(f,h) = S(h,f) \quad (11)$$

$$\begin{array}{l} \forall (f,h,j) \in \mathcal{S}^3 \\ \min(S(f,h), S(h,j)) \leq S(f,j) \end{array} \quad (12)$$

These properties mean that S has to be reflexive (Eq. 10), symmetrical (Eq. 11) and max-min transitive (Eq. 12).

Definition 4.2. The connectivity degree [8] between two points p and p' of Ω^2 is $c_\mu(p,p') = \max\limits_{L \in \{Path_{p,p'}\}} (\min\limits_{p_i \in L} \mu(p_i))$

where $\{Path_{p,p'}\}$ denotes the set of all paths from p to p' according to the connectivity on Ω.

Definition 4.3. The connected component associated to a point $p \in \Omega$ is expressed [8] as $\forall p' \in \Omega \quad \Gamma_\mu^p(p') = c_\mu(p,p')$.

Considering a Ψ_λ SEFCO, the notion of fuzzy persistence of a fuzzy connected component at a point of Ω in a FUI can be introduced using the following definition:

Definition 4.4. The fuzzy persistence operator $pers : \mathcal{F} \times E \times \Omega \rightarrow \mathcal{K}$ is defined as:

$$\begin{array}{l} \forall F \in \mathcal{F}, \forall g \in E, \forall p \in \Omega, \forall \lambda \in \mathbb{R}^+ \in \Omega \\ pers(F,g,p)(\lambda) = S(\Gamma^p_{F(*,g)}, \Gamma^p_{\psi_\lambda^{g,F}(F(*,g))}) \end{array}$$

with $F(*,g)$ the fuzzy subset ($f \in \mathcal{S}$) verifying $\forall p \in \Omega \quad f(p) = F(p,g)$.

For a gray level g, $pers(F,g,p)(\lambda)$ corresponds to the degree to which the component associated with p is still present after filtering by ψ_λ. The resulting fuzzy quantity $pers(F,g,p)$ represents how the component vanishes according to λ (see Figure 2). This formulation aims at representing the same concept as classical extinction functions.

5 Fuzzy connected operators on fuzzy gray scale images

Now, using the former definitions of connected operators for fuzzy sets, we propose a way to process fuzzy grayscale images.

Definition 5.1. Let Ψ be a SWFCO, the operator $\delta_\Psi : \mathcal{F} \rightarrow \mathcal{F}$ associated to Ψ working on fuzzy images is defined as $\forall F \in \mathcal{F}, \forall p \in \Omega, \forall g \in E \; \delta_\Psi(F)(p,g) = \psi^{g,F}(F(*,g))(p)$.

Such a filter processes separately the fuzzy subsets extracted from each gray level of the umbra image. Nonetheless, thanks to the indices g and F, corresponding to external data,

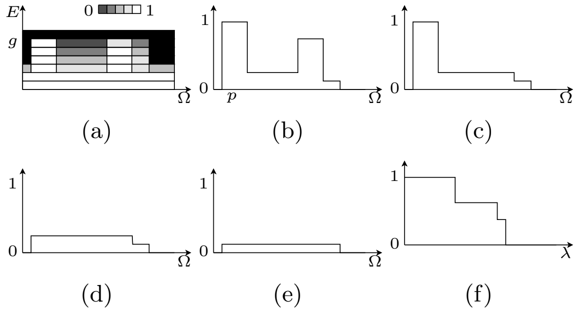


Figure 2: Persistence (f) of a fuzzy connected component $\Gamma^p_{F(*,g)}$ (c) from the set $F(*,g)$ (b) extracted from an umbra image (a). (d) and (e) represent $\Gamma^p_{\psi^{g,F}_{\lambda}(F(*,g))}$ for increasing values of λ.

information about the whole image can still be retrieved.

Using the properties Ψ may have, let us discuss some properties such an operator δ_Ψ inherits.

It would also be interesting to define another type of operator like $\psi^{g,F'}(F(*,g))$ with F' an image different from the one to be filtered. In this case F' would represent some real external data.

Theorem 5.2. *A filter associated to a Ψ SOFCO returns a FUI if its input is a FUI.*

Proof. This results from the anti-extensivity (Eq. 4) and the increasingness (Eq. 6) of the operators $\psi^{g,F}$. □

Theorem 5.3. *A filter δ constructed from a Ψ SBFCO is increasing.*

Proof. This is because the operators $\psi^{g,F}$ are increasing. □

Theorem 5.4. *A filter δ constructed from a Ψ SWFCO verifying $\psi^{l,F}(f) = \psi^{l,F'}(\psi^{l,F}(f))$ is idempotent.*

Proof. This comes directly from $\psi^{l,F}(f) = \psi^{l,F'}(\psi^{l,F}(f))$. □

The hypothesis about the lack of link between F and F' is strong. Actually, idempotent filters are usually not relying on l and F. An instance based on component cardinal will be detailed later.

Theorem 5.5. *A δ_Ψ based on a Ψ SBFCO that verifies $\forall(F,F') \in \mathcal{F}^2, \forall l \in E, \forall f \in \mathcal{S}$ $\psi^{l,F}(f) = \psi^{l,F'}(\psi^{l,F}(f))$ is a morphological filter (i.e. idempotent and increasing).*

Theorem 5.6. *A δ_Ψ based on a Ψ SWFCO is anti-extensive.*

Proof. Anti-extensivity is inherited from the anti-extensivity of the $\psi^{g,F}$. □

Theorem 5.7. *A δ_Ψ based on a Ψ SBFCO that verifies $\forall(F,F') \in \mathcal{F}^2, \forall l \in E, \forall f \in \mathcal{S}$ $\psi^{l,F}(f) = \psi^{l,F'}(\psi^{l,F}(f))$ is an algebraic opening (i.e. an anti-extensive morphological filter).*

In order to interpret filtered images that are not FUI, an aggregation operator can be used. This is a suitable way to pass from the umbra image to the domain of the image, and allows to deal with problems of interpretation that can arise with thinning-like [3] operators (e.g. creation of artificial edges).

Definition 5.8. The operator $agg : \mathcal{F} \to \mathcal{S}$ is defined as:

$$\begin{aligned} &\forall F \in \mathcal{F}, p \in \Omega \\ &agg(F)(p) = \mathop{\perp}_{g \in E} F(p,g) \end{aligned}$$

with $\perp : [0,1] \times [0,1] \to [0,1]$ a t-conorm.

Figure 3 shows how agg works. In this example, only components which lie in a given size range are kept in the filtered image. The aggregation operator enables to see where in Ω the objects that verify this property are located.

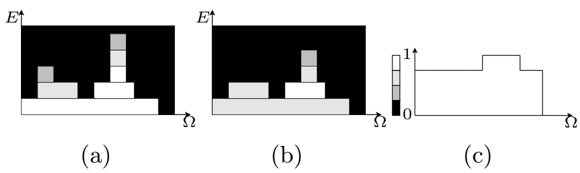


Figure 3: Aggregation (c) of a FUI image (a) filtering result (b).

6 Link with crisp connected operators

We will now express how regular filters (e.g. crisp attribute opening [11]) are expressed in this new framework.

Definition 6.1. For any crisp set N, the connected component associated to a point is defined as:

$$\forall (p,p') \in \Omega^2$$
$$^b\Gamma_N^p(p') = \begin{cases} 1 & \textit{if } p \textit{ and } p' \textit{ are connected in} \\ & N \textit{ or } \overline{N} \\ 0 & \textit{otherwise} \end{cases}$$

The connectivity used in N and $\overline{N}$ can be different. For instance, using the 4-connectivity in N will force to use use the 8-connectivity in the background.

Definition 6.2. $\delta : \mathcal{F} \to \mathcal{F}$ is a fuzzy extension of the operator $G : \mathcal{I} \to \mathcal{I}$ iff: $\forall I \in \mathcal{I} \quad im_1(\delta(um(I))) = G(I)$.

Let $A : \Omega \times E \times \mathcal{I} \times 2^\Omega \to \mathbb{R}^+$ verifying:

$$\begin{array}{l} \forall g \in E, \forall p \in \Omega, \forall I \in \mathcal{I}, \forall N \subseteq \Omega, \forall p' \in \Omega \\ p' \in {}^b\Gamma_N^p \Rightarrow A_{p'}^{g,I}(N) = A_p^{g,I}(N) \end{array} \tag{13}$$

$$\begin{array}{l} \forall (g_1,g_2) \in E^2, \forall p \in \Omega, \forall I \in \mathcal{I}, \forall N \subseteq \Omega, \\ g_1 \leq g_2 \Rightarrow A_p^{g_1,I}(N) \geq A_p^{g_2,I}(N) \end{array} \tag{14}$$

$$\begin{array}{l} \forall g \in E, \forall p \in \Omega, \forall (I,I') \in \mathcal{I}^2, \\ \forall N \subseteq \Omega, M \subseteq \Omega, \\ (I \leq I') \wedge (N \subseteq M) \Rightarrow A_p^{g,I}(N) \leq A_p^{g,I'}(M) \end{array} \tag{15}$$

An operator A can be interpreted as a measure performed on a connected component of a set N or $\overline{N}$ that contains p. This measure is relying on data provided by the image I and the gray level g.

Let $A' : \Omega \times E \times \mathcal{I} \to \mathbb{R}^+$ defined as:

$$\begin{array}{l} \forall g \in E, \forall p \in \Omega, \forall I \in \mathcal{I}, \\ A_p'^g(I) = A_p^{g,I}(X_g^+(I)) \end{array} \tag{16}$$

with X_g^+ the threshold operator.

A' represents a *restricted* version of A. Actually, a relation between N, g and I is introduced: N is the thresolding of I at level g.

Finally, the operator $G : \mathbb{R}^+ \times \mathcal{I} \to \mathcal{I}$ is introduced and defined as: $\forall \lambda \in \mathbb{R}^+, \forall I \in \mathcal{I}, \forall p \in \Omega \quad G_\lambda(I)(p) = sup\{g \in E / A_p'^g(I) \geq \lambda\}$. Obviously, using Eq. 16 this can be rewritten as: $G_\lambda(I)(p) = sup\{g \in E / A_p^{g,I}(X_g^+(I)) \geq \lambda\}$.

G_λ represents a connected filter that removes maxima of I that do not satisfy a criterion modeled by A. It will be shown later that filters like volume levelling or area opening can be expressed this way.

Theorem 6.3. *An operator δ_{Ψ_λ} based on a $\Psi_\lambda = \left\{\psi_\lambda^{g,F} : \mathcal{S} \to \mathcal{S} / g \in E, F \in \mathcal{F}, f \in \mathcal{S}\right\}$, $\lambda \in \mathbb{R}^+$ with $\forall p \in \Omega$:*

$$\psi_\lambda^{g,F}(f)(p) = sup\left\{ \begin{array}{l} \alpha \in [0; f(p)] / \\ A_p^{g, im_1(F_\alpha)}(f_\alpha) \geq \lambda \end{array} \right\}$$

is a fuzzy extension of the operator G_λ expressed as:

$$\begin{array}{l} \forall I \in \mathcal{I}, \forall p \in \Omega \\ G_\lambda(I)(p) = sup\{g \in E / A_p'^g(I) \geq \lambda\} \end{array}$$

Proof. $im_1(\delta_\Psi^\lambda(um(I)))(p)$ can be rewritten as: $sup\left\{g / sup\left\{ \begin{array}{l} \alpha \in [0,1] / \\ A_p^{g, im_1(um(I)_\alpha)}(X_g^+(I)) = \lambda \end{array} \right\} \geq 1 \right\}$.

Furthermore, using Eq. 15, it can be shown that: $\forall p \in \Omega, \forall g \in E \; A_p^{g, im_1(um(I)_\alpha)}(X_g^+(I)) \leq A_p^{g, im_1(um(I)_0)}(X_g^+(I))$. Thus (because $\forall \alpha \in]0;1] \quad um(I)_\alpha = um(I)_1$, since $um(I)$ is crisp): $im_1(\delta_\Psi^\lambda(um(I)))(p) = sup\{g / A_p^{g,I}(X_g^+(I)) \geq \lambda\} = G_\alpha(I)$. □

Each element of this set Ψ can be interpreted as the filtering of each α-cut of f and F. Here, this is not F_α that is directly used but rather $im_1(F_\alpha)$. This can be seen as a way to generate an image in $\mathcal{I}$ from the α-cut of F, which will be used as external data (as it is the case for g) to perform the measures on the set f. The quantity $\psi_\lambda^{g,F}(f)(p)$ represents the maximal membership degree α for which a measure on f_α is greater or equal to a given threshold λ. A concrete instance based on the cardinal of a set will illustrate this formulation later.

Theorem 6.4. *Ψ built from a A is SEFCO.*

Proof. Using an α-cut decomposition, Eq. 13 leads to Eq. 1. Furthermore, Eq. 2 is obviously verified (definition of $\psi_\lambda^{g,F}$). Then

Eq. 15 can be used to show Eq. 4. This results in a Ψ SBFCO. Eq. 8 can be proved using Eq. 14. And finally, by definition, we have Eq. 9. For those reasons, Ψ is a SEFCO. $\square$

Using the former theorems, the extensions of two regular filters (volume levelling and area opening) can be introduced.

Let us define $Vol : \Omega \times E \times \mathcal{I} \times 2^{\Omega} \rightarrow \mathbb{R}^+$ as:

$$\begin{aligned}&\forall p \in \Omega, \forall g \in E, \forall I : \mathcal{I}, \forall N \subseteq \Omega\\ &Vol_p^{g,I}(N) = \sum_{y \in {}^{b}\Gamma_N^p} \max(0, (I(y) - g))\end{aligned}$$

This attribute corresponds to the volume inside $\max(0, I - g)$ on the domain defined by the connected component of N that contains p (see Figure 4).

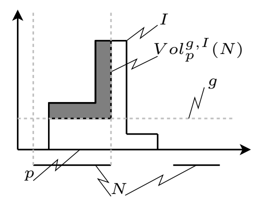


Figure 4: Computation of $Vol_p^{g,I}$.

Theorem 6.5. $\delta_{\Psi_\lambda^{Vol}}$ *is a fuzzy extension of the volume levelling* $G_\lambda^{Vol} : \mathcal{I} \rightarrow \mathcal{I}$, $\lambda \in \mathbb{R}^+$ *defined as:* $\forall I \in \mathcal{I}, \forall p \in \Omega$ $G_\lambda^{Vol}(I)(p) = sup\{g \in E / Vol_p^{\prime g}(I) \geq \lambda\}$.
with:

$$\begin{aligned}&Vol' : \Omega \times E \times \mathcal{I} \rightarrow \mathbb{R}^+ / \forall g \in E, \forall p \in \Omega,\\ &\forall I \in \mathcal{I} \quad Vol_p^{\prime g}(I) = Vol_p^{g,I}(X_g^+(I))\end{aligned}$$

Proof. The operator Vol' verifies the properties (13), (14) and (15). Using Theorem 6.3, we derive that $\delta_{\Psi_\lambda^{Vol}}$ is a fuzzy extension of G^{Vol}. $\square$

Using the same approach, a similar result can be obtained for the area opening.

Let us define $Card : \Omega \times E \times \mathcal{I} \times 2^{\Omega} \rightarrow \mathbb{R}^+$ as:

$$\begin{aligned}&\forall p \in \Omega, \forall g \in E, \forall I \in \mathcal{I}, \forall N \subseteq \Omega\\ &Card_p^{g,I} = \sum_{y \in {}^{b}\Gamma_N^p} 1\end{aligned}$$

Theorem 6.6. $\delta_{\Psi_\lambda^{Card}}$ *is a fuzzy extension of the area opening* $G_\lambda^{Card} : \mathcal{I} \rightarrow \mathcal{I}$, $\lambda \in \mathbb{R}^+$ *defined as:* $\forall I \in \mathcal{I}, \forall p \in \Omega$ $G_\lambda^{Card}(I)(p) = sup\{g \in E / Card_p^{\prime g}(I) \geq \lambda\}$.
with:

$$\begin{aligned}&Card' : \Omega \times E \times \mathcal{I} \rightarrow \mathbb{R}^+ / \forall g \in E, \forall p \in \Omega,\\ &\forall I \in \mathcal{I}, \quad Card_p^{\prime g}(f) = Card_p^{g,I}(X_g^+(I))\end{aligned}$$

Proof. $Card'$ also verifies the properties (13), (14) and (15), thus $\delta_{\Psi_\lambda^{Card}}$ is also a fuzzy extension of G^{Card}. $\square$

7 Practical usage of fuzzy connected operators

Now, we present a new filter modeled to mark circumscribed masses in DBT volumes.

Definition 7.1. The set of the fuzzy connected components of a fuzzy set is expressed using $Q : \mathcal{S} \rightarrow [0,1]^{\mathcal{S}}$ defined as: $\forall f \in \mathcal{S}$ $Q(f) = \{\Gamma_f^p / p \in \Omega \wedge$ *p is a point of a regional maximum of f*$\}$.

Using this definition there are as much fuzzy connected component as maxima in the image (see Figure 5).

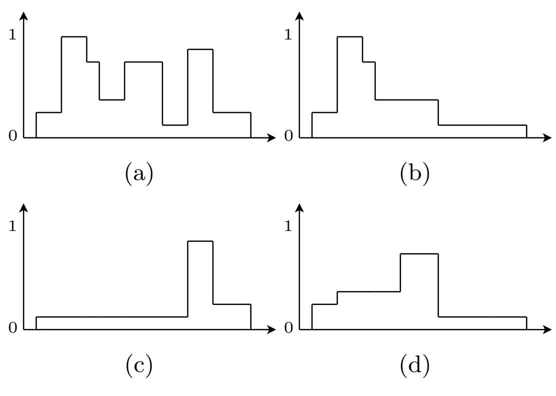


Figure 5: Extraction of fuzzy connected components (b, c, d) from a fuzzy subset (a).

Definition 7.2. Using the definition provided by [1], the fuzzy dilation $D : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ can be expressed as:

$$\begin{aligned}&\forall (f, h) \in \mathcal{S}^2, \forall p \in \Omega\\ &D_h(f)(p) = \sup_{p' \in \Omega}\{h(p - p') \top f(p')\}\end{aligned}$$

Definition 7.3. The fuzzy mean $fmean : \mathcal{F} \times \mathcal{S} \rightarrow \mathbb{R}^{+}$ is computed as:

$$\forall F \in \mathcal{F}, \forall f \in \mathcal{S}$$
$$fmean(F, f) = \frac{\sum_{p \in \Omega} f(p) \sum_{g \in E} F(p,g)}{fcard(f)}$$

with $fcard$ the fuzzy cardinal of a fuzzy set as defined in [7].

This can be interpreted as counting the number of elements inside F weighted by their membership to f (see Figure 6).

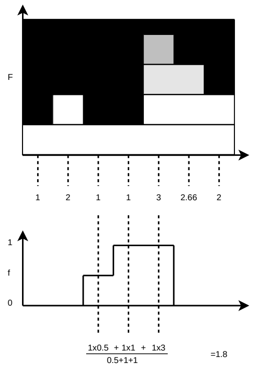


Figure 6: Fuzzy mean computation.

Definition 7.4. The fuzzy contrast $fctrast : \Omega \times \mathcal{F} \times \mathcal{S} \rightarrow \mathbb{R}$ is defined as:

$$\begin{aligned} &\forall F \in \mathcal{F}, \forall f \in \mathcal{S}, \forall h \in \mathcal{S} \\ &fctrast_h(F, f) = \; fmean(F, f) - \\ &\qquad\qquad fmean(F, D_h(f) \cap \overline{f}) \end{aligned}$$

with $\forall f \in \mathcal{S}, \forall p \in \Omega \quad \overline{f}(p) = 1 - f(p)$.

The idea is here to subtract the mean inside and outside the component. *Inside* the component is modeled by f, and *outside* is described by the intersection of $\overline{f}$ and the dilation of f.

Let $\Psi_{mass} = \{\psi^F_{mass} : \mathcal{S} \rightarrow \mathcal{S} / F \in \mathcal{F}\}$ be an operators set such that $\forall f \in \mathcal{S}, \forall p \in \Omega$:

$$\begin{aligned} &\psi^F_{mass}(f)(p) = \\ &\quad \max_{h \in Q(f)} (\min(h(p), t_{u_1}(fcard(h))) \\ &\quad \top \max_{h \in Q(f)} (\min(h(p), r_{u_2}(fcomp(h))) \\ &\quad \top \max_{h \in Q(f)} (\min(h(p), r_{u_3}(fctrast(F, h))) \end{aligned}$$

with $fcomp$ the fuzzy compacity of a fuzzy set as defined in [7], $t : \mathbb{R}^4 \times \mathbb{R} \rightarrow [0, 1]$ the trapeze function, $r : \mathbb{R}^2 \times \mathbb{R} \rightarrow [0, 1]$ the ramp function and where $u_1 \in \mathbb{R}^4$, $u_2 \in \mathbb{R}^2$ and $u_3 \in \mathbb{R}^2$ are some constants.

This Ψ_{mass} is composed of ψ_{mass} independant to g. Each operator can be interpreted as the aggregation of membership degrees corresponding to measures computed on the objects (fuzzy connected components) of f using data of F. Here, the wanted objects that are compact, contrasted and within a given size range.

The gray scale operator $\delta_{\Psi_{mass}}$ associated to this Ψ_{mass} can be defined as:

$$\begin{aligned} &\forall F \in \mathcal{F}, \forall p \in \Omega, \forall g \in E \\ &\delta_{\Psi_{mass}}(F)(p, g) = \psi^F_{mass}(F(*, g))(p) \end{aligned}$$

Theorem 7.5. *Ψ_{mass} is a SWFCO.*

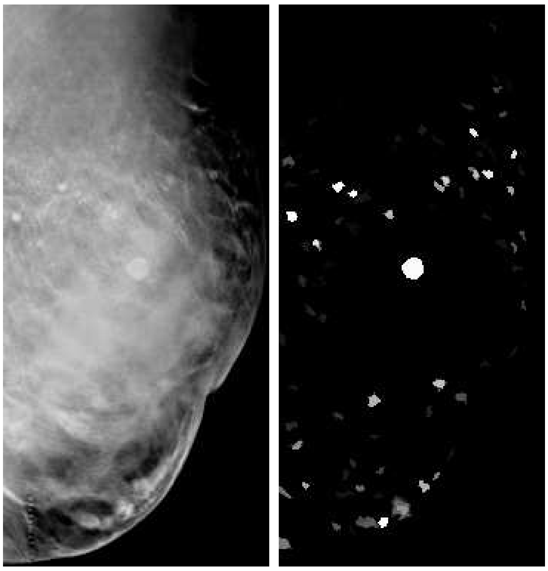
Proof. It can be shown that the $\psi^{g,F}$ are fuzzy connected using the fact that the max and min are FC operators. Then the anti-extensivity can be shown using the t-norm monotony. $\square$

This family of operators provides weak properties compared to other families (SBFCO, SOFCO or SEFCO), nonetheless, it can define an operator working on fuzzy images ($\mathcal{F}$) that is looking for objects verifying geometrical properties in the image.

A practical usage of this filter, to mark circumscribed masses in a 3D mammography I can be done using the following scheme:

- conversion of the mammography I to a fuzzy umbra image F: processing using um, and imprecision introduction in the umbra image for instance (note that while the last step enables to use a modeling of the image imperfections, this one was skipped in the proposed example),
- computation of $\delta_{\Psi_{mass}}(F)$,
- partial defuzzyfication [5] using the operator agg.

Figure 7 illustrates a result from such a process. Here, a slice (instead of the whole volume) of a breast reconstructed with iterative techniques is filtered. The resulting image can be interpreted for each pixel p of Ω as a mem-



a b

Figure 7: Filtering (b) of a digital breast tomosynthesis slice (a) using $\delta_{\Psi_{mass}}$.

bership degree to the class *circumscribed object*. Furthermore, because of the fuzzy sets usage, these degrees enable to evaluate how much the objects verify the given criteria, whereas it would not be possible with a crisp formulation of the same filter (all or nothing response). Thus the output of the filter may be a suitable input for other processing like classification using fuzzy decision trees [6].

8 Conclusion

A possible extension of connected filters into the fuzzy set framework was proposed. This formulation relies on the concept of fuzzy umbra images, which is a possible straightforward extension of umbra images which allows expressing images with imprecision on their gray levels.

The formulation proposed allows expressing classical filters like area opening or volume levelling, and thus seems to be a coherent extension for fuzzy images. Furthermore, using the flexibility provided by the fuzzy sets, new filters can be designed like for instance a detector of circumscribed objects, which may be a good marker for circumscribed lesions in DBT images.

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