Explanatory Relations Based on Mathematical Morphology

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Abstract. Using mathematical morphology on formulas introduced recently by Bloch and Lang (*Proceedings of IPMU'2000*) we define two new explanatory relations. Their logical behavior is analyzed. The results show that these natural ways for defining preferred explanations are robust because these relations satisfy almost all postulates of explanatory reasoning introduced by Pino-Pérez and Uzcátegui (*Artificial Intelligence,* 111:131–169, 1999). Actually, the first explanatory relation is Explanatory-Rational. The second one is not even Explanatory-Cumulative but it satisfies new weak postulates.

1 Introduction

The process of inferring the best explanation of an observation is usually known as *abduction.* In the logic-based approach to abduction, the background theory is given by a consistent set of formulas Σ. The notion of a *possible explanation* is defined by saying that a formula γ is an explanation of α if $\Sigma \cup \{\gamma\} \vdash \alpha$. An explanatory relation is a binary relation $\rhd$ where the intended meaning of $\alpha \rhd \gamma$ is "γ is a *preferred explanation* of α".

In [4], a set of postulates that should be satisfied by preferred explanatory relations is proposed and discussed.

The aim of this work is at least threefold. First, to propose very natural explanatory relations that in some cases are computationally practicable. Second, to examine the adequacy of logical postulates proposed in [4] and third, the discovery of new logical properties for the explanatory reasoning.

In order to accomplish our goals we propose concrete definitions of preferred explanations based on mathematical morphology. The starting point is a very general setting: a relation between worlds that in most of the cases can be viewed as a graph connecting worlds.

Mathematical morphology operators on logical formulas have been introduced recently in [1]. These ideas allow us to define the *most central part* of a formula, according to the fundamental principles of this theory (see e.g. [6,7]). Using this notion we define two explanatory relations. The first one, $\rhd^{\ell ne}$, has

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the following intended meaning: γ is a preferred explanation of α if γ is a formula entailing the *most central part* of the conjunction of Σ with α. For the second one, $\rhd^{\ell c}$, we define a sequence which approximates the most central part of Σ; then we say that γ is a preferred explanation of α if γ implies the conjunction of α with the closest element of the sequence which is consistent with α.

2 Preliminaries

Let us recall here the basic principles of morpho-logics. Let PS be a finite set of propositional symbols. The language is generated by PS and the usual connectives. Well-formed formulas will be denoted by Greek letters φ, ψ... Worlds will be denoted by ω, ω'... and the set of all worlds by Ω. $Mod(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$ is the set of all worlds where φ is satisfied. Dilation and erosion (the two fundamental operations of mathematical morphology [6]) of a formula φ by a structuring element B have been defined in [1] as follows:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset\}, \tag{1}$$

$$Mod(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}. \tag{2}$$

In these equations, the structuring element B represents a relationship between worlds, i.e. $\omega' \in B(\omega)$ iff ω' satisfies some relationship with ω. The condition in Equation 1 expresses that the set of worlds in relation to ω should be consistent with φ, i.e.: $\exists \omega' \in B(\omega),\ \omega' \models \varphi$. The condition in Equation 2 is stronger and expresses that φ should be satisfied in all worlds which stand in relation to ω .

2.1 Properties

The properties of these basic operations and of other derived operations are detailed in [1]. The fundamental properties of erosion, that will be used intensively in the following, can be summarized as:

- Independence of the syntax (follows directly from the definition through the models).
- Monotonicity: erosion is increasing with respect to φ, i.e.

$$\varphi \vdash \psi \ \Rightarrow\ E_B(\varphi) \vdash E_B(\psi), \tag{3}$$

  for any structuring element B. Erosion is decreasing with respect to the structuring element, i.e.

$$\forall \omega \in \Omega, B_\omega \subset B'_\omega \ \Rightarrow\ E_{B'}(\varphi) \vdash E_B(\varphi). \tag{4}$$

- Anti-extensivity[1]: if B is derived from a reflexive relation, i.e. such that $\forall \omega \in \Omega, \omega \in B_\omega$, the erosion is anti-extensive, i.e.

$$E_B(\varphi) \vdash \varphi. \tag{5}$$

[1] In set theoretical mathematical morphology an operation Ψ is said anti-extensive iff for any set X, $\Psi(X) \subset X$.

We will only deal with such cases in what follows. We will also consider symmetrical relations, i.e. $\forall(\omega,\omega') \in \Omega^2, \omega \in B_{\omega'} \Leftrightarrow \omega' \in B_\omega$.

- Iteration: Erosion satisfies an iteration property, which is expressed for symmetrical structuring elements as:

$$E_B[E_{B'}(\varphi)] = E_{D_B(B')}(\varphi). \tag{6}$$

For instance if $B = B'$, and if we denote by E^n the erosion by B dilated $(n-1)$ times by itself (this is typically the case for distance based operations where the structuring element is a ball of distance, as will be seen in Section 2.2), we have:

$$E^{n+n'}(\varphi) = E^{n'}[E^n(\varphi)] = E^n[E^{n'}(\varphi)], \tag{7}$$

where n, n' denote the size of the erosion (i.e. the "radius" of the structuring element).

- Commutativity with conjunction:

$$E_B(\wedge_{i=1}^m \varphi_i) = \wedge_{i=1}^m E_B(\varphi_i). \tag{8}$$

- Erosion of a disjunction: erosion and disjunction do not commute, but we have a partial relation:

$$E_B(\varphi) \vee E_B(\psi) \vdash E_B(\varphi \vee \psi). \tag{9}$$

2.2 Illustrative Example

In all what follows, we will consider as an illustrative example the case where the structuring element is defined as a ball of the Hamming distance between worlds d_H, where $d_H(\omega,\omega')$ is the number of propositional symbols that are instantiated differently in both worlds. Then dilation and erosion of size n are defined from Equations 1 and 2 by using the distance balls of radius n as structuring elements:

$$Mod(D^n(\varphi)) = \{\omega \in \Omega \mid \exists \omega' \in \Omega, \omega' \models \varphi \textit{ and } d_H(\omega,\omega') \leq n\}, \tag{10}$$

$$Mod(E^n(\varphi)) = \{\omega \in \Omega \mid \forall \omega' \in \Omega, d_H(\omega,\omega') \leq n \Rightarrow \omega' \models \varphi\}. \tag{11}$$

We make use of a graph representation of worlds, where each node represents a world and a link represents an elementary connection between two worlds, i.e. being at distance 1 from each other. A ball of radius 1 centered at ω is constituted by ω and the extremities of the arcs originating in ω. This allows for an easy visualization of the effects of transformations.

Let us consider an example with three propositional symbols a, b, c. The possible worlds are represented in Figure 1.

Let us consider $\varphi = \neg a \wedge b \wedge c$. Then we have:

$$D^1(\varphi) = (\neg a \wedge b) \vee (\neg a \wedge c) \vee (b \wedge c),$$

$$D^2(\varphi) = \neg a \vee b \vee c = \neg(a \wedge \neg b \wedge \neg c).$$

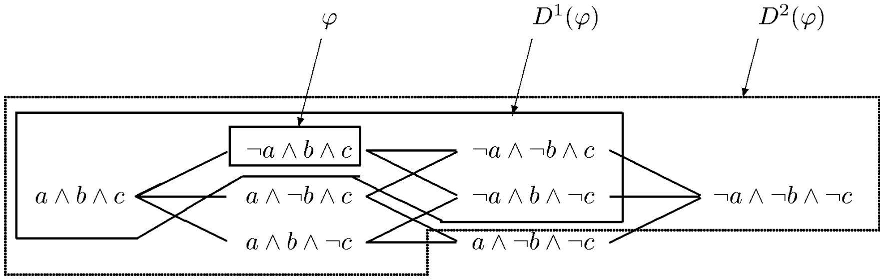

Fig. 1. Graph representation of possible worlds with 3 symbols and an example of φ and two successive dilations. An arc between two nodes means that the corresponding nodes are at a distance to each other equal to 1.

These results are illustrated in Figure 1. Notice that in this kind of figures the formula defined by a border is the disjunction of the formulas in the interior of the border.

Erosion can be computed very easily from any conjunctive normal form. Indeed, if φ is a disjunction of literals, i.e., $\varphi = l_1 \vee l_2 \vee ... \vee l_n$, then we have:

$$E^1(\varphi) = \wedge_{j=1}^{n}(\vee_{i \neq j} l_i). \tag{12}$$

This property, along with the commutativity of erosion with conjunction, allows to compute easily the erosion of any formula expressed as a CNF.

3 Explanatory Relations Based on Erosion

In this section we define precisely the concept of *most central part* of a formula with the help of the erosion operator. Then, based on this concept, we define two explanatory relations.

3.1 Last Non-empty Erosion

We denote by $E_\ell(\varphi)$ the last erosion of φ, i.e. the erosion of φ of the largest possible size such that the set of worlds where $E_\ell(\varphi)$ is satisfied is not empty:

$$E_\ell(\varphi) = E^n(\varphi) \Leftrightarrow \begin{cases} E^n(\varphi) \nvdash \bot, \\ \text{and } \forall m > n,\ E^m(\varphi) \vdash \bot. \end{cases} \tag{13}$$

By convention, we set $E^0(\varphi) = \varphi$. Note that last erosion is different from the classical notion of ultimate erosion in mathematical morphology[2]. We define the most central part of a formula as its last erosion. This concept is similar to one used in preference modeling in [3].

[2] The ultimate eorsion is obtained by successive erosions, and is defined as the union of the connected components that disappear from one step to the other.

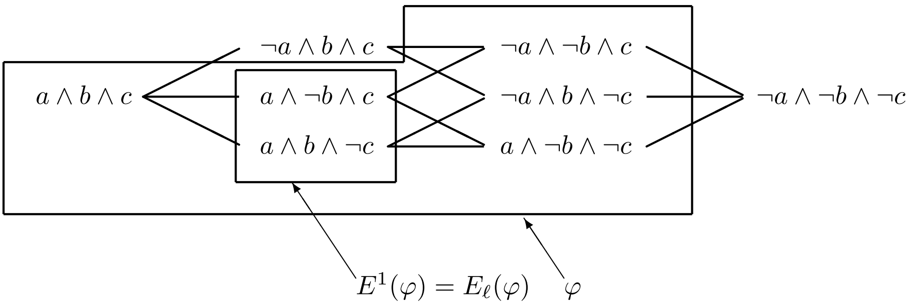

Fig. 2. An example of φ and its last erosion.

Let us consider the illustrative example of Section 2.2. Let us take (see Figure 2):

$$\varphi = (a \vee \neg b \vee \neg c) \wedge (a \vee b \vee c).$$

Using Equations 8 and 12, we derive:

$$E^1(\varphi) = (a \vee \neg b) \wedge (a \vee \neg c) \wedge (\neg b \vee \neg c) \wedge (a \vee b) \wedge (a \vee c) \wedge (b \vee c) = (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c).$$

Since $E^2(\varphi) \vdash_\Sigma \bot$, we have $E^1(\varphi) = E_\ell(\varphi)$.

A preferred explanation of α is then defined from this operator applied on $\Sigma \wedge \alpha$, more precisely:

$$\alpha \rhd^{\ell ne} \gamma \quad \overset{def}{\Leftrightarrow} \quad \gamma \vdash E_\ell(\Sigma \wedge \alpha). \tag{14}$$

The idea of taking the last erosion of $\Sigma \wedge \alpha$ can be interpreted in terms of robustness. An erosion of size n of a formula is a formula that can be changed while still proving the initial formula. If at most n symbols are changed in $E^n(\varphi)$ then φ is always satisfied. Here, considering $E_\ell(\Sigma \wedge \alpha)$ means that we are looking at the most reduced formula that satisfies $\Sigma \wedge \alpha$, i.e. the one that can be changed the most while satisfying $\Sigma \wedge \alpha$.

Let us take $\Sigma \wedge \alpha = \varphi$ where φ is defined as in the previous example (Figure 2). For Definition 14, if we denote $PE_{\rhd^{\ell ne}}(\alpha) = \{\gamma \ : \ \alpha \rhd^{\ell ne} \gamma\}$ (the preferred explanations of α), we have:

$$PE_{\rhd^{\ell ne}}(\alpha) = \{(a \wedge \neg b \wedge c), (a \wedge b \wedge \neg c), (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)\}.$$

One potential problem with last erosion is that it does not represent all "parts" of a formula. Let us take for instance: $\Sigma \wedge \alpha = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ and $\Sigma \wedge \beta = ((a \vee b) \wedge (a \vee c) \wedge (b \vee c)) \vee (\neg a \wedge \neg b \wedge \neg c)$. Then we have: $E_\ell(\Sigma \wedge \alpha) = E_\ell(\Sigma \wedge \beta) = a \wedge b \wedge c$ and $PE_{\rhd^{\ell ne}}(\alpha) = PE_{\rhd^{\ell ne}}(\beta)$. The set of worlds satisfying $\Sigma \wedge \beta$ is disconnected, and the connected component containing only $(\neg a \wedge \neg b \wedge \neg c)$ is not represented in the explanations of β. If this is considered to be a problem, it can be avoided by considering the ultimate erosion instead of the last erosion.

3.2 Last Consistent Erosion

Another idea consists in eroding Σ as much as possible but still under the constraint that it remains consistent with α:

$$E_{\ell c}(\Sigma, \alpha) = E^n(\Sigma) \text{ where } n = \max\{k : E^k(\Sigma) \wedge \alpha \not\vdash \perp\}. \quad (15)$$

From this operator, we define the following explanatory relation:

$$\alpha \rhd^{\ell c} \gamma \overset{def}{\Leftrightarrow} \gamma \vdash E_{\ell c}(\Sigma, \alpha) \wedge \alpha, \quad (16)$$

This definition has a different interpretation. Here we consider erosion of Σ alone, which means that we are looking at the formulas that satisfy α while being the most in the theory, i.e. that can be changed while remaining in the theory (but not necessary satisfying α after the changes).

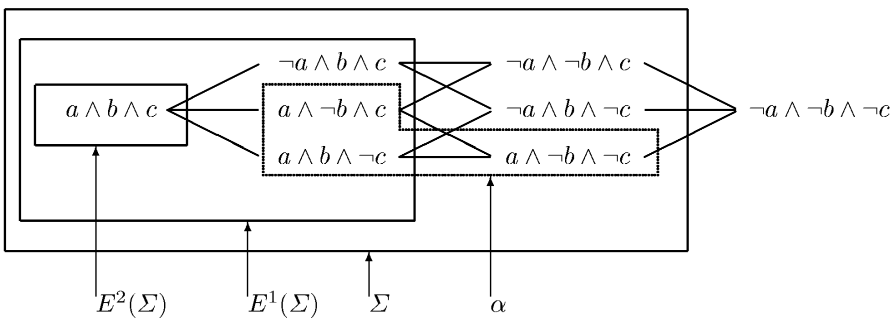

Fig. 3. An example of last consistent erosion.

Let us come back to the illustrative example, and take (see Figure 3): $\Sigma = a \vee b \vee c$, and $\alpha = (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c)$. We have: $E^1(\Sigma) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$, $E^2(\Sigma) = a \wedge b \wedge c$, and finally $E^3(\Sigma) \vdash \perp$. Therefore:

$$E^1(\Sigma) \wedge \alpha = (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$$

and $E^2(\Sigma) \wedge \alpha \vdash \perp$. Therefore the value of n in Definition 16 is equal to 1. For Definition 16, γ can be anything in the set

$$PE_{\rhd^{\ell c}}(\alpha) = \{(a \wedge \neg b \wedge c), (a \wedge b \wedge \neg c), (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)\}.$$

There is an alternative way of looking at $\rhd^{\ell c}$ which will be particularly useful in the next section. The iteration of the erosion operator provides a method of linearly pre-ordering the models of Σ. Consider the following relation among models.

$$\omega \leq \omega' \overset{def}{\Leftrightarrow} \forall k \ (\omega' \in E^k(\Sigma) \to \omega \in E^k(\Sigma)).$$

It is clear that $\leq$ is a total pre-order and it is not difficult to verify that the following holds:

$$\alpha \rhd^{\ell c} \gamma \iff \text{mod}(\Sigma \cup \{\gamma\}) \subseteq \min(\text{mod}(\Sigma \cup \{\alpha\}), \leq). \quad (17)$$

4 Rationality Postulates

In this section we study the properties of the two proposed explanatory relations according to the postulates introduced in [4]. The basic rationality postulates for explanatory relations are the following (we use the notation $\alpha \vdash_\Sigma \beta$ instead of $\Sigma \cup \{\alpha\}$):

LLE$_\Sigma$:
$$\frac{\vdash_\Sigma \alpha \leftrightarrow \alpha' \ , \ \ \alpha \rhd \gamma}{\alpha' \rhd \gamma}$$

RLE$_\Sigma$:
$$\frac{\vdash_\Sigma \gamma \leftrightarrow \gamma' \ ; \ \ \alpha \rhd \gamma}{\alpha \rhd \gamma'}$$

E-CM:
$$\frac{\alpha \rhd \gamma \ ; \ \gamma \vdash_\Sigma \beta}{(\alpha \wedge \beta) \rhd \gamma}$$

E-C-Cut:
$$\frac{(\alpha \wedge \beta) \rhd \gamma \ , \ \ \forall \delta \, [\alpha \rhd \delta \ \Rightarrow \ \delta \vdash_\Sigma \beta \,]}{\alpha \rhd \gamma}$$

RA:
$$\frac{\alpha \rhd \gamma \ ; \ \gamma' \vdash_\Sigma \gamma \ ; \ \gamma' \not\vdash_\Sigma \bot}{\alpha \rhd \gamma'}$$

E-RW:
$$\frac{\alpha \rhd \gamma \ ; \ \alpha \rhd \delta}{\alpha \rhd (\gamma \vee \delta)}$$

LOR:
$$\frac{\alpha \rhd \gamma \ ; \ \beta \rhd \gamma}{(\alpha \vee \beta) \rhd \gamma}$$

E-DR:
$$\frac{\alpha \rhd \gamma \ ; \ \beta \rhd \delta}{(\alpha \vee \beta) \rhd \gamma \ \text{ or } \ (\alpha \vee \beta) \rhd \delta}$$

E-R-Cut:
$$\frac{(\alpha \wedge \beta) \rhd \gamma \ ; \ \exists \delta \, [\alpha \rhd \delta \ \& \ \delta \vdash_\Sigma \beta]}{\alpha \rhd \gamma}$$

E-Reflexivity :
$$\frac{\alpha \rhd \gamma}{\gamma \rhd \gamma}$$

E-Con$_\Sigma$: $\not\vdash_\Sigma \neg\alpha$ iff there is γ such that $\alpha \rhd \gamma$

The intended meaning and motivation for these postulates can be found in [4].

It is immediate from the definition of $\rhd^{\ell c}$ and $\rhd^{\ell ne}$ that LLE$_\Sigma$, RLE$_\Sigma$, RA, E-RW, and E-Con$_\Sigma$ are satisfied. Moreover, from the representation of $\rhd^{\ell c}$ given by equation 17 and some general results of [4] we get the following proposition.

Proposition 1. $\rhd^{\ell c}$ *is a causal E-rational explanatory relation. In particular, it satisfies* LLE$_\Sigma$, RLE$_\Sigma$, RA, E-RW, E-Con$_\Sigma$, E-CM *and* E-R-Cut.

From the results in [4] we also know that by being E-rational, $\rhd^{\ell c}$ also satisfies E-C-Cut, E-Reflexivity, E-DR and LOR. However, the situation for $\rhd^{\ell ne}$ is quite different since, as we will see below, the basic postulates E-CM and E-C-Cut do not hold.

We will provide now a counter-example of E-CM for $\rhd^{\ell ne}$. Let us consider our illustrative example (see Section 2.2), and take the following formulas (see Figure 4):

$$\Sigma \wedge \alpha = \neg a \vee b \vee c,$$

$$\Sigma \wedge \alpha \wedge \beta = \neg[(a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge \neg c)] = (\neg a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee b \vee c).$$

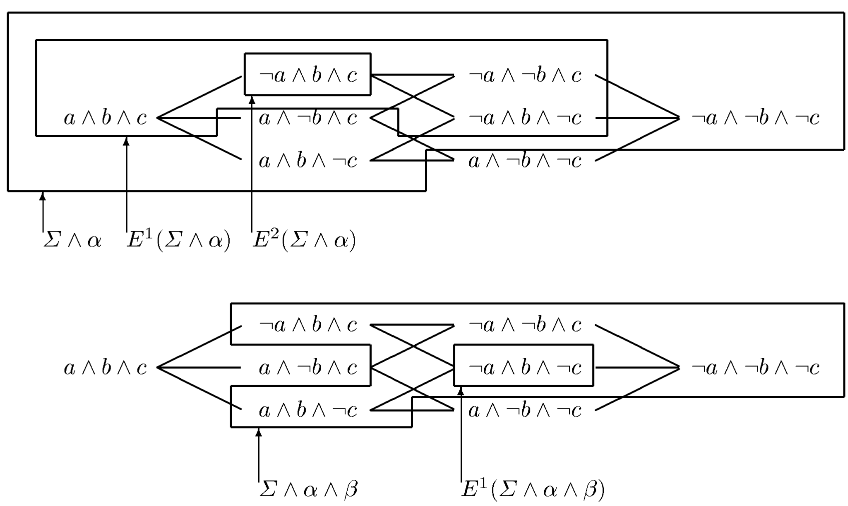

Fig. 4. A counter-example for **E-CM**.

Using the computation formulas for erosion of a formula under CNF (Equations 8 and 12), we get:

$$E^1(\Sigma \wedge \alpha) = (\neg a \vee b) \wedge (\neg a \vee c) \wedge (b \vee c),$$

$$E^2(\Sigma \wedge \alpha) = \neg a \wedge b \wedge c = E_\ell(\Sigma \wedge \alpha).$$

A unique world satisfies this formula, and therefore no further erosion can be performed ($E^3(\Sigma \wedge \alpha) \vdash_\Sigma \bot$). Similarly, we have:

$$E^1(\Sigma \wedge \alpha \wedge \beta) = \neg a \wedge b \wedge \neg c = E_\ell(\Sigma \wedge \alpha \wedge \beta)$$

which is the last non-empty erosion. It follows that $\alpha \rhd^{\ell ne} (\neg a \wedge b \wedge c)$ but clearly $\neg a \wedge b \wedge c$ is not a preferred explanation of $\alpha \wedge \beta$.

Now we will present a counterexample of E-C-Cut for $\rhd^{\ell ne}$. Consider

$$\Sigma \wedge \alpha = a \vee b \vee c,$$

$$\Sigma \wedge \beta = a \vee \neg b \vee \neg c.$$

We have then:

$$E^1(\Sigma \wedge \alpha) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c),$$

$$E^2(\Sigma \wedge \alpha) = a \wedge b \wedge c = E_\ell(\Sigma \wedge \alpha),$$

$$E^1(\Sigma \wedge \beta) = (a \vee \neg b) \wedge (a \vee \neg c) \wedge (\neg b \vee \neg c),$$

$$E^2(\Sigma \wedge \beta) = a \wedge \neg b \wedge \neg c = E_\ell(\Sigma \wedge \beta),$$

$$\Sigma \wedge \alpha \wedge \beta = (a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c),$$

$$E(\Sigma \wedge \alpha \wedge \beta) = (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c) = E_\ell(\Sigma \wedge \alpha \wedge \beta).$$

Let us now put $\gamma = (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c)$, then $(\alpha \wedge \beta) \rhd^{\ell ne} \gamma$. Then it is clear that $\alpha \not\rhd^{\ell ne} \gamma$. On the other hand, we have that $\alpha \rhd^{\ell ne} \delta$ iff $\delta \equiv a \wedge b \wedge c$. Thus if $\alpha \rhd^{\ell ne} \delta$, then $\delta \vdash_\Sigma \beta$.

We introduce a weaker form of these postulates:

E-W-CM: $$\frac{\alpha \rhd \gamma \ ; \ \beta \rhd \gamma}{(\alpha \wedge \beta) \rhd \gamma}$$

E-W-C-Cut: $$\frac{(\alpha \wedge \beta) \rhd \gamma \ , \ \forall \delta \, [\alpha \rhd \delta \Rightarrow \beta \rhd \delta \,]}{\alpha \rhd \gamma}$$

These new postulates might look even more natural than the original version E-CM and E-C-Cut. However, $\rhd^{\ell ne}$ is the first natural non trivial example known in the literature that satisfies E-W-CM and E-W-C-Cut but neither E-CM nor E-C-Cut[3]. There is a natural weakening of E-R-Cut which can be considered but we do not have any example for it in which the preferred explanations are not unique.

The next proposition collects all the facts we know about $\rhd^{\ell ne}$.

Proposition 2. *The explanatory relation* $\rhd^{\ell ne}$ *satisfies* LLE$_\Sigma$, RLE$_\Sigma$, RA, E-RW, E-W-CM, E-W-C-Cut, E-Reflexivity *and* E-Con$_\Sigma$*.*

Proof: (i) E-W-CM. Let us assume that $\gamma \vdash_\Sigma E_\ell(\Sigma \wedge \alpha)$ with $E_\ell(\Sigma \wedge \alpha) = E^n(\Sigma \wedge \alpha)$ and $\gamma \vdash_\Sigma E_\ell(\Sigma \wedge \beta)$ with $E_\ell(\Sigma \wedge \beta) = E^m(\Sigma \wedge \beta)$. Let us assume that the last non-empty erosion of $\Sigma \wedge \alpha \wedge \beta$ is obtained for k. We have, due to Equation 8: $E_\ell(\Sigma \wedge \alpha \wedge \beta) = E^k(\Sigma \wedge \alpha \wedge \beta) = E^k(\Sigma \wedge \alpha) \wedge E^k(\Sigma \wedge \beta)$.

[3] E-W-CM in fact was already considered by Flach [2] but he did not provide any example for it not satisfying already the stronger version E-CM

We necessarily have $k \leq n$ and $k \leq m$ since otherwise either $E^k(\Sigma \wedge \alpha)$ or $E^k(\Sigma \wedge \beta)$ would be inconsistent. This implies, due to the monotonicity property of erosion (Equation 4) that: $\vdash_\Sigma E^n(\Sigma \wedge \alpha) \rightarrow E^k(\Sigma \wedge \alpha)$ and $\vdash_\Sigma E^m(\Sigma \wedge \beta) \rightarrow E^k(\Sigma \wedge \beta)$ from which we derive:

$$\vdash_\Sigma E_\ell(\Sigma \wedge \alpha) \wedge E_\ell(\Sigma \wedge \beta) \rightarrow E_\ell(\Sigma \wedge \alpha \wedge \beta).$$

This interesting general result proves that $\gamma \vdash_\Sigma E_\ell(\Sigma \wedge \alpha \wedge \beta)$.

(ii) E-W-C-Cut. Let $\gamma \vdash_\Sigma E_\ell(\Sigma \wedge \alpha \wedge \beta) = E^n(\Sigma \wedge \alpha \wedge \beta)$. For all δ such that $\alpha \rhd \delta$, $\delta \vdash_\Sigma E_\ell(\Sigma \wedge \alpha) = E^m(\Sigma \wedge \alpha)$. Since $\Sigma \wedge \alpha \wedge \beta \vdash_\Sigma \Sigma \wedge \alpha$ we have:

$$E^n(\Sigma \wedge \alpha \wedge \beta) \not\vdash_\Sigma \bot \Rightarrow E^n(\Sigma \wedge \alpha) \not\vdash_\Sigma \bot.$$

Therefore $n \leq m$.

Let us first assume that $n < m$. For all δ such that $\alpha \rhd \delta$, we have $\beta \rhd \delta$, i.e. $\delta \vdash_\Sigma E_\ell(\Sigma \wedge \beta) = E^k(\Sigma \wedge \beta)$. For the same reason as before, we necessarily have $n \leq k$. Since the set of preferred explanations of α is included in the one of β, we have: $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \beta)$. Since $m > n$, we have:

$$E^m(\Sigma \wedge \alpha \wedge \beta) = E^m(\Sigma \wedge \alpha) \wedge E^m(\Sigma \wedge \beta) \vdash_\Sigma \bot.$$

Let us now assume $n < k$. Then similarly, we have:

$$E^k(\Sigma \wedge \alpha \wedge \beta) = E^k(\Sigma \wedge \alpha) \wedge E^k(\Sigma \wedge \beta) \vdash_\Sigma \bot.$$

If $k > m$, we have: $E^m(\Sigma \wedge \beta) \not\vdash_\Sigma \bot$, and, due to Equation 4: $E^k(\Sigma \wedge \beta) \vdash_\Sigma E^m(\Sigma \wedge \beta)$. Therefore: $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \beta) \vdash_\Sigma E^m(\Sigma \wedge \beta)$, which implies: $E^m(\Sigma \wedge \alpha \wedge \beta) \not\vdash_\Sigma \bot$ which leads to a contradiction.

Similarly, if $k < m$, we have: $E^k(\Sigma \wedge \alpha) \not\vdash_\Sigma \bot$, and $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \alpha)$. Therefore, since we had $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \beta)$, we have:

$$E^k(\Sigma \wedge \alpha \wedge \beta) = E^k(\Sigma \wedge \alpha) \wedge E^k(\Sigma \wedge \beta) \not\vdash_\Sigma \bot$$

which also leads to a contradiction. From these two contradictions, we can conclude that necessarily $k = m$. Then $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \beta)$ becomes $E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^m(\Sigma \wedge \beta)$ and therefore we have:

$$E^m(\Sigma \wedge \alpha \wedge \beta) = E^m(\Sigma \wedge \alpha) \not\vdash_\Sigma \bot$$

which is in contradiction with $n < m$. Therefore we also have $n = m$.

Finally the only possibility is to have $k = n = m$. In this case, we have:

$$E^n(\Sigma \wedge \alpha \wedge \beta) \vdash_\Sigma E^n(\Sigma \wedge \alpha) = E^m(\Sigma \wedge \alpha) \vdash_\Sigma E^k(\Sigma \wedge \beta),$$

and therefore:

$$\gamma \vdash_\Sigma E^n(\Sigma \wedge \alpha \wedge \beta) \Rightarrow \gamma \vdash_\Sigma E^n(\Sigma \wedge \alpha),$$

i.e. $\alpha \rhd \gamma$.

(iii) E-Reflexivity. The definition of $\rhd^{\ell ne}$ is based on the notion of largest possible erosion, and therefore no further erosion can be performed. More precisely, let $\alpha \rhd^{\ell ne} \gamma$ and suppose that the last non empty erosion of $\Sigma \wedge \alpha$ is $E^n(\Sigma \wedge \alpha)$. Then we have:

$$E^0(\Sigma \wedge \gamma) = \Sigma \wedge \gamma = \gamma$$

and

$$E^1(\Sigma \wedge \gamma) = E^{n+1}(\Sigma \wedge \alpha)$$

which is inconsistent. Therefore $\gamma \rhd^{\ell ne} \gamma$. □

We end this section by considering the postulate LOR. We will give a counter-example of it for $\rhd^{\ell ne}$. Consider

$$\Sigma \wedge \alpha = (a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c)$$

and

$$\Sigma \wedge \beta = (\neg a \vee \neg b \vee c) \wedge (a \vee \neg b \vee c) \wedge (a \vee b \vee c).$$

We have:

$$E^1(\Sigma \wedge \alpha) = (a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c) = E_\ell(\Sigma \wedge \alpha),$$
$$E^1(\Sigma \wedge \beta) = a \wedge \neg b \wedge c = E_\ell(\Sigma \wedge \alpha),$$
$$\Sigma \wedge (\alpha \vee \beta) = a \vee b \vee c,$$
$$E^1(\Sigma \wedge (\alpha \vee \beta)) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c),$$
$$E^2(\Sigma \wedge (\alpha \vee \beta)) = a \wedge b \wedge c = E_\ell(\Sigma \wedge (\alpha \vee \beta)).$$

Let $\gamma = a \wedge \neg b \wedge c$. Then $\alpha \rhd^{\ell ne} \gamma$ and $\beta \rhd^{\ell ne} \gamma$, but $(\alpha \vee \beta) \not\rhd^{\ell ne} \gamma$.

Since E-DR implies LOR [4], then we already know that E-DR fails for $\rhd^{\ell ne}$.

Table 1 summarizes the results we obtained so far.

5 Conclusion

We have proposed in this paper two definitions of explanatory relations based on morphological erosion. Several other definitions could be developed based on mathematical morphology. For instance if we replace $\vdash$ by $=$ in Equations 14 and 16, we come up with definitions that have slightly different properties (in particular RA is not satisfied). More importantly, it is natural to use other morphological operators instead of erosion, for example the ultimate erosion.

It is important to observe that erosion provides a geometrical way to totally pre-order the models of a formula and this is the underlying idea behind the definition of $\rhd^{\ell c}$.

Another interesting feature of this work is that it reveals new properties as E-W-C-Cut and new aspects of E-W-CM. These two postulates are very natural; they are the weakening of the well known E-CM and E-C-Cut. But until now the methods used to define explanatory relations always yield relations satisfying the strongest ones. So the method presented here to construct $\rhd^{\ell ne}$ is indeed a new way of approaching the problem of selecting preferred explanations of an observation.

Table 1. Properties of the proposed relations.

Property	$\rhd^{\ell ne}$ (Equation 14)	$\rhd^{\ell c}$ (Equation 16)
LLE	$\surd$	$\surd$
RLE	$\surd$	$\surd$
E-CM	$\times$	$\surd$
E-W-CM	$\surd$	$\surd$
E-C-Cut	$\times$	$\surd$
E-R-Cut	$\times$	$\surd$
E-W-C-Cut	$\surd$	$\surd$
E-Reflexivity	$\surd$	$\surd$
E-RW	$\surd$	$\surd$
RA	$\surd$	$\surd$
LOR	$\times$	$\surd$
E-DR	$\times$	$\surd$
E-Con$_{\Sigma}$	$\surd$	$\surd$

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