From Structuring Elements to Structuring Neighborhood Systems

Alexandre Goy[1(✉)], Marc Aiguier[1], and Isabelle Bloch[2]

[1] MICS, CentraleSupélec, Université Paris-Saclay, Gif-sur-Yvette, France
{alexandre.goy,marc.aiguier}@centralesupelec.fr
[2] LTCI, Télécom ParisTech, Université Paris-Saclay, Paris, France
isabelle.bloch@telecom-paristech.fr

Abstract. In the context of mathematical morphology based on structuring elements to define erosion and dilation, this paper generalizes the notion of a structuring element to a new setting called structuring neighborhood systems. While a structuring element is often defined as a subset of the space, a structuring neighborhood is a subset of the subsets of the space. This yields an extended definition of erosion; dilation can be obtained as well by a duality principle. With respect to the classical framework, this extension is sound in many ways. It is also strictly more expressive, for any structuring element can be represented as a structuring neighborhood but the converse is not true. A direct application of this framework is to generalize modal morpho-logic to a topological setting.

Keywords: Structuring element · Neighborhood · Filter · Morpho-logic · Topology

1 Introduction

The motivation of this paper is to apply mathematical morphology in logic (*morpho-logic*), in particular for spatial reasoning. Morpho-logic was initially introduced for propositional logic [5], and proved useful to model knowledge, beliefs or preferences, and to model classical reasoning methods such as revision, fusion or abduction [6,7]. Extensions to modal logic [8] and first-order logic [10] were then proposed. The framework of satisfaction systems and stratified institutions was then proposed as a more general setting encompassing many logics [1–3]. In modal logic, morphological operators can be seen as modalities, with generally strong properties. However, the modalities of *topological* modal logic cannot be obtained when considering the usual definition of structuring element as a set or a binary relation; furthermore, the properties of these modalities are not those of erosion and dilation but closer to those of opening and closing, because of the double quantification $\forall/\exists$ in their definition. The starting point of this paper is to try to see these modalities as weaker forms of erosion and dilation, derived from a lax notion of structuring element.

B. Burgeth et al. (Eds.): ISMM 2019, LNCS 11564, pp. 16–28, 2019.
https://doi.org/10.1007/978-3-030-20867-7_2

Let X denote a set and $\mathcal{P}(X)$ its powerset. The context we consider is the one of deterministic mathematical morphology, with dilations and erosions defined on $\mathcal{P}(X)$ from structuring elements. The main idea of this paper is to consider a structuring element not as an element of $\mathcal{P}(X)$, or a function from X into $\mathcal{P}(X)$, but as a function taking values in $\mathcal{P}(\mathcal{P}(X))$. We call it structuring neighborhood system, in accordance with the topological flavor of the considered setting. The aim of this paper is then to study the structure of the set of such structuring neighborhoods, and to establish their properties. In particular we show that many properties are satisfied, but some, mostly related to adjunctions, may be lost.

The paper is organized as follows. In Sect. 2, we recall some results on dilations and erosions defined using classical structuring elements. In Sect. 3 we introduce the proposed definition of structuring neighborhood systems. Their set can be endowed with a lattice structure and with a monoid structure, as in the classical setting [11,15]. Properties are then derived. We also introduce a weaker form of erosion, which is proved to occur exactly when the structuring neighborhood system is a filter (in the sense of logic). Finally in Sect. 4 we show that the proposed framework leads to good results on topological modal logic, thus achieving our initial aim.

2 Structure on Structuring Elements

2.1 Mathematical Morphology Based on Structuring Elements

In the context of deterministic mathematical morphology, a class of basic operators is often defined based on the notion of structuring element (see e.g. [4,12,16]). A general definition of a structuring element is the following.

Definition 1 (Structuring element). *A* structuring element *is a function* $b : X \to \mathcal{P}(X)$.

Example 1 (Translations in $\mathbb{R}^n$). Let B be a subset of $\mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be defined by $b(x) = B_x$ (the translated of B at position x). This is a well-known structuring element in translation invariant morphological image processing.

The two following examples come from the morphological study of logic, also called morpho-logic. See [8] for the modal case and [3] for a more general account of morpho-logic. A wider class of logical systems will be presented in Sect. 4.

Example 2 (Modal morpho-logic). Let $\langle W, R\rangle$ be a Kripke frame, i.e. W is a set of worlds, and $R \subseteq W \times W$ is a binary relation often called accessibility relation. One can define a structuring element $b : W \to \mathcal{P}(W)$ by $b(w) = \{w' \in W \mid (w, w') \in R\}$, i.e. the set of worlds accessible from w.

Example 3 (First-order morpho-logic). Let $\mathsf{Var} = \{x, y, z, ...\}$ be a set of variables and M be a set. A function $f : \mathsf{Var} \to M$ is called a *variable affectation.* For any $x \in \mathsf{Var}$, one can define the x-structuring element $b_x : M^{\mathsf{Var}} \to \mathcal{P}(M^{\mathsf{Var}})$ by $b(f) = \{g \in M^{\mathsf{Var}} \mid \forall y \neq x, g(y) = f(y)\}$.

The two following simple examples will be useful in what follows.

Example 4 (Singleton). The singleton structuring element $\mathsf{sgt} : X \to \mathcal{P}(X)$ is defined by $\mathsf{sgt}(x) = \{x\}$.

Example 5 (Symmetric). Let $b : X \to \mathcal{P}(X)$ be a structuring element. Its symmetric structuring element $b^\dagger$ is defined by $b^\dagger(x) = \{y \in X \mid x \in b(y)\}$.

Let $b : X \to \mathcal{P}(X)$ be a structuring element. Erosion $\varepsilon[b]$ and dilation $\delta[b]$ are two operators $\mathcal{P}(X) \to \mathcal{P}(X)$ defined for all $U \in \mathcal{P}(X)$ by

$$\varepsilon[b](U) = \{x \in X \mid b(x) \subseteq U\} \tag{1}$$

$$\delta[b](U) = \{x \in X \mid b(x) \cap U \neq \emptyset\} \tag{2}$$

Remark 1. Usually, dilation is rather defined using the symmetric structuring element by $\delta[b](U) = \{x \in X \mid b^\dagger(x) \cap U \neq \emptyset\}$. This yields good properties, especially adjunction [11]. Report to Remark 2 to understand the choice of not using $b^\dagger$.

Let $\mathsf{StEl}(X) = \mathcal{P}(X)^X$ denote the set of all structuring elements on X, and let $\mathsf{Op}(X) = \mathcal{P}(X)^{\mathcal{P}(X)}$ be the set of all operators on $\mathcal{P}(X)$, which contains all erosions and dilations.

2.2 Lattice Structure

One can define a partial order on $\mathsf{StEl}(X)$ using pointwise inclusion: $b \leq c$ iff for all $x \in X$, $b(x) \subseteq c(x)$. This order endows $\mathsf{StEl}(X)$ with a complete lattice structure, where for all $(b_i)_{i \in I} \in \mathsf{StEl}(X)^I$,

$$\left(\bigwedge_{i \in I} b_i\right)(x) = \bigcap_{i \in I} b_i(x) \tag{3}$$

$$\left(\bigvee_{i \in I} b_i\right)(x) = \bigcup_{i \in I} b_i(x) \tag{4}$$

The greatest element is the full structuring element, defined by $\mathsf{ful}(x) = X$, and the least element is the empty structuring element, defined by $\mathsf{emp}(x) = \emptyset$. There is also a similar complete lattice structure on $\mathsf{Op}(X)$.

2.3 Monoid Structure

One can define an internal composition law $\star$ on $\mathsf{StEl}(X)$. Let $b, c : X \to \mathcal{P}(X)$ be structuring elements. Let $(b \star c)(x) = \{z \in X \mid \exists y \in b(x), z \in c(y)\}$. The operation $\star$ is associative, with neutral element sgt. This turns $(\mathsf{StEl}(X), \star, \mathsf{sgt})$ into a monoid. There is also a monoid structure $(\mathsf{Op}(X), \circ, \mathsf{id})$ where $\circ$ is the usual composition of functions and $\mathsf{id}(U) = U$ for all $U \in \mathcal{P}(X)$. Note that $\varepsilon[\mathsf{sgt}] = \delta[\mathsf{sgt}] = \mathsf{id}$.

We will need the following result in the next section.

isabelle.bloch@telecom-paristech.fr

Proposition 1. *For any* $b, c \in \mathsf{StEl}(X)$*, one has, for all* $x \in X, U \in \mathcal{P}(X)$*,* $(b \star c)(x) \subseteq U$ *iff* $b(x) \subseteq \varepsilon[c](U)$*.*

Proof. First note by unfolding the definitions that $(b \star c)(x) = \delta[c^\dagger](b(x))$. The wanted property then becomes $\delta[c^\dagger](b(x)) \subseteq U \iff b(x) \subseteq \varepsilon[c](U)$. This is true, by the standard result of adjunction between $\varepsilon[c]$ and $\delta[c^\dagger]$. □

2.4 Usual Properties

Table 1 contains a review of classical properties of erosion and dilation that will be generalized in the following section. We also give new interpretations of some of these properties with respect to the lattice and monoid structures of $\mathsf{StEl}(X)$ and $\mathsf{Op}(X)$.

Table 1. Properties of classical erosion and dilation

Usual name	Statement	Interpretation
Duality	$\varepsilon[b](X \setminus U) = X \setminus \delta[b](U)$	/
Monotonicity	$U \subseteq V \Rightarrow \varepsilon[b](U) \subseteq \varepsilon[b](V)$	$\varepsilon[b]$ isotone
	$U \subseteq V \Rightarrow \delta[b](U) \subseteq \delta[b](V)$	$\delta[b]$ isotone
	$b \leq c \Rightarrow \varepsilon[c](U) \subseteq \varepsilon[b](U)$	$\varepsilon[-]$ antitone
	$b \leq c \Rightarrow \delta[b](U) \subseteq \delta[c](U)$	$\delta[-]$ isotone
Anti-extensivity	$\mathsf{sgt} \leq b \Leftrightarrow \varepsilon[b](U) \subseteq U$	$\varepsilon[-]$ antitone
Extensivity	$\mathsf{sgt} \leq b \Leftrightarrow U \subseteq \delta[b](U)$	$\delta[-]$ isotone
Preservation of X	$\varepsilon[b](X) = X$	$\varepsilon[b]$ erosion
Preservation of $\emptyset$	$\delta[b](\emptyset) = \emptyset$	$\delta[b]$ dilation
Commutation	$\varepsilon[b](\bigcap_{i\in I} U_i) = \bigcap_{i\in I} \varepsilon[b](U_i)$	$\varepsilon[b]$ erosion
with inf	$\varepsilon[b](U \cap V) = \varepsilon[b](U) \cap \varepsilon[b](V)$	/
Commutation	$\delta[b](\bigcup_{i\in I} U_i) = \bigcup_{i\in I} \delta[b](U_i)$	$\delta[b]$ dilation
with sup	$\delta[b](U \cup V) = \delta[b](U) \cup \delta[b](V)$	/
Associativity	$\varepsilon[b \star c] = \varepsilon[b] \circ \varepsilon[c]$	$\varepsilon[-]$ monoid homomorphism
of dilation	$\delta[b \star c] = \delta[b] \circ \delta[c]$	$\delta[-]$ monoid homomorphism
Adjunction	$U \subseteq \varepsilon[b](V) \Leftrightarrow \delta[b^\dagger](U) \subseteq V$	Galois connection

3 Structuring Neighborhoods

3.1 Introducing Neighborhoods Systems

In this section, we will generalize all notions of Sect. 2 using a lax notion of structuring element, called structuring neighborhood system, or structuring neighborhood for short.[1]

[1] While $b(x)$ is often considered as a neighborhood of x according to a given topology on X, here our notion of neighborhood system refers to a set of subsets of X.

isabelle.bloch@telecom-paristech.fr

Definition 2 (Structuring neighborhood). *A* structuring neighborhood *is a function $b : X \to \mathcal{P}(\mathcal{P}(X))$.*

Example 6 (Topology). Assume that X is endowed with a topology τ. The associated structuring neighborhood is $b(x) = \{U \in \mathcal{P}(X) \mid \exists O \in \tau, x \in O \text{ and } O \subseteq U\}$ (the set of topological neighborhoods of x).

Definition 3 (From element to neighborhood). *Let $b : X \to \mathcal{P}(X)$ be a structuring element in the sense of the previous section. Then one can define a structuring neighborhood $\overline{b} : X \to \mathcal{P}(\mathcal{P}(X))$ by $\overline{b}(x) = \{U \in \mathcal{P}(X) \mid b(x) \subseteq U\}$ (the set of all supersets of $b(x)$).*

Example 7 (Singleton). The structuring element sgt becomes the structuring neighborhood $\overline{\mathsf{sgt}}$, which is given explicitly by $\overline{\mathsf{sgt}}(x) = \{U \in \mathcal{P}(X) \mid x \in U\}$.

Let $b : X \to \mathcal{P}(\mathcal{P}(X))$ be a structuring neighborhood.

Definition 4. *Define the operator $\overline{\varepsilon}[b] : \mathcal{P}(X) \to \mathcal{P}(X)$ by*

$$\overline{\varepsilon}[b](U) = \{x \in X \mid U \in b(x)\} \tag{5}$$

Note that for the moment, nothing guarantees that the operator $\overline{\varepsilon}[b]$ is an erosion. One can also define another operator $\overline{\delta}[b]$ using a duality principle, namely $\overline{\delta}[b](U) = X \setminus \overline{\varepsilon}[b](X \setminus U)$. This leads to

$$\overline{\delta}[b](U) = \{x \in X \mid X \setminus U \notin b(x)\} \tag{6}$$

Remark 2. When it comes to define $\overline{\delta}[b]$, we could not have used a *symmetric* structuring neighborhood. Indeed, there is no obvious definition of such an object. This is also why, for consistency purposes, no symmetric was involved in our definition of dilation in Sect. 2. As the symmetric structuring element is a crucial component to get the adjunction-related properties, structuring neighborhood will lack these results.

Let $\mathsf{StNb}(X) = \mathcal{P}(\mathcal{P}(X))^X$ denote the set of all structuring neighborhoods on X. Definition 3 induces a map $\overline{} : \mathsf{StEl}(X) \to \mathsf{StNb}(X)$ which will be called the *plunge map* because it is injective. The following proposition shows that this extension of erosion and dilation is sound with respect to the classical case:

Proposition 2. *For any $b \in \mathsf{StEl}(X)$, $\overline{\varepsilon}[\overline{b}] = \varepsilon[b]$ and $\overline{\delta}[\overline{b}] = \delta[b]$.*

Proof. By definition, $x \in \overline{\varepsilon}[\overline{b}](U) \Leftrightarrow U \in \overline{b}(x) \Leftrightarrow b(x) \subseteq U \Leftrightarrow x \in \varepsilon[b](U)$. For the other one, use duality: $\overline{\delta}[\overline{b}](U) = X \setminus \overline{\varepsilon}[\overline{b}](X \setminus U) = X \setminus \varepsilon[b](X \setminus U) = \delta[b](U)$. $\square$

Remark 3. This yields immediately $\overline{\varepsilon}[\overline{\mathsf{sgt}}] = \varepsilon[\mathsf{sgt}] = \mathsf{id}$ and $\overline{\delta}[\overline{\mathsf{sgt}}] = \delta[\mathsf{sgt}] = \mathsf{id}$.

3.2 Lattice Structure

One can define a partial order on $\mathsf{StNb}(X)$ using pointwise *reversed* inclusion. More precisely, for any $b, c \in \mathsf{StNb}(X)$, define $b \preceq c$ if and only if for all $x \in X$, $b(x) \supseteq c(x)$. The reason for which the inclusion is reversed in the definition of $\preceq$ lies in the following result.

Proposition 3. *For any $b, c \in \mathsf{StEl}(X)$, $b \leq c$ if and only if $\overline{b} \preceq \overline{c}$.*

Proof. Assume $b \leq c$, take $x \in X$, $U \in \mathcal{P}(X)$, and assume that $U \in \overline{c}(x)$ i.e. $c(x) \subseteq U$. As $b \leq c$, we have $b(x) \subseteq c(x)$, hence $b(x) \subseteq U$ i.e. $U \in \overline{b}(x)$. This completes the proof that $\overline{b} \preceq \overline{c}$. For the other implication, assume $\overline{b} \preceq \overline{c}$. Let $x \in X$ and show that $b(x) \subseteq c(x)$. As $c(x) \in \overline{c}(x)$ and $\overline{c}(x) \subseteq \overline{b}(x)$, one has $c(x) \in \overline{b}(x)$, so by definition of $\overline{b}$ this yields $b(x) \subseteq c(x)$. Hence, $b \leq c$. □

Then, the set $\mathsf{StNb}(X)$ has a complete lattice structure, with, for any family $(b_i)_{i \in I}$ in $\mathsf{StNb}(X)$:

$$\left(\bigwedge_{i \in I} b_i \right)(x) = \bigcup_{i \in I} b_i(x) \tag{7}$$

$$\left(\bigvee_{i \in I} b_i \right)(x) = \bigcap_{i \in I} b_i(x) \tag{8}$$

The empty structuring element emp gives rise to a structuring neighborhood $\overline{\mathsf{emp}}$, defined by $\overline{\mathsf{emp}}(x) = \mathcal{P}(X)$, which turns out to be least element of this lattice. The greatest structuring neighborhood is called the *void* and is given by $\mathsf{void}(x) = \emptyset$. It is not the image of the full structuring element by the plunge function; indeed, $\overline{\mathsf{ful}}(x) = \{X\}$. Actually, void is the image of no structuring element by the plunge map, because for any structuring element b, $b(x) \in \overline{b}(x)$; henceforth, the plunge map is not surjective.

3.3 Monoid Structure

One can define an internal composition law in $\mathsf{StNb}(X)$. Given $b, c \in \mathsf{StNb}(X)$, let $(b \star c)(x) = \{U \in \mathcal{P}(X) \mid \overline{\varepsilon}[c](U) \in b(x)\}$. The symbol $\star$ is used for both the composition in $\mathsf{StEl}(X)$ and the one in $\mathsf{StNb}(X)$ because its meaning can be determined unambiguously from the type of the maps b and c.

Proposition 4. *Let $b, c \in \mathsf{StNb}(X)$ be structuring neighborhoods. Let $\overline{\varepsilon}$ and $\overline{\delta}$ be the two operators introduced in Definition 4. Then, one has $\overline{\varepsilon}[b \star c] = \overline{\varepsilon}[b] \circ \overline{\varepsilon}[c]$ and $\overline{\delta}[b \star c] = \overline{\delta}[b] \circ \overline{\delta}[c]$.*

Proof. Concerning the first operator, $x \in \overline{\varepsilon}[b \star c](U) \iff U \in (b \star c)(x) \iff \overline{\varepsilon}[c](U) \in b(x) \iff x \in \overline{\varepsilon}[b]\left(\overline{\varepsilon}[c](U)\right)$. The case of the other operator is obtained by duality. □

Proposition 5. *The operation $\star$ is associative and $\overline{\mathsf{sgt}}$ is its neutral element.*

Proof. The element $\overline{\mathsf{sgt}}$ is neutral because $U \in (b \star \overline{\mathsf{sgt}})(x) \iff \overline{\varepsilon}[\overline{\mathsf{sgt}}](U) \in b(x) \iff U \in b(x)$, and $U \in (\overline{\mathsf{sgt}} \star b)(x) \iff \overline{\varepsilon}[b](U) \in \overline{\mathsf{sgt}}(x) \iff x \in \overline{\varepsilon}[b](U) \iff U \in b(x)$. For associativity, use Proposition 4:

$$
\begin{aligned}
U \in (b \star (c \star d))(x) &\iff \overline{\varepsilon}[c \star d](U) \in b(x) && (9)\\
&\iff (\overline{\varepsilon}[c] \circ \overline{\varepsilon}[d])(U) \in b(x) && (10)\\
&\iff \overline{\varepsilon}[d](U) \in (b \star c)(x) && (11)\\
&\iff U \in ((b \star c) \star d)(x) && (12)
\end{aligned}
$$

□

Proposition 5 yields that $(\mathsf{StNb}(X), \star, \overline{\mathsf{sgt}})$ is a monoid. According to Proposition 4 and Remark 3, the functions $\overline{\varepsilon}[-], \overline{\delta}[-] : \mathsf{StNb}(X) \to \mathsf{Op}(X)$ are then monoid homomorphisms. This is analogous to the structuring element case, where $\varepsilon[-], \delta[-] : \mathsf{StEl}(X) \to \mathsf{Op}(X)$ are monoid homomorphisms. We have the following additional result.

Proposition 6. *The plunge map* $\overline{-} : \mathsf{StEl}(X) \to \mathsf{StNb}(X)$ *is a monoid homomorphism.*

Proof. It does obviously preserve the neutral element, for sgt is neutral in $\mathsf{StEl}(X)$ and $\overline{\mathsf{sgt}}$ is neutral in $\mathsf{StNb}(X)$. Let $b, c \in \mathsf{StEl}(X)$. What remains to show is that $\overline{b} \star \overline{c} = \overline{b \star c}$. Computations result in

$$
\begin{aligned}
U \in (\overline{b} \star \overline{c})(x) &\iff \overline{\varepsilon}[\overline{c}](U) \in \overline{b}(x) && (13)\\
&\iff \varepsilon[c](U) \in \overline{b}(x) && (14)\\
&\iff b(x) \subseteq \varepsilon[c](U) && (15)\\
&\iff (b \star c)(x) \subseteq U && (16)\\
&\iff U \in (\overline{b \star c})(x) && (17)
\end{aligned}
$$

where we used Proposition 1 for the equivalence between Eqs. 15 and 16. □

Proposition 2 and all monoid homomorphisms that have been discussed so far can be summed up in the commutative diagram of Fig. 1.

$$
\begin{tikzcd}
(\mathsf{StEl}(X),\star,\mathsf{sgt}) \arrow[rr, "\overline{-}"] \arrow[dr, "\varepsilon[-]"'] & & (\mathsf{StNb}(X),\star,\overline{\mathsf{sgt}}) \arrow[dl, "\overline{\varepsilon}[-]"] \\
& (\mathsf{Op}(X),\circ,\mathsf{id}) & \\
(\mathsf{StEl}(X),\star,\mathsf{sgt}) \arrow[rr, "\overline{-}"] \arrow[ur, "\delta[-]"] & & (\mathsf{StNb}(X),\star,\overline{\mathsf{sgt}}) \arrow[ul, "\overline{\delta}[-]"']
\end{tikzcd}
$$

Fig. 1. Overview of the monoid-preserving structure of morphological operators

3.4 Usual Properties: Towards Filters

Many properties of classical erosion and dilation can be found back at the price of some necessary and sufficient conditions on the structuring neighborhood. Table 2 sums them up.

Definition 5 (Upper family). *Let* $b \in \mathsf{StNb}(X)$*. It is an* upper family *if for all* x *in* X*, all* V *in* $\mathcal{P}(X)$ *and all* U *in* $b(x)$*,* $V \supseteq U$ *implies* $V \in b(x)$*.*

Definition 6 (Preservation). *Let* $b \in \mathsf{StNb}(X)$*. It* preserves intersections *if for all* $x \in X$ *and all* $(U_i)_{i \in I} \in \mathcal{P}(X)$*,* $\bigcap_{i \in I} U_i \in b(x) \iff \forall i \in I, U_i \in b(x)$*. It* preserves finite intersections *if for all* $x \in X$ *and all* $U, V \in \mathcal{P}(X)$*,* $U \cap V \in b(x) \iff U, V \in b(x)$*.*

Table 2. Properties of $\overline{\varepsilon}[b]$ and $\overline{\delta}[b]$ with respect to b

Usual name	Statement	Equivalent condition on b
Duality	$\overline{\varepsilon}[b](X \setminus U) = X \setminus \overline{\delta}[b](U)$	True
Monotonicity	$U \subseteq V \Rightarrow \overline{\varepsilon}[b](U) \subseteq \overline{\varepsilon}[b](V)$	b upper family
	$U \subseteq V \Rightarrow \overline{\delta}[b](U) \subseteq \overline{\delta}[b](V)$	b upper family
	$b \preceq c \Rightarrow \overline{\varepsilon}[c](U) \subseteq \overline{\varepsilon}[b](U)$	True
	$b \preceq c \Rightarrow \overline{\delta}[b](U) \subseteq \overline{\delta}[c](U)$	True
Anti-extensivity	$\overline{\mathsf{sgt}} \preceq b \Leftrightarrow \overline{\varepsilon}[b](U) \subseteq U$	True
Extensivity	$\overline{\mathsf{sgt}} \preceq b \Leftrightarrow U \subseteq \overline{\delta}[b](U)$	True
Preservation of X	$\overline{\varepsilon}[b](X) = X$	$b \preceq \overline{\mathsf{ful}}$
Preservation of $\emptyset$	$\overline{\delta}[b](\emptyset) = \emptyset$	$b \preceq \overline{\mathsf{ful}}$
Commutation	$\overline{\varepsilon}[b](\bigcap_{i \in I} U_i) = \bigcap_{i \in I} \overline{\varepsilon}[b](U_i)$	b preserves $\cap$
with inf	$\overline{\varepsilon}[b](U \cap V) = \overline{\varepsilon}[b](U) \cap \overline{\varepsilon}[b](V)$	b preserves finite $\cap$
Commutation	$\overline{\delta}[b](\bigcup_{i \in I} U_i) = \bigcup_{i \in I} \overline{\delta}[b](U_i)$	b preserves $\cap$
with sup	$\overline{\delta}[b](U \cup V) = \overline{\delta}[b](U) \cup \overline{\delta}[b](V)$	b preserves finite $\cap$
Associativity	$\overline{\varepsilon}[b \star c] = \overline{\varepsilon}[b] \circ \overline{\varepsilon}[c]$	True
of dilation	$\overline{\delta}[b \star c] = \overline{\delta}[b] \circ \overline{\delta}[c]$	True

Preservation of intersections does confer $\overline{\varepsilon}[b]$ the name of erosion, but this is a very strong requirement. Following the path of Example 6 (topological neighborhoods are preserved by finite intersections, but not by all intersections), it would be more reasonable to only ask b to preserve *finite* intersections. This is why we define hereafter a notion that is weaker than erosion.

Definition 7 (Weak erosion). *Let* $E \in \mathsf{Op}(X)$*. It is called a* weak erosion *if* $E(X) = X$ *and for all* $U, V \in \mathcal{P}(X)$*,* $E(U \cap V) = E(U) \cap E(V)$*.*

Proposition 7. *If b preserves finite intersections, then b is an upper family.*

Proof. Assume that b preserves finite intersections. Let $U, V \in \mathcal{P}(X)$ such that $V \supseteq U$ and $U \in b(x)$. Then $U \cap V = U \in b(x)$. By hypothesis, this yields $V \in b(x)$, so that b is an upper family. □

We are naturally led to a mathematical construction that is very important in topology and logic: filters.

Definition 8 (Filter). *A* filter *on X is an element $\mathfrak{f}$ of $\mathcal{P}(\mathcal{P}(X))$ such that*

1. $X \in \mathfrak{f}$,
2. $U, V \in \mathfrak{f} \Rightarrow U \cap V \in \mathfrak{f}$,
3. $V \supseteq U, U \in \mathfrak{f} \Rightarrow V \in \mathfrak{f}$.

Proposition 8. *The operator $\overline{\varepsilon}[b]$ is a weak erosion if, and only if, the set $b(x)$ is a filter for every $x \in X$.*

Proof. Assume that $\overline{\varepsilon}[b]$ is a weak erosion. As $\overline{\varepsilon}[b](X) = X$, one has $b \preceq \overline{\mathsf{ful}}$ so $X \in b(x)$ for all $x \in X$, and then Condition *(1)* for a filter is true. As $\overline{\varepsilon}[b]$ distributes over finite intersections, b preserves finite intersections: this gives both Condition *(2)* and (according to Proposition 7) the upper family property, i.e. Condition *(3)* is true. Conversely, assume that for every $x \in X$, $b(x)$ is a filter. Then $X \in b(x)$ so $\overline{\varepsilon}[b](X) = X$. Furthermore, Condition *(2)* of filters implies that $\overline{\varepsilon}[b](U \cap V) \supseteq \overline{\varepsilon}[b](U) \cap \overline{\varepsilon}[b](V)$; the other inclusion comes directly from Condition *(3)*. □

Remark 4. A variant of Proposition 8 is that $\overline{\varepsilon}[b]$ is an erosion if and only if the set $b(x)$ is *an Alexandrov filter* for every $x \in X$, i.e. $b(x)$ is a filter that is stable under arbitrary intersections. It turns out that in this case, defining $\cap b(x) = \bigcap_{U \in b(x)} U$ yields $\overline{\varepsilon}[b] = \varepsilon[\cap b]$. In other words, any erosion obtained via a structuring neighborhood can already be obtained via a well-chosen structuring element. Asking only for weak erosions allows us to strictly enhance the expressiveness of this framework (see Example 8).

Example 8 (Interior in $\mathbb{R}$). Take $X = \mathbb{R}$ and τ be its usual topology. As it does not commute with infinite intersections, the interior operator is not an erosion. However, it is a weak erosion and can be modeled in our framework using the structuring element given in Example 6.

4 Application: Morpho-Logic

The fact that erosions commute with *infinite* infima is particularly important in mathematical morphology. However, some interesting applications are still within reach of weak erosions, which only commute with *finite* infima. The most noticeable of them is maybe logic. Indeed, in logic, the infimum of two formulas is given by their conjunction. As formulas are built through an inductive process in a finite number of steps, no infinite conjunctions will ever arise. Weak erosions obtained from structuring neighborhoods turn out to be useful to model logical phenomena that were previously beyond the scope of mathematical morphology. In this section we will discuss how our framework handles neighborhood logic, which is based on structuring neighborhoods. For the sake of succintness, sometimes only erosion will be addressed, but all dual expected results also hold for dilation.

4.1 Neighborhood Modal Logic

The logical study of neighborhood structures has ancient origins (see e.g. [9]). A pretty exhaustive account is given in the recent book of Pacuit [13]. Neighborhood models are a generalization of Kripke models (see Example 2) towards a second-order semantics of modalities.

Syntax. Let P be a countable set whose elements will be called *propositional variables* and denoted by letters $p, q, r \dots$ The set $\mathcal{F}$ of formulas of modal logic is defined by the following grammar:

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_i \varphi \tag{18}$$

where i runs through some index set I and p runs through P. There are thereby $|I|$ different modal operators $\Box_i$. Other connectives like $\vee$, $\rightarrow$, $\Diamond_i$ are all expressible from those above: $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi)$ and $\Diamond_i \varphi = \neg\Box_i\neg\varphi$. Given an arbitrary $p \in P$, define also the logical constants $\top = p \vee \neg p$ and $\bot = p \wedge \neg p$.

Semantics. A *model* is a triple $\langle W, (N_i)_{i \in I}, V \rangle$ where W is a set whose elements are called *worlds* or *states*, $N_i \in \mathsf{StNb}(W)$ is a structuring neighborhood and $V : P \rightarrow \mathcal{P}(W)$ is a function called the *valuation*. The semantics of formulas with respect to a model $\langle W, N, V \rangle$ consists of a relation $\models$ included in $W \times \mathcal{F}$. The assertion $w \models \varphi$ intuitively means that the formula φ is true at state w. Its negation is denoted $w \not\models \varphi$, i.e. the formulat φ is not true at w. The relation $\models$ is defined by structural induction on formulas as follows:

- $w \models p$ if and only if $w \in V(p)$,
- $w \models \neg\varphi$ if and only if $w \not\models \varphi$,
- $w \models \varphi \wedge \psi$ if and only if $w \models \varphi$ and $w \models \psi$,
- $w \models \Box_i \varphi$ if and only if $\{w' \in W \mid w' \models \varphi\} \in N_i(w)$.

The set $\{w' \in W \mid w' \models \varphi\}$ will also be denoted by $[\![\varphi]\!]$. Note that the definition of the satisfaction relation amounts to:

- $[\![p]\!] = V(p)$,
- $[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$,
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$,
- $[\![\Box_i \varphi]\!] = \overline{\varepsilon}[N_i]([\![\varphi]\!])$.

isabelle.bloch@telecom-paristech.fr

Define an equivalence relation $\equiv$ on $\mathcal{F}$ by $\varphi \equiv \psi$ iff $[\![\varphi]\!] = [\![\psi]\!]$. Table 2 then gives rise to the logical rules listed in Table 3, where a model $\langle W, (N, N', N''), V\rangle$ is fixed, with $N'' = N \star N'$. The modalities of N, N', N'' are denoted respectively by $\Box, \Box', \Box''$, and also $\Diamond, \Diamond', \Diamond''$. Note that $\varphi \equiv \top$ means that φ is true at every state; hence $\varphi \equiv \top$ will be simply abbreviated to φ in the left column of Table 2.

Table 3. Properties of logical systems with respect to N

Logical rule	Equivalent condition on N
$\Box\neg\varphi \equiv \neg\Diamond\varphi$	True
$(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	N upper family
$(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$	N upper family
$N \preceq N' \Rightarrow \Box'\varphi \to \Box\varphi$	True
$N \preceq N' \Rightarrow \Diamond\varphi \to \Diamond'\varphi$	True
$\overline{\mathsf{sgt}} \preceq N \Leftrightarrow \Box\varphi \to \varphi$	True
$\overline{\mathsf{sgt}} \preceq N \Leftrightarrow \varphi \to \Diamond\varphi$	True
$\Box\top \equiv \top$	$N \preceq \overline{\mathsf{ful}}$
$\Diamond\bot \equiv \bot$	$N \preceq \overline{\mathsf{ful}}$
$\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$	N preserves finite $\cap$
$\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$	N preserves finite $\cap$
$\Box''\varphi \equiv \Box\Box'\varphi$	True
$\Diamond''\varphi \equiv \Diamond\Diamond'\varphi$	True

The case of classical modal logic was studied by Bloch [8]. It corresponds to the case $N = \overline{b}$ where b is given in Example 2; also all properties on the left of the above table are satisfied because $\overline{\varepsilon}[N] = \overline{\varepsilon}[\overline{b}] = \varepsilon[b]$ is then an erosion.

4.2 Topological Modal Logic

A more specific case is the one of topological modal logic, where the space W is endowed with a topology τ. The set $[\![\Box\varphi]\!]$ (resp. $[\![\Diamond\varphi]\!]$) is defined to be the topological interior (resp. closure) of $[\![\varphi]\!]$ with respect to τ. The question of whether these modalities can be represented as a pair erosion/dilation was raised in [3]. By extending the notion of a structuring element, the framework developed in this paper brings a positive answer to this issue. Indeed, the topological $\Box$ operator can be obtained as $\overline{\varepsilon}[N]$ using the structuring neighborhood of Example 6:

$$N(w) = \{U \in \mathcal{P}(W) \mid \exists O \in \tau, w \in O, O \subseteq U\} \tag{19}$$

It is a standard fact that the interior operator distributes over binary intersections and that the interior of the whole set is itself, making this operator a weak erosion. Consistently with Proposition 8, this reflects the other standard fact that the set $N(w)$ of topological neighborhoods is a filter for every $w \in W$.

5 Conclusion

In this paper, we have shown that moving from $\mathcal{P}(X)$ to $\mathcal{P}(\mathcal{P}(X))$ to define structuring neighborhoods leads to strong algebraic structures (lattice and monoid), as well as to a large set of properties of dilations and erosions, at the price of losing the adjunction property. The restriction to finite intersections resulted in a weaker definition of erosion, equivalently asking the structuring neighborhood to be a filter. This setting was then applied to morpho-logic, e.g. topological modal logic. The logic inherits the properties of the morphological operators. A potential application could then be spatial logics based on spatial relations, and their use for image understanding.

For further extensions, for instance fuzzy systems, a coalgebraic treatment might be relevant. Specifically, the theory of coalgebras has identified the concept underlying modal operators semantics as *predicate liftings* [14]. General erosion could be defined by $\varepsilon[b](U) = b^{-1}(\lambda(U))$ where the predicate lifting λ shapes the type of the system; modal properties can then be studied at the level of λ.

Other perspectives include replacing $\mathcal{P}(X)$ or $\mathcal{P}(\mathcal{P}(X))$ by any complete lattice, or further moving to a categorical setting by generalizing structuring elements as coalgebras, dilation and erosion as adjoint functors, and so on.

References

1. Aiguier, M., Atif, J., Bloch, I., Hudelot, C.: Belief revision, minimal change and relaxation: a general framework based on satisfaction systems, and applications to description logics. Artif. Intell. **256**, 160–180 (2018)
2. Aiguier, M., Atif, J., Bloch, I., Pino Pérez, R.: Explanatory relations in arbitrary logics based on satisfaction systems, cutting and retraction. Int. J. Approx. Reason. **102**, 1–20 (2018)
3. Aiguier, M., Bloch, I.: Dual logic concepts based on mathematical morphology in stratified institutions: applications to spatial reasoning. CoRR abs/1710.05661 (2017). http://arxiv.org/abs/1710.05661
4. Bloch, I., Heijmans, H., Ronse, C.: Mathematical morphology (Chap. 13). In: Aiello, M., Pratt-Hartman, I., van Benthem, J. (eds.) Handbook of Spatial Logics, pp. 857–947. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-5587-4_14
5. Bloch, I., Lang, J.: Towards mathematical morpho-logics. In: 8th International Conference on Information Processing and Management of Uncertainty in Knowledge based Systems IPMU 2000, Madrid, Spain, vol. III, pp. 1405–1412 (2000)
6. Bloch, I., Lang, J., Pino Pérez, R., Uzcátegui, C.: Morphologic for knowledge dynamics: revision, fusion, abduction. Technical report arXiv:1802.05142, arXiv cs.AI, February 2018
7. Bloch, I., Pérez, R.P., Uzcategui, C.: A unified treatment of knowledge dynamics. In: International Conference on the Principles of Knowledge Representation and Reasoning, KR2004, Canada, pp. 329–337 (2004)
8. Bloch, I.: Modal logics based on mathematical morphology for qualitative spatial reasoning. J. Appl. Non-Classical Log. **12**(3–4), 399–423 (2002)
9. Chellas, B.F.: Modal Logic: An Introduction. Cambridge University Press, Cambridge (1980)

10. Gorogiannis, N., Hunter, A.: Merging first-order knowledge using dilation operators. In: Hartmann, S., Kern-Isberner, G. (eds.) FoIKS 2008. LNCS, vol. 4932, pp. 132–150. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-77684-0_11
11. Heijmans, H.J.A.M., Ronse, C.: The algebraic basis of mathematical morphology - part I: dilations and erosions. Comput. Vis. Graph. Image Process. **50**, 245–295 (1990)
12. Najman, L., Talbot, H. (eds.): Mathematical Morphology: From Theory to Applications. ISTE-Wiley, Hoboken (2010)
13. Pacuit, E.: Nijssen: Neighborhood Semantics for Modal Logic. Springer, Heidelberg (2017). https://doi.org/10.1007/978-3-319-67149-9
14. Pattinson, D.: Coalgebraic modal logic: soundness, completeness and decidability of local consequence. Theor. Comput. Sci. **309**(1), 177–193 (2003)
15. Ronse, C.: Lattice-theoretical fixpoint theorems in morphological image filtering. J. Math. Imaging Vis. **4**(1), 19–41 (1994)
16. Serra, J.: Image Analysis and Mathematical Morphology. Academic Press, London (1982)