Hausdorff Distances Between Distributions Using Optimal Transport and Mathematical Morphology

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Abstract. In this paper we address the question of defining and computing Hausdorff distances between distributions in a general sense. We exhibit some links between Prokhorov-Lévy distances and dilation-based distances. In particular, mathematical morphology provides an elegant way to deal with periodic distributions. The case of possibility distributions is addressed using fuzzy mathematical morphology. As an illustration, the proposed approaches are applied to the comparison of spatial relations between objects in an image or a video sequence, when these relations are represented as distributions.

Keywords: Comparison of distributions · Optimal transport · Mathematical morphology · Fuzzy mathematical morphology · Hausdorff · Prokhorov · Lévy distances · Spatial relations

1 Introduction

Comparing distributions is important in image processing and understanding. Typical applications concern the comparison of histograms of gray levels or colors, or of key points [12,21]. At a more structural level, spatial relations between objects, or between instances of objects at different times, are important to assess the spatial arrangement of objects on a scene and its evolution, thus requiring also comparison between representations, e.g. as distributions, of such spatial relations [4].

In this paper we consider the general framework of comparison of distributions in a general sense (related to image information or not), that can have a probabilistic or a possibilistic and fuzzy meaning. We focus on links between dilation-based distances and optimal transport ones.

The Hausdorff distance is a good choice for comparing sets or functions, since it has all the properties of a metric on compact sets. In this paper, we study this distance between distributions, from a mathematical morphology perspective. In particular we highlight links between existing metrics such as Prokhorov and Lévy, and existing or newly proposed expressions of the Hausdorff distance

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derived from morphological dilations. We consider distributions on the real line, as well as periodic distributions, which are important for comparing histograms of colors in some specific color spaces, or directional spatial relations. This problem has been addressed using the Wasserstein distance in [16], but not using the Hausdorff distance.

The Hausdorff distance has been defined between functions in [17], and by considering 1D functions as subsets of $\mathbb{R}^2$ in [18]. We will also investigate a similar approach in this work. This idea was then further studied in [7] by considering truncated umbras and dilations by a half ball, and in [13], where the case of discontinuous functions was also addressed.

When functions are membership functions of fuzzy sets or possibility distributions, different approaches for defining the Hausdorff distance have been proposed. Some of them define the distance as a number, by combining the values of the Hausdorff distances computed between α-cuts (thresholds of the functions, hence sets), either as a weighted sum, or using the extension principle [5,6,15,22]. Several generalizations of the Hausdorff distance have also been proposed under the form of fuzzy numbers [2,8]. Extensions of the Hausdorff distance based on fuzzy mathematical morphology have been developed, either as a number in [10] from the distance from a point to a fuzzy set [3], or as a fuzzy number [3]. This last approach will be exploited in the present work too.

Some preliminaries on periodic and non periodic distributions are first given in Section 2. Several types of dilations are then proposed in Section 3. Then we propose Hausdorff distances on distributions based on optimal transport and morphological methods in Section 4. The links between these two types of approaches allow us to address the case of non periodic distributions in Section 5. This case is illustrated in Section 6 for comparing directional relations between objects and their change in a synthetic video sequence.

2 Preliminaries

Distributions and cumulative distributions. Let f and g denote the distributions (in a broad sense) to be compared, via the computation of a distance between them. We denote by M the definition domain of these distributions. In this paper, we consider only one-dimensional domains, and M can be $\mathbb{R}$ or $\mathbb{R}^+$ for non-periodic distributions, and $[0, \rho]$ for periodic distributions of period ρ (for instance $[0, 2\pi]$ for the example of relative direction in Section 6). We denote points of M by $x, y...$, or $\theta, \alpha...$ when they are angles.

Normalized distributions are assumed in this paper. Two types of normalization are considered: by the sup or max, or by the sum. The first case goes with a fuzzy or possibilistic interpretation, while the second one corresponds to a probabilistic interpretation.

The cumulative distributions of f and g are denoted by F and G. Note that defining a distance between f and g from a distance between F and G actually provides a distance between distributions. For some definitions, we will consider F and G as sets in a 2D space, denoted by SF and SG. Cumulative distributions are right continuous. Jumps correspond to discontinuities in the underlying distributions. In such cases, SF and SG are completed by vertical segments corresponding to these jumps. In the sequel, we always assume that SF and SG are completed graphs. Therefore, the subset SF associated with a cumulative distribution F is the set defined as:

$$SF = \{(x, F(x)) \mid x \in M\} \cup \{(x, y) \mid x \in J(F) \text{ and } \lim_{x' \to x^-} F(x') \leq y \leq F(x)\}$$

where $J(F)$ denotes the set of points at which jumps occur (i.e. where the left limit of F at x is not equal to $F(x)$).

Ground distance. Existing methods for comparing histograms or probability distributions [9] are usually categorized into two classes: (i) bin-to-bin distances, and (ii) cross-bin distances, involving the distance on the support M (or ground-distance) [9,16,21]. In this paper, we only consider distances of the second class, keeping in mind the application to spatial relations. For instance, if two distributions are identical up to a translation and with disjoint supports, the distances of the first class will always provide the same value, while the second ones will differentiate situations with different translations.

Let us denote by d the ground distance on M. Its definition depends on M. If M is equal to $\mathbb{R}$ or $\mathbb{R}^+$, then d is defined from an L_p norm, for instance $d(x, y) = |x - y|$ in 1D. For periodic distributions (or defined on a circle), the geodesic distance is used. If the period is ρ, we will use $d(x, y) = \min(|x-y|, \rho-|x-y|) = \frac{\rho}{2} - ||x - y| - \frac{\rho}{2}|$. In case of distributions on the circle, with $\rho = 2\pi$, this ground distance is expressed as $d(\theta, \theta') = \min(|\theta-\theta'|, 2\pi-|\theta-\theta'|) = \pi-||\theta-\theta'|-\pi|$. This formulation allows us to consider that values close to 0 and 2π, respectively, are at a short distance from each other. The distance values can also be normalized, using for instance $\frac{d(\theta,\theta')}{\pi}$ or $\sin \frac{|\theta-\theta'|}{2}$.

3 Definition of Some Dilations of Distributions

3.1 Morphological Dilation of a Normalized Distribution

We assume in this section that the distributions are normalized by the sup (and we restrict this work to distributions with bounded sup), or at least that they all have the same maximum value. To simplify the presentation, we consider binary structuring elements, defined as subsets of M.

If the distributions are defined on the real line ($M = \mathbb{R}$ or $M = \mathbb{R}^+$), classical mathematical morphology applies and the dilation of f by a structuring element B is expressed by $\forall x \in M, \delta_B(f)(x) = \sup_{y \in B_x} f(y)$, where B_x denotes as usual the translation of B at x ($B_x = x + B$).

If the distributions are periodic, this periodicity should be taken into account in the dilation and the structuring element.

Definition 1. *Let f be a distribution on the unit circle. Its dilation is defined by:*

$$\forall \theta \in M = [0, 2\pi], \delta_{B^\alpha}(f)(\theta) = \sup_{\theta' \in B^\alpha_\theta} f(\theta') \tag{1}$$

where B^α is a structuring element of aperture α, defined as:

– *if* $\alpha \leq \pi$: $B^\alpha_\theta = [\theta - \alpha, \theta + \alpha]$ *if* $\theta - \alpha \geq 0$ *and* $\theta + \alpha \leq 2\pi$,
  $B^\alpha_\theta = [0, \theta + \alpha] \cup [\theta - \alpha + 2\pi, 2\pi]$ *if* $\theta - \alpha \leq 0$ *and* $\theta + \alpha \leq 2\pi$,
  $B^\alpha_\theta = [\theta - \alpha, 2\pi] \cup [0, \theta + \alpha - 2\pi]$ *if* $\theta - \alpha \geq 0$ *and* $\theta + \alpha \geq 2\pi$,
– *if* $\alpha \geq \pi$: $B^\alpha_\theta = [0, 2\pi]$. *(The case* $\theta - \alpha \leq 0$ *and* $\theta + \alpha \geq 2\pi$ *implies* $\alpha \geq \pi$*.)*

Note that Definition 1 extends directly to any periodic function.

The normalization ensures that the core of the distribution (set of points with maximum value) is extended according to the size of the structuring element. In particular, it is always possible to find a size of dilation such that a given point of the support of the distribution belongs to the core of the dilated distribution. This property will be used for Hausdorff distances defined from such dilations. The following proposition is easy to show (proofs are omitted due to lack of space):

Proposition 1. *For all* α, B^α *is a ball or radius* α *of the ground distance* d, *and for all* f *and* α, *we have* $\forall \theta, \delta_{B^\alpha}(f)(\theta) = \sup\{f(\theta') \mid d(\theta, \theta') \leq \alpha\}$.

3.2 Dilations of Cumulative Distributions

In this section we consider a cumulative distribution either as a function F from M into $[0,1]$, or as a subset SF of $M \times [0,1]$.

Let us consider as a structuring element a segment of length 2ε, with $\varepsilon \geq 0$. We denote by $B^\varepsilon_x = [x - \varepsilon, x + \varepsilon] \cap M$ the translation of this structuring element at x, restricted to the support.

Proposition 2. *The dilation of* F *by* B^ε *is expressed as:*

$$\forall x \in M, \delta_{B^\varepsilon}(F)(x) = \sup_{y \in B^\varepsilon_x} F(y) = \begin{cases} F(x+\varepsilon) & \textit{if } x + \varepsilon \in M \\ 1 & \textit{otherwise} \end{cases}$$

Let us now consider the dilation of SF, using different structuring elements, that will prove useful in the following. Let us first consider a ball of radius ε of the L^∞ distance, with a positive proportionality factor λ on M to account for the different scales of the two dimensions (i.e. the structuring element is a rectangle). It is expressed, when translated at (x,y), as:

$$(B^{\varepsilon,\lambda}_1)_{(x,y)} = (\check{B}^{\varepsilon,\lambda}_1)_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon].$$

Proposition 3. *The dilation of any* SF *by* $B^{\varepsilon,\lambda}_1$ *is expressed as:*

$$\delta^{\varepsilon,\lambda}_1(SF) = \{(x,y) \in M \times [0,1] \mid \exists x' \in M, \max(\frac{|x - x'|}{\lambda}, |y - F(x')|) \leq \varepsilon\}. \quad (2)$$

This dilation is illustrated in Figure 1, for $\lambda = 1$.

Let us now consider an asymmetric dilation, with the following structuring element centered at (x,y) and of size ε (still with the factor λ on M): $(B^{\varepsilon,\lambda}_2)_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [y - \varepsilon, 1]$. Its symmetrical with respect to (x,y) is then: $(\check{B}^{\varepsilon,\lambda}_2)_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [0, y + \varepsilon]$.

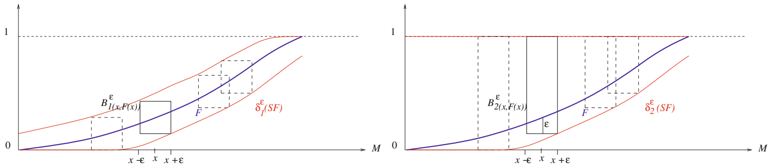

Fig. 1. Dilation with a symmetrical structuring element (left) and with a non-symmetrical one (right)

Proposition 4. *The asymmetric dilation of SF by $B_2^{\varepsilon,\lambda}$ is expressed as:*

$$\delta_2^{\varepsilon,\lambda}(SF) = \{(x,y) \in M \times [0,1] \mid \exists x' \in M, \max(\frac{|x-x'|}{\lambda}, F(x') - y) \leq \varepsilon\}.$$

It is illustrated in Figure 1.

3.3 Dilations of Cumulative Distributions in the Periodic Case

All the definitions introduced above apply also to the periodic case, using the following embedding of F into $\mathbb{R}$:

$$\forall x \in \mathbb{R}, F(x+\rho) = F(x) + 1 \tag{3}$$

and then normalizing the space. For instance if $\rho = 2\pi$, it is sufficient to consider an embedding in $]-\pi, 3\pi[\times[-1,2]$ since for $\lambda\varepsilon \geq \pi$, the dilation would provide the whole space $M \times [0,1]$. The extension of SF then writes:

$$SF^E = SF \cup \{(\theta, F(\theta+2\pi)-1), \theta \in]-\pi, 0]\} \cup \{(\theta, F(\theta-2\pi)+1), \theta \in [2\pi, 3\pi[\}. \tag{4}$$

Dilations can be expressed directly from this set, and we have the following simple form.

Proposition 5. *The dilation of SF with a symmetrical structuring element and $\lambda\varepsilon \leq \pi$ is expressed as:*

$$\delta_{c1}^{\varepsilon,\lambda}(SF) = \{(\theta,y) \in [0,2\pi]\times[0,1] \mid \exists\theta' \in [0,2\pi], |\theta-\theta'| \leq \lambda\varepsilon \text{ and } |F(\theta')-y| \leq \varepsilon\}. \tag{5}$$

For $\lambda\varepsilon > \pi$, then $\delta_{c1}^{\varepsilon,\lambda}(SF) = [0,2\pi] \times [0,1]$.

Note that the simple expression obtained in Proposition 5 corresponds to a geodesic way to process the boundaries of the domain, by truncating the translated structuring element to limit it to the part included in $[0,2\pi] \times [0,1]$. This dilation is illustrated in Figure 2.

Considering now the structuring element $B_2^{\varepsilon,\lambda}$ to dilate only the subgraph (and saturating its complement to 1) leads also to a simple expression:

Proposition 6. *The dilation of SF with an asymmetrical structuring element and $\lambda\varepsilon \leq \pi$ is expressed as:*

$$\delta_{c2}^{\varepsilon,\lambda}(SF) = \{(\theta,y) \in [0,2\pi]\times[0,1] \mid \exists\theta' \in [0,2\pi], |\theta-\theta'| \leq \lambda\varepsilon \text{ and } F(\theta')-y \leq \varepsilon\}. \tag{6}$$

For $\lambda\varepsilon > \pi$, we have $\delta_{c2}^{\varepsilon,\lambda}(SF) = [0,2\pi] \times [0,1]$.

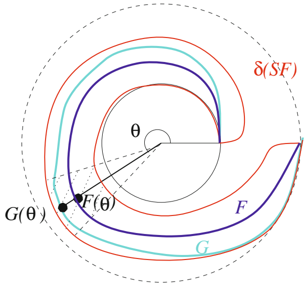

Fig. 2. Dilation in the periodic case, for a symmetrical structuring element. The central circle corresponds to 0 and the larger one to 1. The dashed area is an example of structuring element centered at $(\theta, F(\theta))$. The dilation of SF includes SG.

4 Distances Between Distributions on the Real Line

4.1 Morphological Approach

Haudorff distance from dilations of cumulative distributions. Let us first consider $\delta_1^{\varepsilon,\lambda}$ introduced in Section 3.2, and let us derive a Hausdorff distance from it (see Figure 3, for $\lambda = 1$).

Proposition 7. *The Hausdorff distance associated with δ_1 is:*

$$d_{H1}(F,G) = \max(\sup_{x\in M}\inf_{y\in M}\max(\frac{|x-y|}{\lambda}, |G(x)-F(y)|),$$
$$\sup_{y\in M}\inf_{x\in M}\max(\frac{|x-y|}{\lambda}, |F(y)-G(x)|)). \quad (7)$$

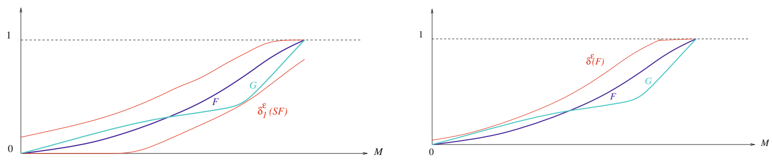

Fig. 3. Left: Minimal size of the dilation of SF such that it contains SG. Right: Computation of the Hausdorff distance by dilating the cumulative distributions considered as functions.

Let us now consider the asymmetric dilation δ_2.

Proposition 8. *The Hausdorff distance derived from δ_2 is:*

$$d_{H2}(F,G) = \max(\sup_{x\in M}\inf_{y\in M}\max(\frac{|x-y|}{\lambda}, G(y)-F(x)),$$
$$\sup_{y\in M}\inf_{x\in M}\max(\frac{|x-y|}{\lambda}, F(x)-G(y))). \quad (8)$$

Finally, let us derive the Hausdorff distance from cumulative distributions considered as functions.

Proposition 9. *We have:*

$$d_H(F,G) = \inf\{\varepsilon > 0 \mid \forall x \in M, G(x) \leq F(x+\varepsilon) \text{ and } F(x) \leq G(x+\varepsilon)\}. \quad (9)$$

This is illustrated in Figure 3.

Proposition 10. *All distances defined in this section are metrics (i.e. positive, separable, symmetrical and satisfy the triangular inequality). If the distributions are Dirac functions (with a unique non zero value at f_0 and g_0), the proposed distances are all equal to $d(f_0, g_0)$, where d is the ground distance.*

Hausdorff distance from dilations of distributions. The idea here is to exploit the link between morphological dilation and some distances, such as minimum and Hausdorff distances, in the case of sets [3,19]. Indeed, the Hausdorff distance between two sets is equal to the minimal size of the ball of the ground distance such that the dilation of each set by this ball contains the other set. We propose to use the same principle on distributions.

Definition 2. *[3] The fuzzy Hausdorff distance is defined from the dilation of the distributions, considered as fuzzy sets, and from an inclusion operator $\Delta_{\subseteq}(f,g)$, expressing the degree to which f is included in g:*

$$\forall \ell \in \mathbb{R}^{+*}, d_H(f,g)(\ell) = t(d'_H(f,g)(\ell), d'_H(g,f)(\ell)) \quad (10)$$

with

$$d'_H(f,g)(\ell) = t(\Delta_{\subseteq}(f, \delta_{B^\ell}(g)), \inf_{0 \leq \ell' < \ell} c(\Delta_{\subseteq}(f, \delta_{B^{\ell'}}(g)))),$$

and $d'_H(f,g)(0) = \Delta_{\subseteq}(f,g)$, with t a t-norm.

The value $d_H(f,g)(\ell)$ expresses the degree to which the Hausdorff distance between f and g is equal to ℓ. A common definition of an inclusion degree in the fuzzy set framework is $\Delta_{\subseteq}(f,g) = \inf_{x \in M} I(f(x), g(x))$ where I is a fuzzy implication. If a crisp number is needed, the center of gravity of this fuzzy number can be used: $\frac{\int_0^\infty d_H(f,g)(\ell) \ell d\ell}{\int_0^\infty d_H(f,g)(\ell) d\ell}$, or the following definition:

$$d_H(f,g) = \inf\{\ell \in \mathbb{R}^+ \mid \forall x \in M, \delta_{B^\ell}(f)(x) \geq g(x) \text{ and } \delta_{B^\ell}(g)(x) \geq f(x)\}, \quad (11)$$

which corresponds to a crisp version of the inclusion. This simplified expression corresponds to the definitions in [7,13] for flat structuring elements.

Proposition 11. *[3] The fuzzy distances introduced in Equations 10 and 11 are positive and symmetrical. The morphological Hausdorff distance between the distributions and computed with a crisp version of the inclusion degree (Equation 11) is separable and satisfies the triangular inequality, while the fuzzy version of the inclusion degree yields a distance (Equation 10) which is a fuzzy number, and separable for Lukasiewicz implication ($I(a,b) = \min(1, 1-a+b)$), but does not satisfy the triangular inequality.*

4.2 Lévy and Prokhorov Distances

An interesting distance between probability distributions, related to optimal transport problems [20] and which involves dilations, is the Prokhorov-Lévy metric $d_{Pr} : \mathcal{P}(M)^2 \to [0, +\infty[$ [14], defined for two distributions f and g as:

$$d_{Pr}(f,g) = \inf\{\varepsilon > 0 \mid \forall Z \in \mathcal{B}(M), f(Z) \leq g(\delta^{\lambda\varepsilon}(Z)) + \varepsilon \text{ and } g(Z) \leq f(\delta^{\lambda\varepsilon}(Z)) + \varepsilon\} \tag{12}$$

where $\delta^{\lambda\varepsilon}(Z)$ is the dilation of size $\lambda\varepsilon$ of Z (see Section 3.1, restricting functions to sets), and $\mathcal{B}(M)$ denotes the set of all Borel sets on M. The definition has been adapted here to introduce λ and thus to account for the potential different scales of M and $[0,1]$, as in [17].

This distance generalizes the Lévy distance (also a metric), defined in 1D between two cumulative distributions F and G as:

$$d_L(F,G) = \inf\{\varepsilon > 0 \mid \forall x \in \mathbb{R}, G(x - \lambda\varepsilon) - \varepsilon \leq F(x) \leq G(x + \lambda\varepsilon) + \varepsilon\}. \tag{13}$$

By restricting the Borel sets of $\mathbb{R}$ to the intervals of the form $Z =]-\infty, x[$ (or equivalently $Z =]x, +\infty[$), which generate $\mathcal{B}(M)$, d_{Pr} is indeed equivalent to d_L in 1D. Note that if all Borel sets are considered, then we only have $d_L \leq d_{Pr}$.

Hausdorffian expression of d_L. The Lévy distance can be expressed in a similar way as the Hausdorff distance [17] and we have:

$$\begin{aligned} d_L(F,G) = \max(&\sup_{x \in M} \inf_{y \in M} \max(\frac{|x-y|}{\lambda}, G(y) - F(x)), \\ &\sup_{y \in M} \inf_{x \in M} \max(\frac{|x-y|}{\lambda}, F(x) - G(y))). \end{aligned} \tag{14}$$

Note that this expression involves explicitly the ground distance on M.

We now exhibit links with Hausdorff distances derived from the dilations proposed in Section 3.2. Note that d_{Pr} already involves a dilation and that the links between d_{Pr}, d_L and its Hausdorff-like expression already suggest that all these notions are closely related.

Proposition 12. *Let F and G be any two cumulative distributions. We have the following equivalences between their distances:*

- *the Lévy distance can be formulated as a Hausdorff-like expression (Equation 14);*
- *Equation 7 is similar to Equation 14, but with absolute values on $G(x)-F(y)$, providing one of the definitions in [17];*
- *Equation 8 is equivalent to Equation 14;*
- *Equation 9 is equivalent to Equation 13;*
- *Equation 11 is similar to d_{Pr} expressed on points.*

All these links make it easier to extend the definitions to the periodic case (next section).

Proposition 13. *d_L is a probability metric [17]. Similarly, the Hausdorff distances defined in Equations 7 and 9 are probability metrics.*

5 Distances Between Periodic Distributions

In this section we now assume periodic distributions. To fix the ideas, we set, without loss of generality, $\rho = 2\pi$.

5.1 Lévy and Prokhorov Distances

Let us start again from d_{Pr}. We propose to express this distance from a circular dilation and by restricting the Borelian sets to $Z = [0, \theta]$ (which are generating all Borelian sets on $[0, 2\pi]$), taking 0 as origin, arbitrarily[1]. Let us define a dilation of size ε, in the positive direction, as: $\delta^{\varepsilon}(Z) = [0, \theta + \varepsilon]$ if $\theta + \varepsilon \leq 2\pi$ and $[0, 2\pi]$ otherwise. This morphological expression allows us to derive easily the following result.

Proposition 14. *The Lévy distance, derived from the Prokhorov distance in 1D in the periodic case, is expressed as:*

$$d_L^c(F, G) = \inf\{\varepsilon > 0 \mid \forall \theta \in [0, 2\pi], F(\theta) \leq G(\theta + \lambda\varepsilon) + \varepsilon \text{ and } G(\theta) \leq F(\theta + \lambda\varepsilon) + \varepsilon\}. \tag{15}$$

by setting $G(\theta + \lambda\varepsilon) = F(\theta + \lambda\varepsilon) = 1$ if $\theta + \lambda\varepsilon \geq 2\pi$.

5.2 Morphological Approach

Haudorff distance from dilations of cumulative distributions. Let us consider symmetrical dilations.

Proposition 15. *The Hausdoff distance derived from δ_{c1} computed with a symmetrical structuring element is:*

$$d_{Hc1}(F, G) = \max\Big(\sup_{\theta \in [0,2\pi]} \inf_{\theta' \in [0,2\pi]} \max\Big(\frac{|\theta - \theta'|}{\lambda}, |F(\theta') - G(\theta)|\Big),$$
$$\sup_{\theta \in [0,2\pi]} \inf_{\theta' \in [0,2\pi]} \max\Big(\frac{|\theta - \theta'|}{\lambda}, |G(\theta') - F(\theta)|\Big)\Big).$$

The asymmetrical dilation δ_{c2} leads to similar results, and the derived Hausdorff distance has a similar expression, without the absolute values:

$$d_{Hc2}(F, G) = \max\Big(\sup_{\theta \in [0,2\pi]} \inf_{\theta' \in [0,2\pi]} \max\Big(\frac{|\theta - \theta'|}{\lambda}, G(\theta) - F(\theta')\Big),$$
$$\sup_{\theta \in [0,2\pi]} \inf_{\theta' \in [0,2\pi]} \max\Big(\frac{|\theta - \theta'|}{\lambda}, F(\theta) - G(\theta')\Big)\Big).$$

[1] If the origin is taken at θ_0, then the cumulative distribution is $\int_{\theta_0}^{\theta} f(t)dt = \int_0^{\theta} f(t)dt - \int_0^{\theta_0} f(t)dt = F(\theta) - F(\theta_0)$ if $\theta_0 \leq \theta \leq 2\pi$, and $\int_{\theta_0}^{2\pi} f(t)dt + \int_0^{\theta} f(t)dt = 1 - F_0(\theta_0) + F_0(\theta)$ if $0 \leq \theta \leq \theta_0$. If we want a distance which is independent of the choice of the origin, then $\inf_{\theta_0} d_L^c(F_{\theta_0}, G_{\theta_0})$ could be considered.

The computation of $d_{Hc1}(F, G)$ is illustrated in Figure 2, where the minimal size of dilation of SF such that it includes SG is shown.

Proposition 16. *As in the non-periodic case, the Hausdorff distance derived from asymmetrical dilation and the Lévy distance are equal:*

$$d_{Hc2}(F, G) = d_L^c(F, G). \tag{16}$$

Hausdorff distance from dilations of distributions. The definitions proposed in Equations 10 and 11 apply directly to periodic distributions, by considering appropriate dilations, taking the periodicity into account, as defined in Section 3.1.

An example of distribution on $[0, 2\pi]$ is given in Figure 4, with three translations. The Hausdorff distances values, computed using morphological dilations of the distributions (using Equation 11), between the first distribution of Figure 4 and the others, correspond to the distance between the cores of the distributions, as expected in this simple case.

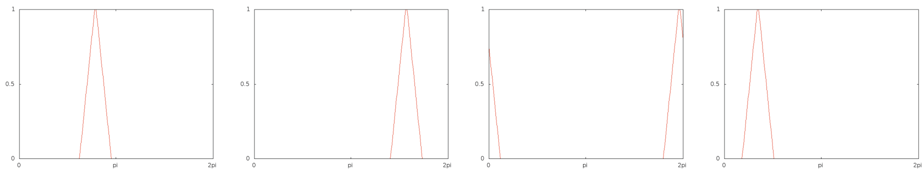

Fig. 4. Example of distribution on $[0, 2\pi]$ and three translations ($T = 2.45, T = 3.68, T = 4.9$). The distances values (in radians) are 0 for $T = 0$, 2.45 for $T = 2.45$, 2.60 for $T = 3.68$, and 1.37 for $T = 4.9$.

6 Comparison Between Spatial Relations

Observing the evolution of a pathology in medical images, or of soil occupation in remote sensing, detecting changes in video sequences, updating a spatial information system are examples that can all benefit from quantification and comparison of spatial relations between objects in the observed scenes. In this paper, to illustrate the proposed approaches, we consider spatial relations represented as distributions or fuzzy numbers, with the typical example of directional relations, represented as a periodic function on $[0, 2\pi]$ via the angle histogram [11]. The normalized angle histogram $ha_{A,B}$ between two 2D objects A and B is defined as: $\forall \theta \in [0, 2\pi], ha_{A,B}(\theta) = \frac{h'_{A,B}(\theta)}{\sup_{\theta' \in [0,2\pi]} h'_{A,B}(\theta')}$, with $h'_{A,B}(\theta) = |\{(a, b), a \in A, b \in B \mid \angle(a, b) = \theta\}|$ and $\angle(a, b)$ the angle modulo 2π between the vector $\boldsymbol{ab}$ and the horizontal axis. This sum is further weighted by the membership values of a to A and of b to B if the objects are fuzzy.

Let us consider, as an example, the application of the proposed approach to quantify the evolution of directional relations between objects in a simulated

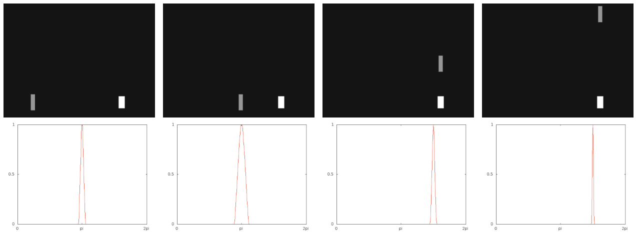

Fig. 5. Simulated video sequence (top, some frames) and angle histograms (bottom).

video sequence (Figure 5). The grey object gets close to the white one in a constant direction, and then changes direction and goes away. The angle histograms ha between these two objects are also illustrated in this figure.

These histograms have been compared using the different proposed measures, by computing the distance between the histogram at time t and the histogram in the first frame. The curves showing the evolution of this distance along time are displayed in Figure 6 for the morphological Hausdorff distance and for the Prokhorov-Lévy distance. In all these curves a jump is observed at the instant where the change in direction occurs, which was expected. We can also notice the strong similarity between these curves, as also observed on other examples.

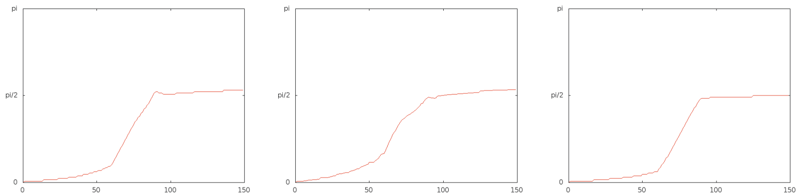

Fig. 6. Morphological Hausdorff distances between the histogram in each frame and the one in the first frame (left). Prokhorov-Lévy distance between the histogram in each frame and the one in the first frame, for histograms normalized by the sup (middle) and by the sum (right).

7 Conclusion

In this paper we have investigated several forms of Hausdorff distances for comparing distributions or cumulative distributions. Based on existing definitions and new ones proposed in this paper, we have exhibited interesting links between optimal transport metrics and morphological ones. In particular, these links have allowed adaptations and extensions to the case of periodic distributions. As an illustration, we have shown that the proposed distances allow comparing spatial relations between objects in images or videos, represented as distributions. This could lead to future applications for detection of ruptures in temporal sequences [1], for comparing different spatial configurations of objects, as a guide for structural recognition and scene understanding, and more generally for spatial reasoning. In our future work we will also go deeper in the formal properties of the proposed distances and their links.

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