Fuzzy and Bipolar Mathematical Morphology, Applications in Spatial Reasoning

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Abstract. Mathematical morphology is based on the algebraic framework of complete lattices and adjunctions, which endows it with strong properties and allows for multiple extensions. In particular, extensions to fuzzy sets of the main morphological operators, such as dilation and erosion, can be done while preserving all properties of these operators. Another, more recent, extension, concerns bipolar fuzzy sets. These extensions have numerous applications, two of each being presented here. The first one concerns the definition of spatial relations, for applications in spatial reasoning and model-based recognition of structures in images. The second one concerns the handling of the bipolarity feature of spatial information.

Keywords: Fuzzy mathematical morphology, bipolar mathematical morphology, spatial relations, bipolar spatial information, spatial reasoning.

1 Algebraic Framework of Mathematical Morphology

Mathematical morphology [1] requires the algebraic framework of complete lattices [2]. Let $(\mathcal{T}, \leq)$ be a complete lattice, $\vee$ the supremum and $\wedge$ the infimum. A dilation is an operator δ on $\mathcal{T}$ which commutes with the supremum: $\forall (x_i) \in \mathcal{T},\ \delta(\vee_i x_i) = \vee_i \delta(x_i)$. An erosion is an operator ε on $\mathcal{T}$ which commutes with the infimum: $\forall (x_i) \in \mathcal{T},\ \varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i)$ [3]. Such operators are called algebraic dilation and erosion. An important property is that they are increasing with respect to $\leq$.

An adjunction on $(\mathcal{T}, \leq)$ is a pair of operators (ε, δ) such that $\forall (x,y) \in \mathcal{T}^2,\ \delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)$. If (ε, δ) is an adjunction, then ε is an algebraic erosion and δ an algebraic dilation. Additionally, the following properties hold: $\varepsilon\delta \geq Id$, where Id denotes the identity mapping on $\mathcal{T}$, $\delta\varepsilon \leq Id$, $\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon\delta = \delta$, $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ et $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ (the compositions $\delta\varepsilon$ and $\varepsilon\delta$ are known as morphological opening and closing, respectively, and can also be formalized in the framework of Moore families [4]).

In the particular case of the lattice of subparts of $\mathbb{R}^n$ or $\mathbb{Z}^n$, denoted by $\mathcal{S}$ in the following, endowed with inclusion as partial inclusion, adding a property of invariance under translation leads to the particular following forms (called morphological dilation and erosion):

$$\forall X \subseteq \mathcal{S}, \delta_B(X) = \{x \in \mathcal{S} \mid \check{B}_x \cap X \neq \emptyset\}, \quad \varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\},$$

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where B is a subset of $\mathcal{S}$ called structuring element, B_x denotes its translation at point x and $\check{B}$ its symmetrical with respect to the origin of space. Opening and closing are defined by composition (using the same structuring element).

These definitions are general and apply to any complete lattice. In the following, we focus on the lattice of fuzzy sets defined on $\mathcal{S}$ and on the lattice of bipolar fuzzy sets. Other works have been done on the lattice of logical formulas in propositional logics [5,6,7,8], with applications to fusion, revision, abduction, mediation, or in modal logics [9], with applications including qualitative spatial reasoning.

Mathematical morphology can therefore be considered as a unifying framework for spatial reasoning, leading to knowledge representation models and reasoning tools in quantitative, semi-quantitative (or fuzzy) and qualitative settings [10].

2 Fuzzy Mathematical Morphology

Extending mathematical morphology to fuzzy sets was proposed in the early 90's, by several teams independently [11,12,13,14,15], and was then largely developed (see e.g. [16,17,18,19,20,21]). An earlier extension of Minkowski's addition (which is directly linked to dilation) was defined in [22].

Let $\mathcal{F}$ be the set of fuzzy subsets of $\mathcal{S}$. For the usual partial ordering $\leq$ ($\mu \leq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x)$), $(\mathcal{F}, \leq)$ is a complete lattice, on which algebraic operations can be defined, as described in Section 1. Adding a property of invariance under translation leads to the following general forms of fuzzy dilation and erosion [12,16]:

$$\forall x \in \mathcal{S},\ \delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} T[\nu(x-y), \mu(y)], \quad \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} S[c(\nu(y-x)), \mu(y)],$$

where ν denotes a fuzzy structuring element in $\mathcal{F}$, μ a fuzzy set, c an involutive negation (or complementation), T a t-norm and S a t-conorm. The adjunction property imposes that S be the t-conorm derived from the residual implication I of T: $\forall (\alpha, \beta) \in [0,1]^2, S(\alpha, \beta) = I(c(\alpha), \beta)$, with $I(\alpha, \beta) = \sup\{\gamma \in [0,1], T(\alpha, \gamma) \leq \beta\}$. The erosion represents the degree to which the translation of the structuring element at point x intersects μ, while the dilation represents the degree to which it is included in μ.

For applications dealing with spatial objects for instance, it is often important to also have a duality property between dilation and erosion, with respect to the complementation. Then T and S have to be dual operators with respect to c. This property, along with the adjunction property, limits the choice of T and S to generalized Lukasiewicz operators [23,24]: $T(\alpha, \beta) = \max(0, \varphi^{-1}(\varphi(\alpha) + \varphi(\beta) - 1))$ and $S(\alpha, \beta) = \min(1, \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)))$ where φ is a continuous strictly increasing function on $[0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$.

The links between definitions obtained for various forms of conjunctions and disjunctions have been presented from different perspectives in [16,20,23,24,25].

Opening and closing are defined by composition, as in the general case. The adjunction property guarantees that these operators are idempotent, and that opening (resp. closing) is anti-extensive (resp. extensive) [16,17,24].

3 Bipolar Fuzzy Mathematical Morphology

Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [26].

A bipolar fuzzy set on $\mathcal{S}$ is defined by a pair of functions (μ, ν) such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. For each point x, $\mu(x)$ defines the membership degree of x (positive information) and $\nu(x)$ the non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness.

Let us consider the set $\mathcal{L}$ of pairs of numbers (a, b) in $[0, 1]$ such that $a + b \leq 1$. It is a complete lattice, for the partial order defined as [27]: $(a_1, b_1) \preceq (a_2, b_2)$ iff $a_1 \leq a_2$ and $b_1 \geq b_2$. The greatest element is $(1, 0)$ and the smallest element is $(0, 1)$. The supremum and infimum are respectively defined as: $(a_1, b_1) \vee (a_2, b_2) = (\max(a_1, a_2), \min(b_1, b_2))$, $(a_1, b_1) \wedge (a_2, b_2) = (\min(a_1, a_2), \max(b_1, b_2))$. The partial order $\preceq$ induces a partial order on the set of bipolar fuzzy sets: $(\mu_1, \nu_1) \preceq (\mu_2, \nu_2)$ iff $\forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$ and $\nu_1(x) \geq \nu_2(x)$, and infimum and supremum are defined accordingly. It follows that, if $\mathcal{B}$ denotes the set of bipolar fuzzy sets on $\mathcal{S}$, $(\mathcal{B}, \preceq)$ is a complete lattice.

Mathematical morphology on bipolar fuzzy sets has been first introduced in [28]. Once we have a complete lattice, it is easy to define algebraic dilations and erosions on this lattice, as described in Section 1, as operators that commute with the supremum and the infimum, respectively. Their properties are derived from general properties of lattice operators.

Let us now consider morphological operations based on a structuring element. A degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as: $\inf_{x \in \mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x)))$, where I is an implication operator. Two types of implication can be defined [29], one derived from a bipolar t-conorm $\perp$[1]: $I_N((a_1, b_1), (a_2, b_2)) = \perp((b_1, a_1), (a_2, b_2))$, and one derived from a residuation principle from a bipolar t-norm $\top$[2]: $I_R((a_1, b_1), (a_2, b_2)) = \sup\{(a_3, b_3) \in \mathcal{L} \mid \top((a_1, b_1), (a_3, b_3)) \preceq (a_2, b_2)\}$, where $(a_i, b_i) \in \mathcal{L}$ and (b_i, a_i) is the standard negation of (a_i, b_i). Two types of t-norms and t-conorms are considered in [29]: operators called t-representable t-norms and t-conorms, which can be expressed using usual t-norms t and t-conorms T, and Lukasiewicz operators, which are not t-representable. A similar approach has been used for intuitionistic fuzzy sets in [30], but with weaker properties (in particular an important property such as the commutativity of erosion with the conjunction may be lost).

[1] A bipolar disjunction is an operator D from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $D((1, 0), (1, 0)) = D((0, 1), (1, 0)) = D((1, 0), (0, 1)) = (1, 0)$, $D((0, 1), (0, 1)) = (0, 1)$ and that is increasing in both arguments. A bipolar t-conorm is a commutative and associative bipolar disjunction such that the smallest element of $\mathcal{L}$ is the unit element.

[2] A bipolar conjunction is an operator C from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $C((0, 1), (0, 1)) = C((0, 1), (1, 0)) = C((1, 0), (0, 1)) = (0, 1)$, $C((1, 0), (1, 0)) = (1, 0)$ and that is increasing in both arguments. A bipolar t-norm is a commutative and associative bipolar conjunction such that the largest element of $\mathcal{L}$ is the unit element.

Based on these concepts, the morphological erosion of $(\mu, \nu) \in \mathcal{B}$ by a bipolar fuzzy structuring element $(\mu_B, \nu_B) \in \mathcal{B}$ is defined as:

$$\forall x \in \mathcal{S}, \varepsilon_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \inf_{y \in \mathcal{S}} I((\mu_B(y - x), \nu_B(y - x)), (\mu(y), \nu(y))).$$

Dilation can be defined based on a duality principle or based on the adjunction property. Applying the duality principle to bipolar fuzzy sets using a complementation c (typically the standard negation $c((a, b)) = (b, a)$) leads to the following definition of morphological bipolar dilation:

$$\delta_{(\mu_B,\nu_B)}((\mu, \nu)) = c[\varepsilon_{(\mu_B,\nu_B)}(c((\mu, \nu)))].$$

Let us now consider the adjunction principle, as in the general algebraic case. The bipolar fuzzy dilation, adjoint of the erosion, is defined as:

$$\begin{aligned}\delta_{(\mu_B,\nu_B)}((\mu, \nu))(x) &= \inf\{(\mu', \nu')(x) \mid (\mu, \nu)(x) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu', \nu'))(x)\} \\ &= \sup_{y \in \mathcal{S}} \top((\mu_B(x - y), \nu_B(x - y)), (\mu(y), \nu(y))).\end{aligned}$$

It has been shown that the adjoint operators are all derived from the Lukasiewicz operator, using a continuous bijective permutation on $[0, 1]$ [29]. Hence equivalence between both approaches can be achieved only for this class of operators.

Properties of these operations are consistent with the the ones holding for sets and for fuzzy sets, and are detailed in [28,31,32,33,34]. Interpretations of these definitions as well as some illustrative examples can also be found in these references.

4 Spatial Relations and Spatial Reasoning

Mathematical morphology provides tools for spatial reasoning at several levels [10]. The notion of structuring element captures the local spatial context, in a fuzzy and bipolar way here, which endows dilation and erosion with a low level spatial reasoning feature. At a more global level, several spatial relations between spatial entities can be expressed as morphological operations, in particular using dilations [35,10], leading to large scale spatial reasoning.

The interest of relationships between objects has been highlighted in very different types of works: in vision, for identifying shapes and objects, in database system management, for supporting spatial data and queries, in artificial intelligence, for planning and reasoning about spatial properties of objects, in cognitive and perceptual psychology, in geography, for geographic information systems. In all these domains, objects, relations, knowledge and questions to be answered may suffer from imprecision, and benefit from a fuzzy modeling, as stated in the 75's [36]. Spatial relations can be intrinsically fuzzy (for instance *close to*, *between*...) or have to be fuzzified in order to cope with imprecisely defined objects.

Fuzzy mathematical morphology has then naturally led to the definition of fuzzy spatial relations (see [35] for a review on fuzzy spatial relations, including morphological approaches). In our previous work, we proposed original definitions for both topological and metric relations (according to the classification of [37]): adjacency, distances, directional relations, and more complex relations such as *between* and *along*. Here we just discuss a few important features (the reader can refer to [35] and the references cited therein for the mathematical developments).

In spatial reasoning, two important questions arise: (i) to which degree is a relation between two objects satisfied? (ii) which is the spatial region in which a relation to a reference object is satisfied (to some degree)? Fuzzy models allow answering these two types of questions. Let us consider the directional relation *to the right of* [38]. Two objects are displayed in Figure 1. Object B is, to some degree, to the right of R. The region of space to the right of R (c) is defined as the dilation of R with a fuzzy structuring element providing the semantics of the relation (b). The membership degree of each point provides the degree to which the relation is satisfied at that point. The definition of the relation as a dilation is a generic model, but the structuring element can be adapted to the context. This type of representation deals with the first type of question. As for the second type, the adequation between B and the fuzzy dilated region can be evaluated. Other fuzzy approaches to this type of relation are reviewed in [39].

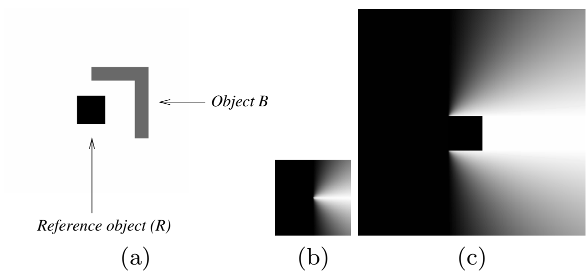


Fig. 1. (a) Two example objects. (b) Fuzzy structuring element defining, in the spatial domain, the semantics of *to the right of*. (c) Fuzzy Dilation of R (black square). Membership values range from 0 (black) to 1 (white).

This example also highlights another important issue, which concerns the representation, for which several forms can be interesting, depending on the raised question. While in the crisp case, a relation between two objects is usually represented by a number (either 0/1 for an all-or-nothing relation, or a numerical value for a distance for instance), in the fuzzy case, several representations are possible. They can typically be intervals, for instance representing necessity and possibility degrees, fuzzy numbers, distributions [40,41], representing actual measurements or the semantics of some linguistic variables. Details can be found in [42,43] in the case of distances. These representations are adequate to answer questions of type 1, since they rely on some computation procedure between two known objects. As for the second type of question, spatial representations are more appropriate, as fuzzy sets in the spatial domain.

Fuzzy representations are also interesting in terms of robustness. For instance set relationships and adjacency are highly sensitive, since in the binary case, the result can depend on one point only [44]. The satisfaction of a relation can be drastically modified depending on the digitization of the space, on the way objects are defined, on errors due to some segmentation process, etc. This is clearly a limitation of binary (all or nothing) definitions. In the fuzzy case, the problem is much less crucial. Indeed, there is no more strict membership, the fuzziness allows dealing with some gradual transition between objects or between object and background, and relations become then a matter of degree. In this respect, the fuzziness, even on digital images, could be interpreted as a partial recovering of the continuity lost during the digitization process.

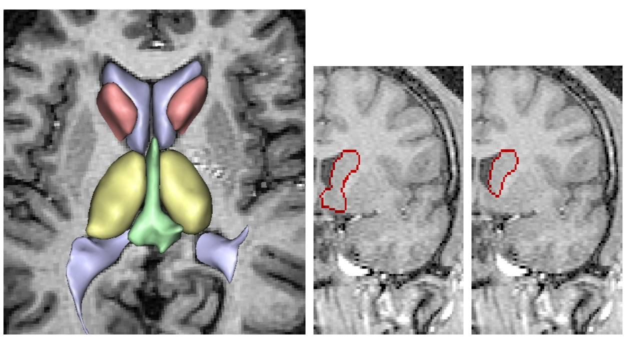
Finally, some relations depend not only on the applicative context, but also on the shape of the considered objects. This is the case for the *between* relation, where the semantics changes depending on whether the objects have similar spatial extensions or very different ones [45]. Here again, fuzzy mathematical morphology leads to models adapted to each situation [46].

Let us now illustrate how these relations can be used in spatial reasoning, in particular for guiding structure recognition and segmentation in medical images. For instance in brain imaging, anatomical knowledge is often expressed as linguistic descriptions of the structures and their spatial arrangement. Spatial relations play a major role in such descriptions. Moreover, they are more stable than shape or size information and are less prone to inter-individual variations, even in the presence of pathologies. Recently, this knowledge was formalized, in particular using ontologies such as the Foundational Model of Anatomy (FMA) [47], just to mention one. However these models do not yet incorporate much structural descriptions. In [48], we proposed an ontology of spatial relations, which has been integrated in the part of the FMA dedicated to brain structures. This ontology has been further enriched by fuzzy models of the spatial relations (defining their semantics). This formalism partially solves the semantic gap issue, by establishing links between symbolic concepts of the ontology and their representation in the image domain (and hence with percepts that can be extracted from images). These links allow using concretely the ontology to help in image interpretation and object recognition. Mathematical morphology is directly involved in these fuzzy representations, but also at the reasoning level, since tools from morphologics can be integrated in the description logics [49].

In our group, we developed two different types of approaches for recognition, working either sequentially or globally. In the sequential approach [50,51], structures are recognized successively according to some order, and the recognition of each structure relies on its relations to previously detected structures. This allows reducing the search space, as in a process of focus of attention. For instance, anatomical knowledge includes statements such as *the right caudate nucleus is to the right of and close to the right lateral ventricle*. The search space for the right caudate nucleus is then defined as the fuzzy region resulting from the conjunctive fusion of the dilations of the right lateral ventricle using fuzzy structuring elements expressing the semantics of each of these relations. The application domain is here very important, since this semantics highly depends on it. It is clear for instance that the semantics of *close to* is not the same for brain structures or for stars. This is actually encoded in the parameters of the membership functions that define the relations, which can be learned from a data base of images [52]. Within the obtained restricted search region, a precise segmentation can be performed, for instance using deformable models integrating spatial relations in the energy functional [51]. The order according to which the structures are processed can also be learned, as proposed in [53,54].

In global approaches [55], several objects are extracted from the image using any segmentation method, generally providing an over-segmentation, and recognition is then based on the relations existing between these segmented regions, in comparison to those expressed in the knowledge base or ontology. Graph-based approaches [55], or constraint satisfaction problems approaches [56] have been developed, implementing these ideas. The ontological modeling allows, using classification tools for instance, filtering the knowledge base so as to keep only the objects that share some given relations. This leads to a reduced combinatorics in the search for possible associations between image regions and structures of the model.

Segmentation results for a few internal brain structures obtained with the sequential approach are illustrated in Figure 2 for a normal case and in Figure 3 for two pathological cases. The original images are 3D magnetic resonance images (MRI). In the pathological cases, the tumors strongly deform the normal structures. In such situations, methods based on shape and size fail, while using spatial relations (with only slight adaptations) leads to correct results.



(a) (b) (c)

Fig. 2. Segmentation results for a few internal structures in a normal case [51]. (a) Results are superimposed to a part of an axial slice of the original 3D MRI image. (b) Segmentation of the caudate nucleus (shown on one slice) without using the spatial relations: the contour does not match the anatomical constraints and leak outside the structure. (c) Result using the spatial relations: anatomical knowledge is respected and the final segmentation is now correct.

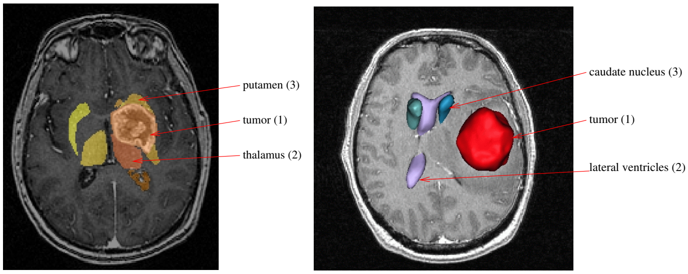


Fig. 3. Segmentation and recognition results in two pathological cases [52]. The order in which structures are segmented is indicated into parentheses.

5 Application of Bipolar Morphology to Spatial Reasoning

Let us provide a few examples where bipolarity occurs when dealing with spatial information, in image processing or for spatial reasoning applications: when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indifference or neutrality. This corresponds to the idea that the union of positive and negative information does not cover all the space. Concerning semantics, it should be noted that a bipolar fuzzy set does not necessarily represent one physical object or spatial entity, but rather more complex information, potentially issued from different sources.

In this section, we illustrate a typical scenario showing the interest of bipolar representations of spatial relations and of morphological operations on these representations for spatial reasoning. An example of a brain image is shown in Figure 4, with a few labeled structures of interest.

Let us first consider the right hemisphere (i.e. the non-pathological one). We consider the problem of defining a region of interest for the RPU, based on a known segmentation of RLV and RTH. An anatomical knowledge base or ontology provides some information about the relative position of these structures: (i) *directional information*: the RPU is exterior (left on the image) of the union of RLV and RTH (positive information) and cannot be interior (negative information); (ii) *distance information*: the RPU is quite close of the union of RLV and RTH (positive information) and cannot be very far (negative information). These pieces of information are represented in the image space based on morphological dilations using appropriate structuring elements (representing the semantics of

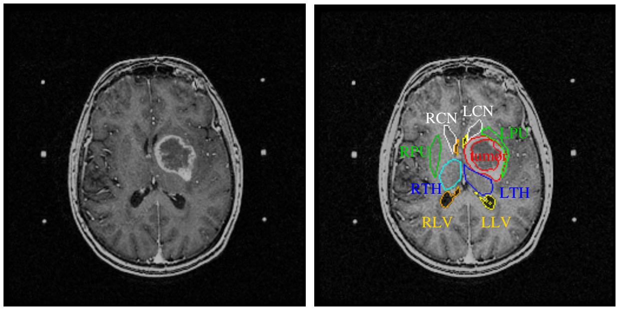


Fig. 4. A slice of a 3D MRI brain image, with a few structures: left and right lateral ventricles (LLV and RLV), caudate nuclei (LCN and RCN), putamen (LPU and RPU) and thalamus (LTH and RTH). A ring-shaped tumor is present in the left hemisphere (the usual “left is right” convention is adopted for the visualization).

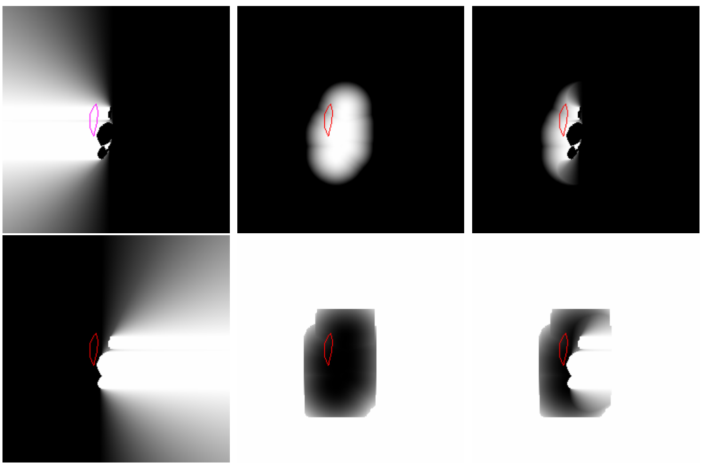


Fig. 5. Bipolar fuzzy representations of spatial relations with respect to RLV and RTH. Top: positive information, bottom: negative information. From left to right: directional relation, distance relation, conjunctive fusion. The contours of the RPU are displayed to show the position of this structure with respect to the region of interest.

the relations) and are illustrated in Figure 5. The neutral area between positive and negative information allows accounting for potential anatomical variability. The conjunctive fusion of the two types of relations is computed as a conjunction of the positive parts and a disjunction of the negative parts. As shown in the illustrated example, the RPU is well included in the bipolar fuzzy region of interest which is obtained using this procedure. This region can then be efficiently used to drive a segmentation and recognition technique of the RPU.

Let us now consider the left hemisphere, where a ring-shaped tumor is present. The tumor induces a deformation effect which strongly changes the shape of the normal structures, but also their spatial relations, to a less extent. In particular

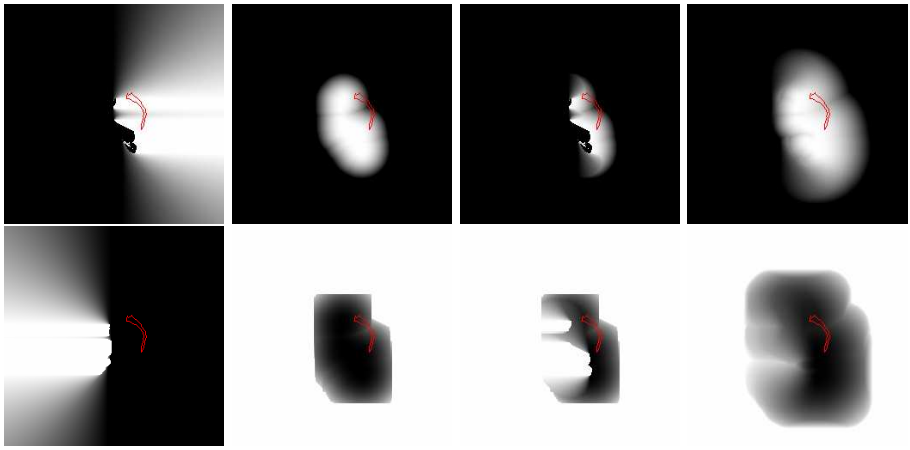


Fig. 6. Bipolar fuzzy representations of spatial relations with respect to LLV and LTH. From left to right: directional relation, distance relation, conjunctive fusion, Bipolar fuzzy dilation. First line: positive parts, second line: negative parts. The contours of the LPU are displayed to show the position of this structure.

the LPU is pushed away from the inter-hemispheric plane, and the LTH is pushed towards the posterior part of the brain and compressed. Applying the same procedure as for the right hemisphere does not lead to very satisfactory results in this case (see Figure 6). The default relations are here too strict and the resulting region of interest is not adequate: the LPU only satisfies with low degrees the positive part of the information, while it also slightly overlaps the negative part. In such cases, some relations (in particular metric ones) should be considered with care. This means that they should be more permissive, so as to include a larger area in the possible region, accounting for the deformation induced by the tumor. This can be easily modeled by a bipolar fuzzy dilation of the region of interest, as shown in the last column of Figure 6. Now the obtained region is larger but includes the right area. This bipolar dilation amounts to dilate the positive part and to erode the negative part.

Other examples are provided in [34]. Exploring further these ideas is planned for future work.

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