

Fuzzy Modeling and Evaluation of the Spatial Relation “Along”

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Abstract. The analysis of spatial relations among objects in an image is an important vision problem that involves both shape analysis and structural pattern recognition. In this paper, we propose a new approach to characterize the spatial relation *along*, an important feature of spatial configuration in space that has been overlooked in the literature up to now. We propose a mathematical definition of the degree to which an object A is along an object B , based on the region *between* A and B and a degree of elongatedness of this region. In order to better fit the perceptual meaning of the relation, distance information is included as well. Experimental results obtained using synthetic shapes and brain structures in medical imaging corroborate the proposed model and the derived measures, thus showing their adequation with the common sense.

1 Introduction

To our knowledge, the only work addressing *alongness* between objects by giving mathematical definitions was developed in the context of geographic information systems (GIS) [1]. In this work, the relation *along* between a line and an object is defined as the length of the intersection of the line and the boundary of the object, normalized either by the length of this boundary (*perimeter alongness*) or by the length of the line (*line alongness*). In these definitions, the boundary can also be extended to a *buffer zone* around the boundary. Crevier [2] addresses the problem of spatial relationships between line segments by detecting collinear chains of segments based on the probability that successive segments belong to the same underlying structure. However this approach cannot be directly extended to any object shape.

Here we consider the more general case where both objects can have any shape, and where they are not necessarily adjacent. For computer vision applications, the considered objects can be obtained for instance from a crisp or fuzzy segmentation of digital images.

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The *along* relation is an intrinsically vague notion. Indeed, in numerous situations even of moderate complexity, it is difficult to provide a definite binary answer to the question “is A *along* B ?”, and the answer should rather be a matter of degree. Therefore fuzzy modeling is appropriate. Now if the objects are themselves imprecisely defined, as fuzzy sets, this induces a second level of fuzziness. In this paper, we propose a fuzzy model of the relation *along*, for both crisp and fuzzy objects. It is based on a measure of elongatedness of the region *between* both objects.

In Section 2 we motivate our work based on a few references to other domains such as psychophysics or linguistics. We propose a mathematical model and a measure of *alongness* between crisp objects in Section 3. Their generalization to fuzzy objects is discussed in Section 4. Experimental results using both synthetic and real objects are shown in Section 5. Some properties and possible extensions are provided in Section 6.

2 Spatial Relations and Motivation for Using Fuzzy Definitions

According to Biederman[3], any object, even the simplest one, may project an infinity of image configurations to the retina considering orientation and, consequently, the bidimensional projection, possible occlusion, texture complexity, or if it is a novel exemplar of its particular category. The hypothesis explored in [3] is that the visual perception may be modeled as a process related to the identification of individual primitive elements, e.g. a finite number of geometrical components. In addition, Biederman claims that the relation between parts is a main feature to the object perception, i.e. two different arrangements of the same components may produce different objects.

Hummel and Biederman, in [4], claim that the majority of the visual recognition models are based on template matching or feature list matching. The two of them present limitations and are not in accordance with the human recognition [3]. In that way, the authors in [4] present a *structural description* to characterize the object as a configuration of features, sensitive to the attribute structure and indifferent to the image overview.

Kosslyn et al, in [5], re-affirm the importance of relative positions for object and scene recognition. They classify those spatial relationships, psychophysically, according to their visuospatial processing, as absolute coordinate representations (i.e. precise spatial localization) and categorical representations (i.e. association of an interval of position to a equivalence class, e.g. *left of*).

The works in this area started mainly with Freeman’s paper [6], and was continued during the 80’s by Klette and Rosenfeld [7]. In [6], Freeman presents mathematical-computational formalisms to represent the semantic context of terms (in English) that codify relationships between objects by underlining the necessity of using fuzzy representations for a number of relations. Then several authors proposed fuzzy representations of some spatial relations (see e.g. [8] for a review).

Moreover, when considering works in psycholinguistics, it appears that even if the objects are crisp, the lack of clarity in the concepts related to the relative positions gives the background to the use of fuzzy definitions of these concepts.

3 Modeling the Spatial Relation *Along* for Crisp Objects

In the example of Fig.1(a), it can be said that A is along B , or that B is along A . The intuitive meaning of the relation is polymorphic: some assumptions can be made or not on the objects (at least one should be elongated, or both should), the distance between them should be reduced with respect to the size of the objects (typically we would not say that A is along B in the example of Fig.1(b)). What is quite clear is that the region between A and B , denoted by β , should be elongated, as is the case in Fig.1(a). In our model, we choose to propose a definition that does not necessarily consider the shape of the objects as a whole, that is symmetrical in both arguments, and that involves the region between the objects and their distance. Moreover, as already advocated in [6], defining such relations in a binary way would not be satisfactory, and a degree of satisfaction of the relation is more appropriate. Finally, we want also to be able to deal with situations where the relation is satisfied locally, between parts of the objects only.

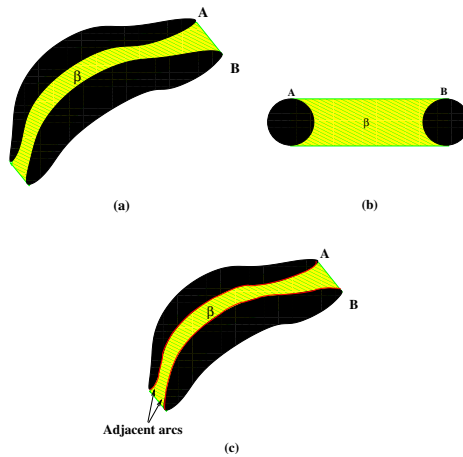


Fig. 1. (a) Example where A is along B , with an elongated region β between A and B . (b) Case where β is elongated but A is not along B . (c) Same example as (a) where adjacent arcs are shown.

Based on these considerations, we propose a mathematical definition of the degree to which an object A is along an object B , based on the region between A and B [9]. The basic idea to characterize to which degree “ A is along B ” is based on two steps:

1. calculate the region β between A and B ;
2. measure how elongated is β , thus defining the degree to which A is along B .

This approach is interesting because it involves explicitly the *between* region, which is also committed in the usual semantics of the *along* relation, and a good technique to calculate the region between A and B is available and used in our approach. Once the region between A and B is obtained, the issue of how elongated is β may be treated by shape analysis, leading to different measures which may be chosen depending on the application, as explained below.

3.1 Definition of the Region Between Two Objects

Since no assumption on the shapes of the objects is made, some classical ways to define the *between* region may not be appropriate. In particular, if the objects have complex shapes with concavities, a simple definition based on the convex hull of the union of both objects does not lead to a satisfactory result. We have addressed this problem in [9], where new methods are proposed in order to cope with complex shapes. We choose here one of these methods, the visibility approach, which provides results adapted to our aim. In particular, concavities of an object that are not visible from the other one are not included in the *between* area. More formally, this approach relies on the notion of admissible segments as introduced in [7]. A segment $]a, b[$, with a in A and b in B (A and B are supposed to be compact sets or digital objects), is said admissible if it is included in $A^C \cap B^C$ [9]. Note that a and b then necessarily belong to the boundary of A and B , respectively. This has interesting consequences from an algorithmic point of view, since it considerably reduces the size of the set of points to be explored. The visible points are those which belong to admissible segments. The region between A and B can then be defined as the union of admissible segments.



Fig. 2. (a) Region between two objects, calculated by the visibility approach; (b) Analogous to (a), but showing that the concavity of one of the objects is properly excluded from the *between* region by the visibility method.

Here, for the second step, we need to keep the extremities (belonging to the boundary of A or B) of the admissible segments in the *between* region. Therefore we slightly modify the definition of [9] as:

$$\beta = \cup\{[a, b], a \in A, b \in B,]a, b[\text{ admissible}\}. \tag{1}$$

This definition is illustrated in Fig.2 for two different cases. Note that, in contrast to the objects in Fig.2(a), in case of Fig.2(b), there is a concavity in one of the shapes not visible from the other object, and which is properly excluded from the *between* region by the visibility approach.

3.2 Definition of the Degree of Elongatedness

There are different possible approaches to measure how elongated is a region. One of the most popular ones is given by the inverse of compactness, i.e. how elongated is the region with respect to a circle. This can be measured in the 2D case by the elongatedness measure $c = P^2/S$, where P and S represent the perimeter and the area of the region. We have $c = 4\pi$ for a perfect disk, and the more elongated is the shape, the larger is c . In order to normalize this measure between 0 and 1, we propose a first *alongness* measure defined as:

$$\alpha_1 = f_a \left(\frac{P^2(\beta)}{S(\beta)} \right), \quad (2)$$

where $S(\beta)$ and $P(\beta)$ denote the area and perimeter of region β , respectively, and f_a is an increasing function, typically a sigmoid, such as $f_a(x) = (1 - \exp(-ax))/(1 + \exp(-ax))$. This measure α_1 tends towards 1 as β becomes more elongated. Although a is a parameter of the method, it preserves the order between different situations, which is the most important property. Absolute values can be changed by tuning a to enhance the difference between different situations.

However the measure α_1 does not lead to good results in all situations. Indeed it considers a global elongatedness, while the elongatedness only in some directions is useful. Let us consider the example in Fig.1(b). The region between A and B is elongated, but this does not mean that A is along B . On the other hand, the situation in Fig.1(a) is good since β is elongated in the direction of its adjacency with A and B . In order to model this, instead of using the complete perimeter of β , the total arc length $L(\beta)$ of the contour portions of β adjacent to A or to B is used (see the adjacent arcs indicated in Fig.1(c)). Here, with the modified definition of β (Equation 1), these lines are actually the intersections between A and β and between B and β . The new elongatedness measure is then defined as:

$$\alpha_2 = f_a \left(\frac{L^2(\beta)}{S(\beta)} \right). \quad (3)$$

Although this measure produces proper results, it presents the drawback of not taking directly into account the distance between A and B , which is useful in some situations. Also, because α_2 is a global measure over A and B , it fails in identifying if there are some parts of A that are along some parts of B , i.e. it lacks the capability of local analysis.

There is an interesting way of incorporating these aspects in the present approach by considering the distance between the two shapes within the *between*

area. Let x be an image point, and $d(x, A)$ and $d(x, B)$ the distances from x to A and B respectively (in the digital case, they can be computed in a very efficient way using distance transforms). Let $D_{AB}(x) = d(x, A) + d(x, B)$. Instead of using the area of β to calculate how elongated it is, we define the volume $V(\beta)$ below the surface $\{(x, D_{AB}(x)), x \in \beta\}$, which is calculated as:

$$V(\beta) = \int_{\beta} D_{AB}(x) dx. \tag{4}$$

In the digital case, the integral becomes a finite sum.

This leads to an *alongness* measure taking into account the distance between A and B :

$$\alpha_3 = f_a \left(\frac{L^2(\beta)}{V(\beta)} \right). \tag{5}$$

The distance $D_{AB}(x)$ may be used in a more interesting way in order to deal with situations where just some parts of A can be considered along some parts of B . In such cases, it is expected that such parts are near each other, thus generating a *between* region with lower values of $D_{AB}(x)$. Let $\beta_t = \{x, D_{AB}(x) < t\}$, where t is a distance threshold. Let $L(\beta_t)$, $S(\beta_t)$ and $V(\beta_t)$ be the total adjacent arc length, area and volume for β_t . Two local *alongness* measures, in the areas which are sufficiently near to each other according to the threshold, are then defined as:

$$\alpha_4(t) = f_a \left(\frac{L^2(\beta_t)}{S(\beta_t)} \right), \tag{6}$$

and

$$\alpha_5(t) = f_a \left(\frac{L^2(\beta_t)}{V(\beta_t)} \right). \tag{7}$$

4 Modeling the Spatial Relation *Along* for Fuzzy Objects

Now we consider the case of fuzzy objects, which may be useful to deal with spatial imprecision, rough segmentation, etc. We follow the same approach in two steps as in the crisp case.

The visibility approach for defining the *between* region can be extended to the fuzzy case by introducing the degree to which a segment is included in $A^C \cap B^C$ (which is now a fuzzy region). Let μ_A and μ_B be the membership functions of the fuzzy objects A and B . The degree of inclusion μ_{incl} of a segment $]a, b[$ in $A^C \cap B^C$ is given by:

$$\mu_{incl}(]a, b[) = \inf_{y \in]a, b[} \min[1 - \mu_A(y), 1 - \mu_B(y)]. \tag{8}$$

Let us denote the support of the fuzzy objects A and B by $\text{Supp}(A)$ and $\text{Supp}(B)$ respectively. The region between A and B , denoted by β_F , is then defined as

$$\beta_F(x) = \sup\{\mu_{incl}(]a, b[); x \in [a, b], a \in \text{Supp}(A), b \in \text{Supp}(B)\}. \tag{9}$$

In order to define *alongness* measures analogous to $\alpha_l, l = 1..5$, it is necessary to calculate the perimeter, area and volume of β_F . Perimeter $P(\beta_F)$ and area $S(\beta_F)$ are usually defined as [10]:

$$P(\beta_F) = \int_{\text{Supp}(\beta_F)} |\nabla \beta_F(x)| dx, \tag{10}$$

where $\nabla \beta_F(x)$ is the gradient of β_F , and

$$S(\beta_F) = \int_{\text{Supp}(\beta_F)} \beta_F(x) dx. \tag{11}$$

The extension of α_2 requires to define the adjacency region R_{adj} between the objects and β . In order to guarantee the consistency with the crisp case, we can simply take the intersection between A and β and between B and β and extend L as:

$$R_{adj}(\beta_F, \mu_{A \cup B}) = (\text{Supp}(\beta_F) \cap \text{Supp}(A)) \cup (\text{Supp}(\beta_F) \cap \text{Supp}(B)), \tag{12}$$

where $\mu_{A \cup B}$ represents the union of the fuzzy objects A and B , and:

$$L(\beta_F, \mu_{A \cup B}) = S(R_{adj}(\beta_F, \mu_{A \cup B})). \tag{13}$$

Finally, it is also necessary to calculate the distance of any point x of the *between* region to A and to B . We propose the use of the length of the admissible segments:

$$D_{AB}(x) = \inf\{\|b-a\|,]a, b[\text{ admissible}, x \in]a, b[\}, \text{ for } x \in (\text{Supp}(A) \cup \text{Supp}(B))^C. \tag{14}$$

Then, we define the volume $V(\beta_F)$ below the surface $\{(x, D_{AB}), x \in \beta_F\}$ by weighting each point by its membership to $\beta_F(x)$, as:

$$V(\beta_F) = \int_{\text{Supp}(\beta_F)} \beta_F(x) D_{AB}(x) dx. \tag{15}$$

In order to keep the fuzzy nature of the model, instead of thresholding the distance function as in the crisp case, we propose to select the closest area based on a decreasing function g of D_{AB} . We thus have $\beta_{F_l}(x) = \beta_F(x)g(D_{AB}(x))$. In our experiments, we have chosen g as:

$$g(t) = 1 - f_{a_1}(t), \tag{16}$$

with $a_1 = 0.3$.

5 Experimental Results

Extensive results with a large number of pairs of shapes have been successfully produced. Some of these results are presented and discussed in this section.

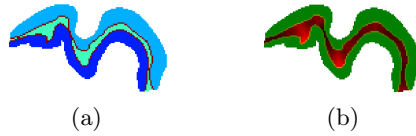


Fig. 3. Results using the visibility approach to calculate β . (a) Synthetic shapes and the region β between them. The adjacent arcs are also indicated. (b) The distance map $D_{AB}(x)$ in β is represented as grey-levels.

Table 1. *Alongness* values for different shape configurations (synthetic shapes) with parameters $a = 0.125$ and $t = 10$

| Shapes | (a) | (b) | (c) |
|----------------|-------|-------|-------|
| α_1 | 0.907 | 0.450 | 0.874 |
| α_2 | 0.885 | 0.431 | 0.340 |
| α_3 | 0.172 | 0.011 | 0.010 |
| $\alpha_4(10)$ | 0.834 | 0.653 | 0.072 |
| $\alpha_5(10)$ | 0.165 | 0.127 | 0.010 |

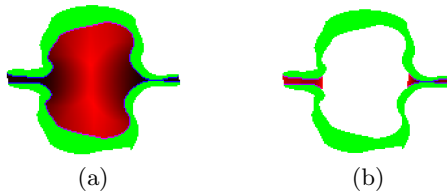


Fig. 4. Results using the visibility approach to calculate β and β_t . (a) The distance map $D_{AB}(x)$ in β is represented as grey-levels. (b) The thresholded *between* region $\beta_t = \{x, D_{AB}(x) < t\}$, indicating that only nearby contour portions are taken into account by this approach.

5.1 Crisp Objects

Table 1 shows some results obtained on synthetic objects illustrating different situations. The adjacent lines and distance values of the object in Table 1(a) are shown in Fig.3 (a) and (b), respectively. High values of $D_{AB}(x)$ correctly indicate image regions where the shapes are locally far from each other.

In the example of Table 1(a), the two objects can be considered as along each other, leading to high values of α_1 , α_2 and α_4 . However some parts of the objects are closer to each other than other parts. When the distance increases, the corresponding parts can hardly be considered as along each other. This is

Table 2. *Alongness* values for different shape configurations (brain structures from medical imaging) with parameters $a = 0.25$ and $t = 10$

| Shapes | (a) | (b) | (c) | (d) |
|----------------|-------|-------|-------|-------|
| α_1 | 0.746 | 0.677 | 0.487 | 0.708 |
| α_2 | 0.746 | 0.677 | 0.438 | 0.289 |
| α_3 | 0.717 | 0.611 | 0.133 | 0.015 |
| $\alpha_4(10)$ | 0.746 | 0.677 | 0.438 | 0.001 |
| $\alpha_5(10)$ | 0.717 | 0.611 | 0.133 | 0.000 |

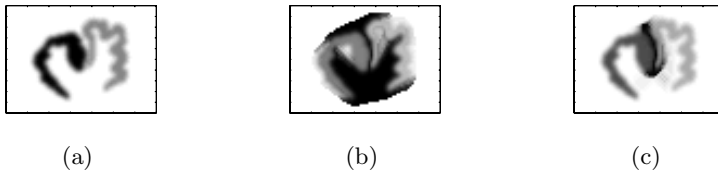


Fig. 5. Results using the fuzzy visibility approach to calculate β_F and β_{F_t} . (a) Original shapes. (b) Shapes and the region β_F between them. (c) Shapes and the thresholded *between* region $\beta_{F_t}(x) = \{x, D_{AB}(x) < t\}$.

well expressed by the lower values obtained for α_3 and α_5 . These effects are even stronger on the example of Table 1(b) where only small parts of the objects can be considered as being along each other. The *between* regions β and β_t (i.e. thresholded) are shown in Fig.4. The third case is a typical example where the region between A and B is elongated, but not in the direction of its adjacency with A and B . This is not taken into account by α_1 , while the other measures provide low values as expected: α_2 is much smaller than α_1 and the other three values are almost 0.

Table 2 shows results obtained on real objects, which are some brain structures extracted from magnetic resonance images. Similar values are obtained for all measures in the two first cases where the relation is well satisfied. The third example shows the interest of local measures and distance information (in particular the similar values obtained for α_2 and α_4 illustrate the fact that only the parts that are close to each other are actually involved in the computation of the *between* region for this example), while the last one is a case where the relation is not satisfied, which is well reflected by all measures except α_1 , as expected.

5.2 Fuzzy Objects

The experiments concerning the fuzzy approach are based on the construction of synthetical fuzzy objects by a Gaussian smoothing of the crisp ones, only

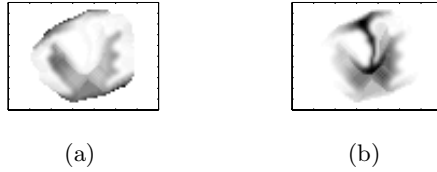


Fig. 6. (a) The distance map $D_{AB}(x)$ in β_F of the objects in Figure 5 (a). (b) The decreasing function g of $D_{AB}(x)$.

Table 3. *Alongness* values for different shape configurations (fuzzy synthetic shapes) with parameters $a = 0.50$ and $a_1 = 0.30$

| Shapes | (a) | (b) | (c) |
|----------------|-------|-------|-------|
| α_{F_1} | 0.990 | 0.815 | 0.982 |
| α_{F_2} | 0.999 | 0.948 | 0.881 |
| α_{F_3} | 0.879 | 0.531 | 0.515 |
| α_{F_4} | 0.975 | 0.755 | 0.572 |
| α_{F_5} | 0.686 | 0.552 | 0.508 |

Table 4. *Alongness* values for different shape configurations (fuzzy brain structures from medical imaging) with parameters $a = 0.25$ and $a_1 = 0.30$

| Shapes | (a) | (b) | (c) | (d) |
|----------------|-------|-------|-------|-------|
| α_{F_1} | 0.996 | 0.997 | 0.980 | 0.997 |
| α_{F_2} | 0.984 | 0.965 | 0.972 | 0.971 |
| α_{F_3} | 0.888 | 0.840 | 0.675 | 0.536 |
| α_{F_4} | 0.812 | 0.764 | 0.781 | 0.544 |
| α_{F_5} | 0.675 | 0.643 | 0.579 | 0.503 |

for the sake of illustration. In real applications, fuzzy objects may be obtained from a fuzzy segmentation of the image, from imprecision at their boundaries, from partial volume effect modeling, etc. Figure 5 illustrates an example of fuzzy objects along with the *between* region and the fuzzy regions β_F and β_{F_i} . The distance map and the selected area are depicted in Figure 6.

Some results obtained on fuzzy synthetic shapes are given in Table 3, while some results on fuzzy real objects are given in Table 4. In these tables, α_{F_i} denotes the fuzzy equivalent of α_i .

Results are again in accordance with what could be intuitively expected. This illustrates the consistency of the proposed extension to fuzzy sets.

Since the computation of L , S and V in the fuzzy case is based on the support of the fuzzy objects, which is larger than the corresponding crisp objects, we have to choose a different value for the parameter a , in order to achieve a better discrimination between the different situations. However a has the same values for all objects in each table, for the sake of comparison. Note that in Table 3 as well as in Table 4, the results obtained on fuzzy synthetic and real objects are qualitatively the same as the results obtained on crisp object: in particular, α_{F_3} and α_{F_5} well reflect the distance constraint on the *alongness* degree.

6 Conclusion

We proposed in this paper an original method to model the relation *along* and to compute the degree to which this relation is satisfied between two objects of any shape. Several measures are proposed, taking into account different types of information: region between the objects, adjacency between the objects and this region, distance, parts of objects. The definitions are symmetrical by construction. They inherit some properties of the visibility method for computing the between area such as invariance under translation and rotation. Measures α_1 , α_2 and α_4 are also invariant under isotropic scaling. Finally, the proposed measures fit well the intuitive meaning of the relation in a large class of situations, and provide a ranking between different situations which is consistent with the common sense. One of the advantages of the proposed approach is the decomposition of the solution in two parts, i.e. to find the region between the objects and to calculate its elongatedness. The inverse of compactness (sometimes called circularity) has been adopted to measure how elongated is the region between the shapes. This is by no means the unique way of characterizing elongatedness. In fact, if the region between the shapes becomes very complex (e.g. Fig.7), the area starts to increase fast with respect to the perimeter (i.e. space-filling property), and circularity-based measures may produce poor results. In such cases, alternative elongatedness measures may be adapted to replace circularity in our proposed approach (e.g. shape measures that characterize thinness of a shape).

Alternative approaches to the computation of length of the adjacencies and distances can be tested. We can restrict, for example, the adjacent region to the

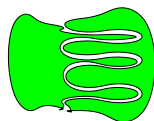


Fig. 7. Complex shapes lead to space-filling *between* region. This may affect the circularity-based elongatedness measure, thus requiring alternative approaches to evaluate how elongated is the *between* region.

watershed line of this intersection, and compute its length in a classical way. On the other hand, instead of using Equation 14, we can calculate $D_{\mu_{A \cup B}}$ with the distances to the α -cuts. The distance $d(x, \mu)$ from a point x to a fuzzy set with membership function μ can indeed be defined by integrating over α the distance from x to each α -cut. Another option is to calculate $d(x, \mu)$ as the distance of x to the support of μ , i.e. $d(x, \mu) = d(x, \text{Supp}(\mu))$. These definitions are useful for implementation purposes since for each α -cut, a fast distance transform can be used.

Extensions to 3D are straightforward: the computation of the *between* relation does not make any assumption on the dimension of space; the measures of elongatedness can be simply performed by replacing lengths by surfaces and surfaces by volumes.

Future work also aims at introducing this relation as a new feature in structural pattern recognition or content-based image retrieval schemes.

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