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FUZZY MATHEMATICAL MORPHOLOGIES: A COMPARATIVE STUDY

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Abstract—Fuzzy set theory has found a promising field of application in the domain of digital image processing, since fuzziness is an intrinsic property of images. For dealing with spatial information in this framework from the signal level to the highest decision level, several attempts have been made to define mathematical morphology on fuzzy sets. The purpose of this paper is to present and discuss the different ways to build a fuzzy mathematical morphology. We will compare their properties with respect to mathematical morphology and to fuzzy sets and interpret them in terms of logic and decision theory.

Fuzzy sets Mathematical morphology Fuzzy mathematical morphology Dilation Erosion Opening Closing Fuzzification Triangular norms and conorms Stochastic geometry Approximate reasoning Decision making Uncertain and imprecise spatial information Data fusion

1. INTRODUCTION

In digital picture processing, fuzzy set theory has found a promising field of application. Fuzziness is an intrinsic quality of images and the natural outcome of most of the picture processing techniques. Let us illustrate these two statements with some examples:

- in any medical image, pathological tissues appear without clear-cut frontiers, as diffuse patches subtly imbedded in sane tissues; in a similar way, irrigated fields continuously evolve towards dry fields in an aerial or satellite image;
- extracting edges in very noisy images, like SAR or ultrasonic images, is only possible using simultaneously low pass filters and edge detectors, resulting in an information spatially inaccurate; similar results exist for most of the local features: corners, vertices, etc.

Fuzzy sets ideally fit our intuitive knowledge of the diffuse localization or limits of the image components due to both uncertainty and imprecision; they are less demanding than their probabilistic counterparts which may also model uncertainty but at the cost of mathematical requirements which necessitate heavy experimental protocols or limiting hypotheses.

Fuzzy sets being able to model uncertainty and imprecision attached both to the image components and the picture processing operations which are performed on it come forward as the comprehensive framework to represent information from the signal level to the highest decision level. Unfortunately, from one point of view at least, the fuzzy set tool-box appears poor: it is when spatial transformation of information is concerned. In order to bridge that gap, several attempts have been made in the last 15 years. Let us make a brief historical review of this topic:

- in 1965, fuzzy sets were introduced by Zadeh;[1] elementary set operations were defined (intersection, union, complementation, inclusion, etc.);
- in 1979, Rosenfeld introduced topology on fuzzy sets;[2]
- since 1982, geometrical operations were developed, and are summarized by Rosenfeld;[3]
- in 1980, elementary mathematical morphology operations (dilation and erosion) were applied on grey-level images interpreted as fuzzy sets,[4] using max. and min. operators; only binary structuring elements were considered;
- in 1984, shrinking and expanding were defined,[3] which exactly corresponded to classical erosion and dilation with a binary structuring element;
- in 1988, Kaufmann provided a definition of the Minkowski addition for two fuzzy sets[5] (the first really fuzzy approach) by means of the α-cuts of the two fuzzy sets: $(\mu \oplus \nu)_\alpha = \mu_\alpha \oplus \nu_\alpha$, where μ_α denotes the α-cut of μ and $\oplus$ denotes the Minkowski addition; it is given briefly in an appendix without any reference to mathematical morphology and so it has not been exploited in that way;
- in 1988, operations varying continuously from classical erosion to classical dilation with a binary structuring element (support of the structuring element) were proposed;[6] they are not really fuzzy dilation and fuzzy erosion and, therefore, this approach will not be considered in the following;

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• since 1989, several attempts have been made to use directly grey-level mathematical morphology on images with grey levels interpreted as fuzzy sets,[7-12] instead of developing a fuzzy approach.

However, although these developments solve many limited problems associated with information diffusion and control, they do not provide the universal basis which is needed. A model for such a basis is provided by mathematical morphology which created a coherent set of operations able to process grey-level images. However, mathematical morphology cannot be directly extended to fuzzy sets, since it is not internal in the [0, 1] segment. Several attempts have been made to adapt mathematical morphology or to mimic it. They will constitute the backbone of this communication. One family of works emphasizes a mathematical morphology approach. The second family is more representing the fuzzy set and decision theory aspect of the problem. In the first family, we find the works by Sinha and Dougherty, and by de Baets and Kerre, and in the second family, the works by Bandemer and Näther and by the authors, which have been independently developed.

The purpose of this communication is to present and discuss the different ways to build a mathematical morphology in order to process fuzzy sets. We will first give a presentation, without discussion, of the existing definitions (part 2). In part 3, the required properties of the basic operations will be reviewed from the two different points of view of mathematical morphology and of decision theory. Some ways to reduce the required properties will be explored. In part 4, from the previous requirements, two major construction principles are exhibited. One relies on a fuzzification principle. The second is based on translating set equations into functional ones and involves the theoretical framework of triangular norms and conorms, which appears to be well adapted in this context. In part 5, the existing definitions of part 2 are examined under the light of the requirements. In particular, it will be shown that the various definitions may have slightly different properties. The second construction principle leads to an infinity of definitions for the basic operators. An important result is that this family of fuzzy mathematical morphology operators is partially ordered. Part 6 is devoted to a critical comparison between the six definitions with respect to mathematical morphology and fuzzy sets, and their properties are interpreted in terms of logic and decision theory.

Table 1.

Set	Structuring element	Classical MM	ΦMM
Binary	Binary	BMM	Possibly compatible
Fuzzy	Binary	GMM	Possibly compatible
Fuzzy	Fuzzy	FMM	Not compatible

Attention will be paid here only to the four basic operations of mathematical morphology (erosion, dilation, opening, closing), but it should be clear for the reader that for every definition, a complete set of morphological operations could be derived.

For the sake of clarity, we will denote binary mathematical morphology as BMM, the morphology which uses binary sets and binary structuring elements; grey-level mathematical morphology (GMM) will represent classical morphology with a function (or a grey-level image) and a binary structuring element, whereas functional mathematical morphology (FMM) will represent classical mathematical morphology on functions with functional structuring element. We are interested in developing a fuzzy morphology (ΦMM). As mentioned previously, considering a fuzzy set as a grey-level function does not allow us to use FMM as a ΦMM (see Table 1) since, for instance, the dilation of a set defined in [0, 1] by a set defined in [0, 1] provides a new set defined in [0, 2] (see Fig. 1). As binary sets are special fuzzy sets, we may expect ΦMM to be compatible with classical mathematical morphology, when at least one of the two sets is binary, because starting from one fuzzy set (with values in [0, 1]) we may obtain another fuzzy set. On the contrary, FMM cannot guarantee this internal property.

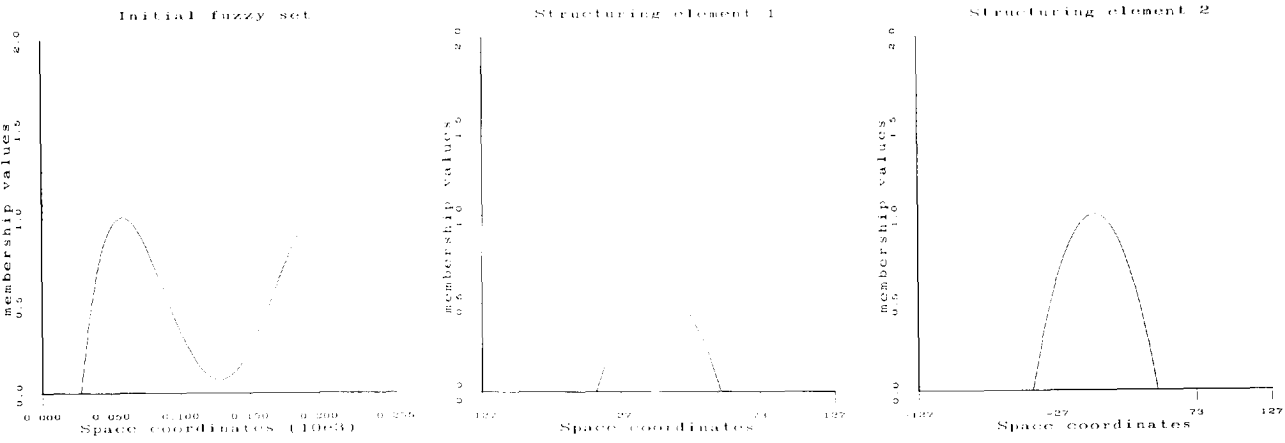
Seen from a mathematical morphology point of view, these attempts to specify an adequate mathematical morphology for a given application are comparable to the developments proposed for statistical,[13] soft[14] and topographical[15] morphologies.

2. FUZZY EROSION AND DILATION: SEVERAL DEFINITIONS

The attempts to build a morphology relying on true fuzzy approaches and dealing with fuzzy sets and fuzzy structuring elements will be briefly recalled in this section. Several definitions can be found in the literature for fuzzy dilation and erosion. They will be numbered in a non-chronological way, according to an order which will become clear later in this publication. We proposed definitions 1 and 2,[16] and then a third.[17] Sinha and Dougherty gave definition 4[18,19] and its generalization (definition 5).[20] We proposed[21] a more general definition (6), which includes the previous ones, except definition 1.

In the following, fuzzy sets are represented by their membership functions, defined over a space S of objects or points (without restriction on its dimension) and taking their values in the interval [0, 1]. The space of all fuzzy sets (or, equivalently, of all membership functions) is denoted by M. All definitions are given for any fuzzy set μ of M, any fuzzy structuring element ν of M, and any point x of S. μ_α denotes the α-cut of μ, i.e. the binary set with characteristic function defined as:

$$\mu_\alpha(x) = \begin{cases} 0 & \text{if } \mu(x) < \alpha, \\ 1 & \text{if } \mu(x) \geq \alpha. \end{cases}$$



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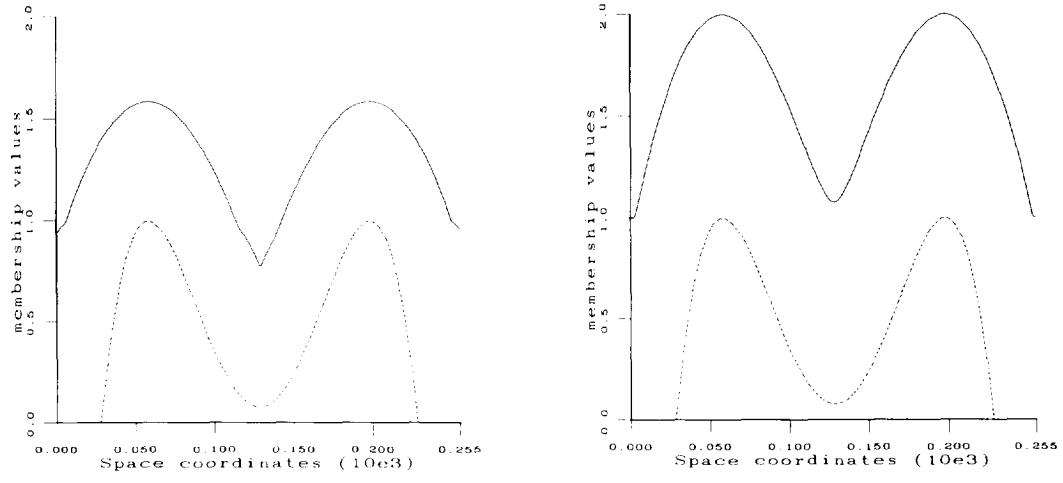

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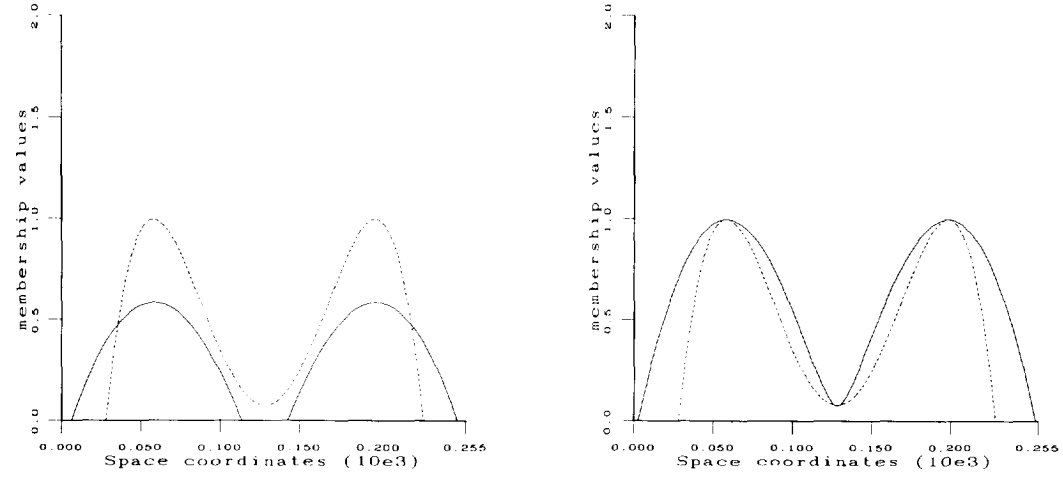


(c)

Fig. 1. Why FMM cannot be used as a ΦMM: (a) initial fuzzy set μ, structuring elements v_1 and v_2; (b) functional dilation provides a function taking values in [0, 2] (left: dilation of μ by v_1, right: dilation of μ by v_2, the initial function is dashed); (c) translating and truncating the result in order to obtain a membership function (i.e. in [0, 1]) is not satisfactory because information is lost (left) and the result does not reflect the shape and the size of the structuring element (right).

Definition 1:[16,22,23]

$$D1_{v_S}(\mu)(x) = \int_0^1 \sup_{y \in (v_S)_x} \mu(y)\, d\alpha,$$

$$E1_{v_S}(\mu)(x) = \int_0^1 \inf_{y \in (v_S)_x} \mu(y)\, d\alpha,$$

where $v_\alpha = \{x \in S / v(x) \geq \alpha\} = \{x \in S / v_\alpha(x) = 1\}$, in order to simplify the notations.

Definition 2:[16,24,25]

$$D2_v(\mu)(x) = \sup_{y \in S} \min[\mu(y), v(y - x)],$$

$$E2_v(\mu)(x) = \inf_{y \in S} \max[\mu(y), 1 - v(y - x)].$$

Definition 3:[17]

$$D3_v(\mu)(x) = \sup_{y \in S} [\mu(y) v(y - x)],$$

$$E3_v(\mu)(x) = \inf_{y \in S} [\mu(y) v(y - x) + 1 - v(y - x)].$$

Definition 4:[18,19]

$$D4_v(\mu)(x) = \sup_{y \in S} \max[0, \mu(y) + v(y - x) - 1],$$

$$E4_v(\mu)(x) = \inf_{y \in S} \min[1, 1 + \mu(y) - v(y - x)].$$

Definition 5:[20]

$$D5_v(\mu)(x) = \sup_{y \in S} \max[0, 1 - \lambda(\mu(y)) - \lambda(v(y - x))],$$

$$E5_v(\mu)(x) = \inf_{y \in S} \min[1, \lambda(1 - \mu(y)) + \lambda(v(y - x))],$$

with λ a function from $[0, 1]$ to $[0, 1]$ satisfying the six following conditions:

- $\lambda(z)$ is a non-increasing function of z;
- $\lambda(0) = 1$;
- $\lambda(1) = 0$;
- the equation $\lambda(z) = 0$ has a single solution;
- $\forall \alpha \in [0.5, 1]$, the equation $\lambda(z) = \alpha$ has a single solution;
- $\forall z \in [0, 1]$, $\lambda(z) + \lambda(1 - z) \geq 1$.

The case where $\lambda(z) = 1 - z = \lambda_0(z)$ corresponds to definition 4.

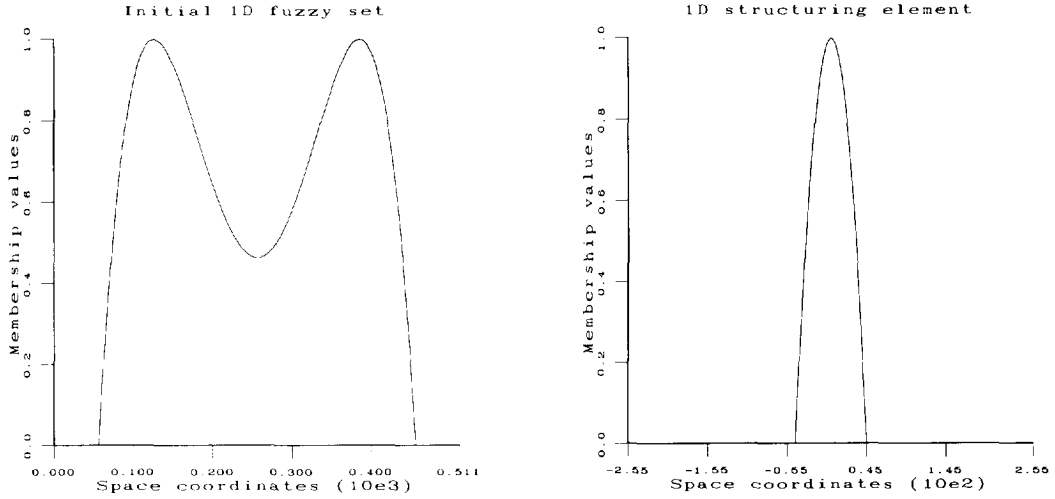
Definition 6:[21,24,26]

$$D6_v(\mu)(x) = \sup_{y \in S} i[v(y - x), \mu(y)],$$

$$E6_v(\mu)(x) = \inf_{y \in S} u[c(v(y - x)), \mu(y)],$$

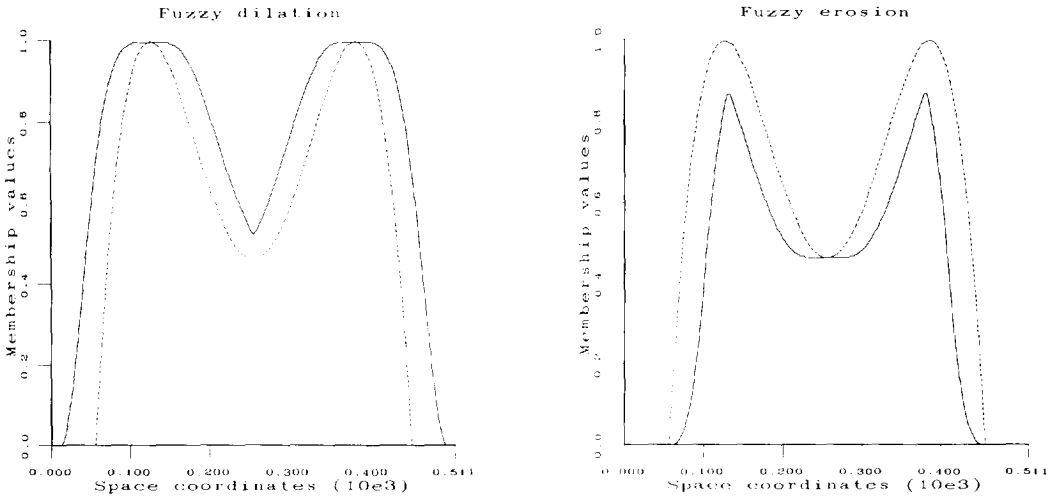
where i is any T-norm and u the associated T-conorm with respect to a complementation c (see definitions in Section 4.2.2).

We will see (Section 4.2.2) that this definition unifies the preceding ones except the first: definitions 2–4 are obtained with particular T-norms, definition 5 corresponds to a weak T-norm. Figure 2(a)–(f) (respectively, g-l) illustrates these definitions for S being a one-dimensional (respectively, two-dimensional) space.



(a)

Fig. 2. Illustrations of definitions 1–6: (a) initial one-dimensional fuzzy set and fuzzy structuring element; (b) dilation and erosion for definition 1; (c) dilation and erosion for definition 2 [definition 6 with $i(x, y) = \min(x, y)$ and $u(x, y) = \max(x, y)$]; (d) dilation and erosion for definition 3 [definition 6 with $i(x, y) = xy$ and $u(x, y) = x + y - xy$; (e) dilation and erosion for definition 4 [definition 6 with $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$]; (f) dilation and erosion for definition 5 with $\lambda = 1 - x^3$ (definition 6 with the weak T-norm $i(x, y) = \max[0, 1 - \lambda(x) - \lambda(y)]$ and the weak T-conorm $u(x, y) = \min[1, \lambda(1 - x) + \lambda(1 - y)]$); (g) initial two-dimensional fuzzy set and two-dimensional fuzzy structuring element; (h) dilation and erosion for definition 1; (i) dilation and erosion for definition 2 [definition 6 with $i(x, y) = \min(x, y)$ and $u(x, y) = \max(x, y)$]; (j) dilation and erosion for definition 3 [definition 6 with $i(x, y) = xy$ and $u(x, y) = x + y - xy$]; (k) dilation and erosion for definition 4 [definition 6 with $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$]; (l) dilation and erosion for definition 5 with $\lambda = 1 - x^3$ [definition 6 with the weak T-norm $i(x, y) = \max[0, 1 - \lambda(x) - \lambda(y)]$ and the weak T-conorm $u(x, y) = \min[1, \lambda(1 - x) + \lambda(1 - y)]$].



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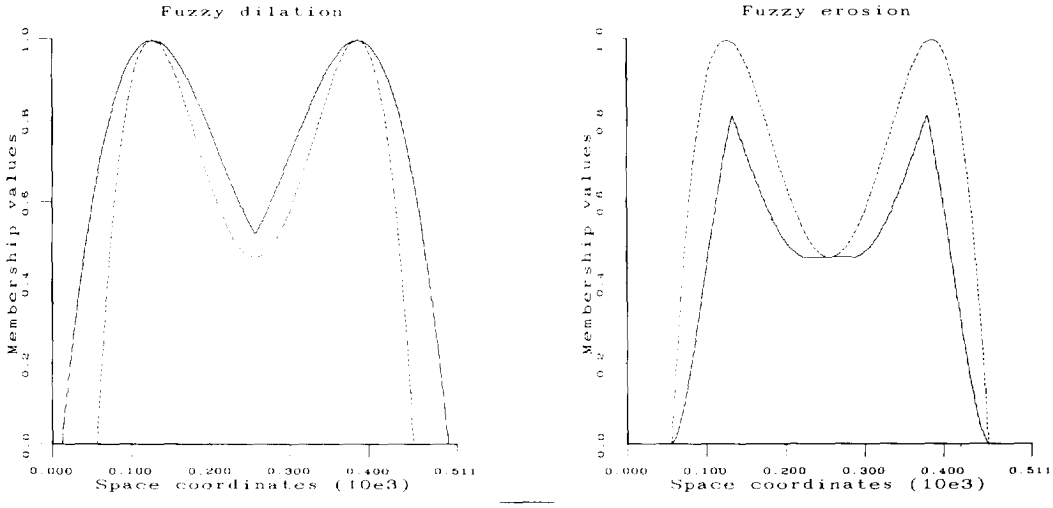

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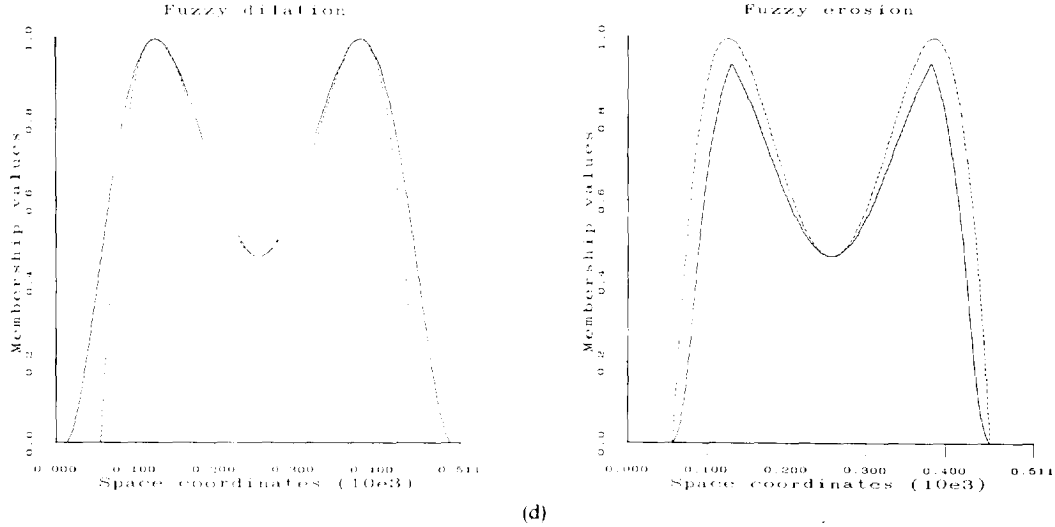
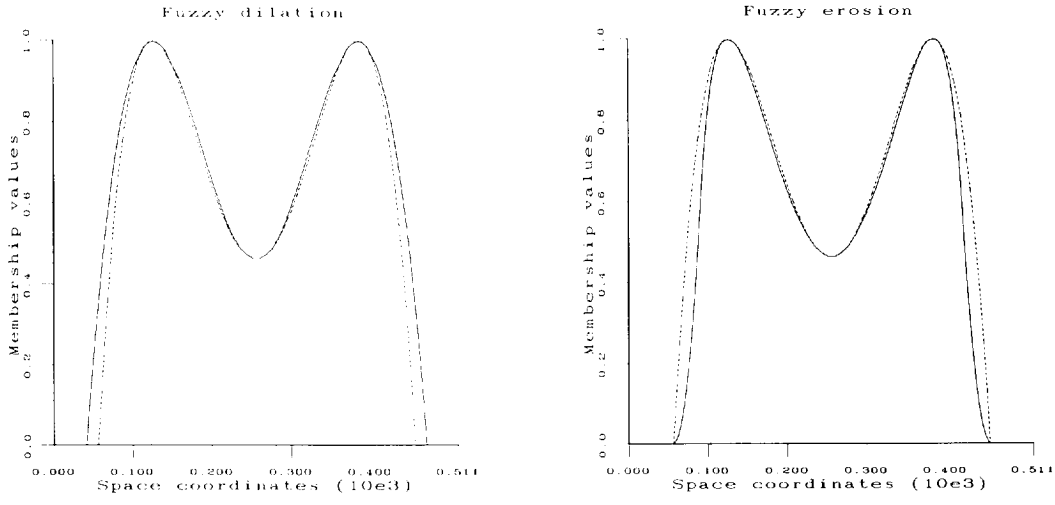


(d)

Fig. 2 (*Continued*)



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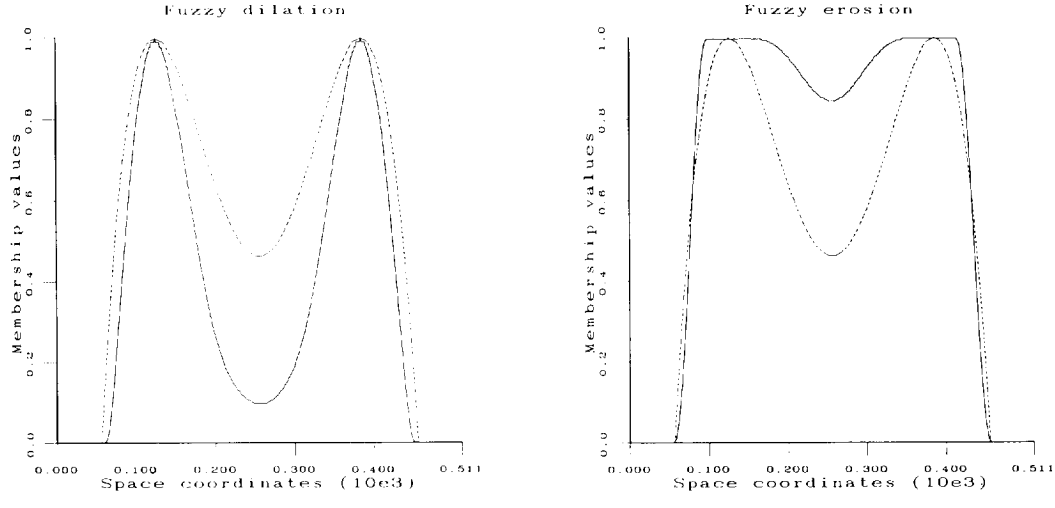

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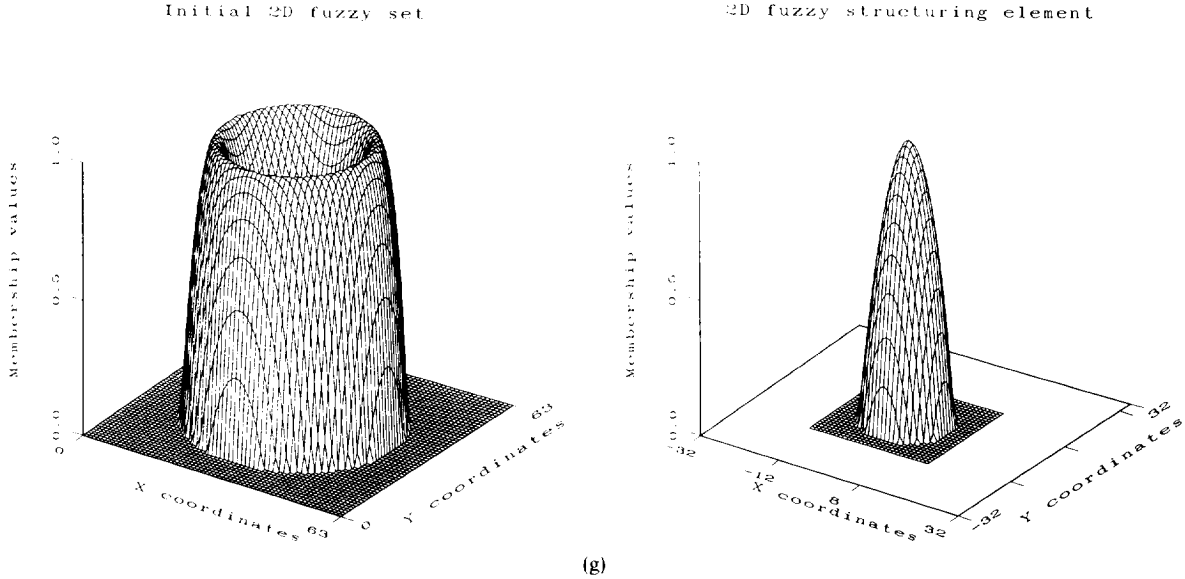

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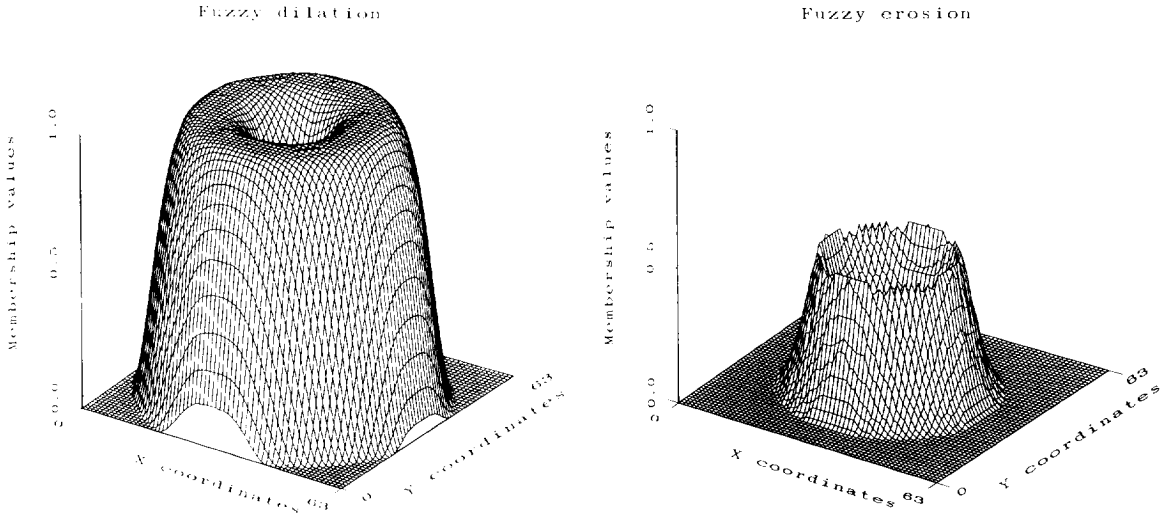

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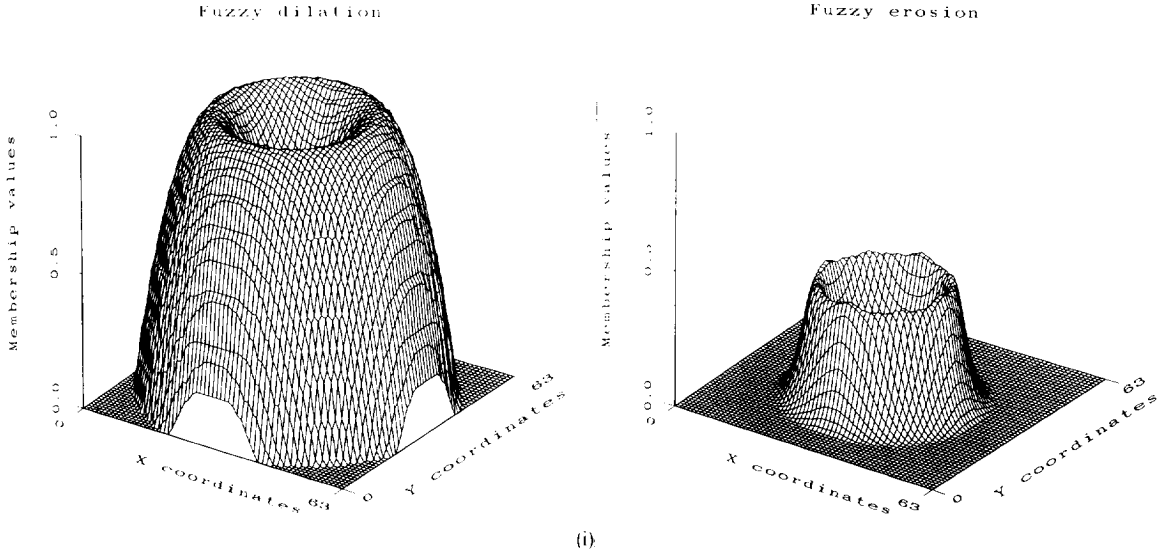

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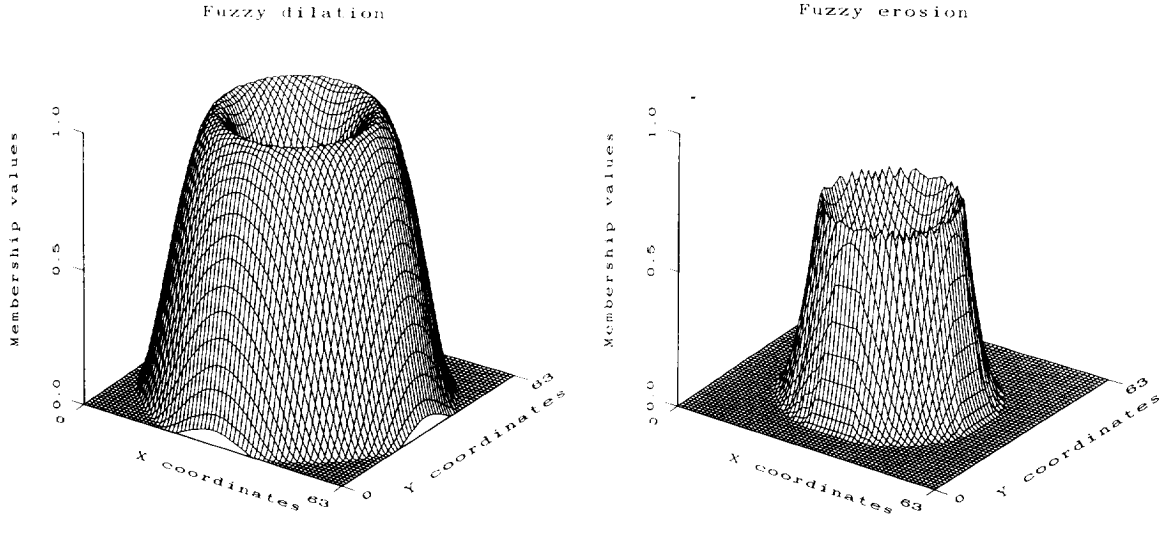

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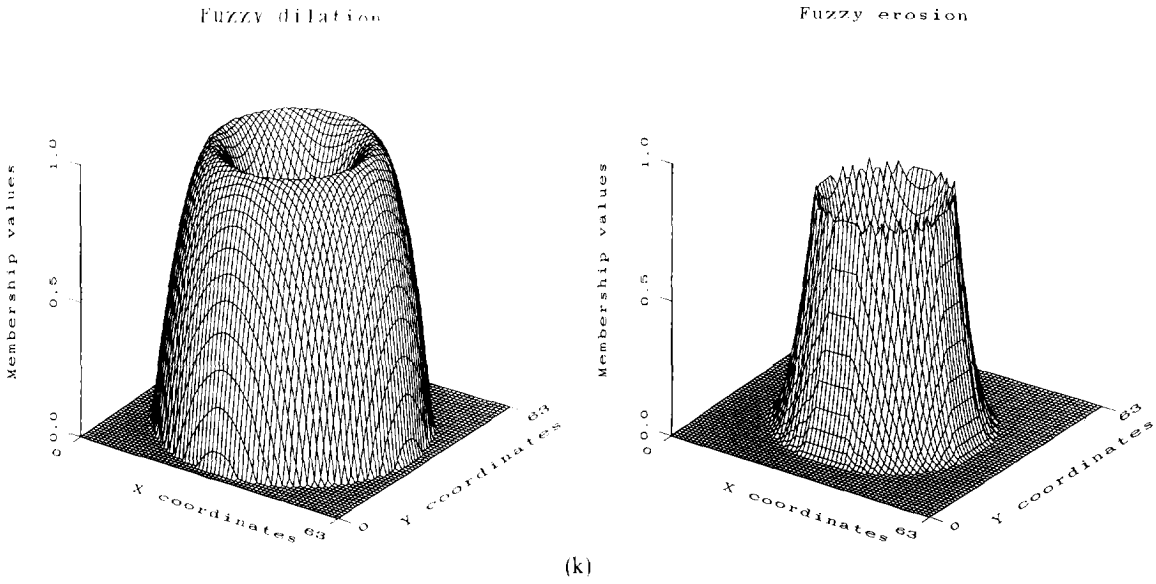

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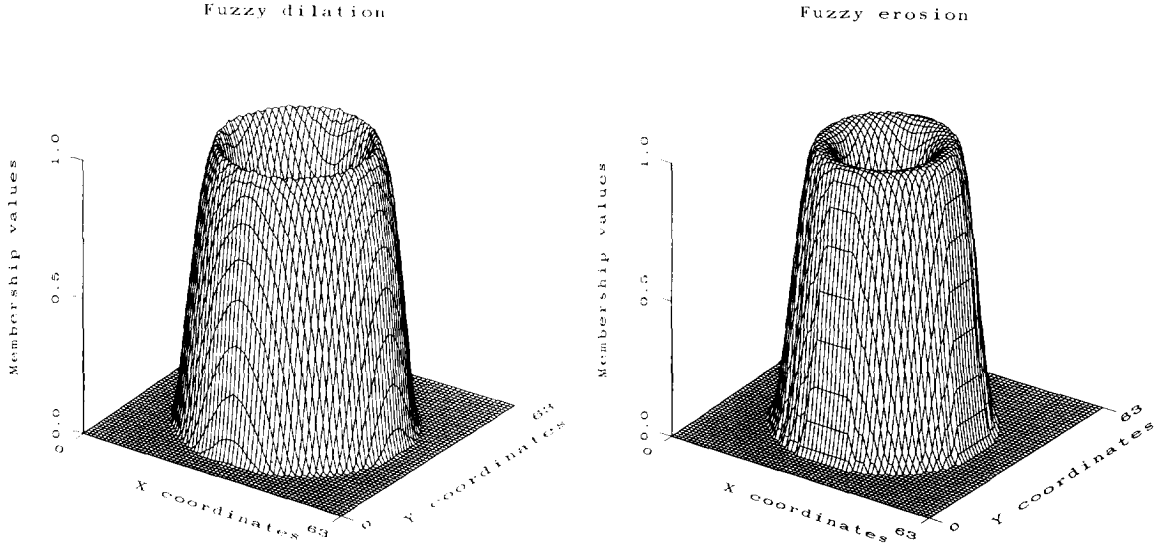


(l)

Fig. 2 (*Continued*)

For a better understanding of these definitions and of their similarities or differences, we now examine the principles governing their construction and the underlying concepts.

3. REQUIRED PROPERTIES FOR CONSTRUCTING A FUZZY MORPHOLOGY

A logical construction of a fuzzy mathematical morphology has to follow the sequence of three steps:

- defining requirements;
- stating principles governing the construction;
- deriving basic definitions (dilation and erosion) based on these principles and satisfying as many requirements as possible.

This very general construction can be tracked in Bloch *et al.*[16,17,21] as well as in Sinha *et al.*,[18,20] but the details of these steps are different. They will be explained in the next sections: Sections 3.1, 3.2 and 3.3 aim at defining requirements for ΦMM, two principles governing its construction are stated in Section 4, and the derivation of definitions for fuzzy dilation and erosion is achieved in Sections 4.2.1 and 4.2.2.

What has to be expected from fuzzy morphological operations? Intuitive requirements on the effects of the transformations such as expanding, contracting and filtering correspond to algebraic properties: extensivity is a mathematical property to express that a transformation expands a set, anti-extensivity formalizes contracting, increasingness, idempotence, and extensivity

or anti-extensivity are the mathematical properties representing filtering, etc..[27] Other requirements correspond to analytical properties (e.g. robustness to small variations expressed by continuity properties of the operators). As the properties of classical morphology are well known from a theoretical as well as from a practical point of view, they suggest requirements for ΦMM. In Section 3.1, a set of properties will be given, directly derived from classical morphology, and thus mainly seen from a morphological point of view. However, on the other side, in order to remain connected to fuzzy set theory, we will have to take note of some other properties, discussed in Section 6, which are inherited from decision theory.

Before we start constructing a ΦMM, we have to keep in mind that the extension of a theory to cover, new fields will probably not be done preserving all the nice properties of the initial theory. In Section 3.3, the requirements which can be relaxed or even lost will thus be discussed.

3.1. *Properties suggested by classical mathematical morphology*

In this section, the main properties of classical mathematical morphology are listed that could be imposed to ΦMM. They will be numbered for a later use and are only summarized here. Precise definitions and interpretations are given in Appendix 1, where for each property the operations (dilation, erosion, morphological opening and morphological closing) for which they are required, in order that ΦMM inherits this property, are also specified.

3.1.1. *Four fundamental principles.* In the framework of mathematical morphology, four fundamental principles are assumed.[27] Here, we translate them in terms of fuzzy sets.

Property 1. Translation invariance.
Property 2. Compatibility with homotheties.
Property 3. Local knowledge.
Property 4. Semi-continuity.

3.1.2. *Algebraic properties.* In addition to these basic properties, classical mathematical morphology operations have important algebraic properties which are effectively used for the applications. They are given below.

Property 5. Duality with respect to complementation
Property 6. Increasingness.
Property 7. Extensivity or anti-extensivity.
Property 8. Idempotence.
Property 9. Pseudo-commutativity of dilation.
Property 10. Fitting characterization.
Property 11. Compatibility with union and intersection, or with "max" and "min" on functions (with either equalities or inequalities as in classical morphology).
Property 12. Iteration and combination.

3.2. *Properties related to fuzzy sets*

As our purpose is to extend mathematical morphology to fuzzy sets, another requirement can be asserted, expressing that, in degenerate cases, fuzzy morphology coincides with classical:

Property 13. Compatibility with BMM and GMM. If $\Psi_v(\mu)$ denotes a morphological operation on μ with respect to the structuring element v, and $\tilde{\Psi}$ the classical corresponding operation, compatibility means:

$$v \text{ is binary} \Rightarrow \Psi_v(\mu) = \tilde{\Psi}_v(\mu).$$

Compatibility has to be satisfied by the four operations.

We cannot require the same property for v not being binary, since FMM on functions taking their values in $[0, 1]$ does not provide results in $[0, 1]$, as seen before.

A last possible required property is the following:

Property 14. Relationship with cuts. As fuzzy sets may be interpreted as stacks of binary sets (cuts at level α, for $\alpha \in [0, 1]$), it could be expected that a transformation can be obtained from operations on the cuts of the fuzzy set and/or of the fuzzy structuring element, or that the cuts of the results are related to the cuts of the initial fuzzy sets. We will later see that this property could indeed be used as a constructing rule (Section 4.2.1).

3.3. *Reducing the requirements?*

In order to obtain an operational method to construct the basic operations of ΦMM, it would be appreciated to reduce the number of selected requirements. This can be done in two ways: at first, some properties described in the previous sections are redundant, and the suppression of some of them provides an equivalent set of properties; the second consists of suppressing properties which are considered of less importance for the application to fuzzy sets. These two ways will now be explored.

3.3.1. *Reducing redundancy.* Property 11 can be reduced if property 5 is satisfied. The duality between erosion and dilation allows us to deduce the two sets of properties (P.11.1 P11.4) and (P11.5–P11.8) from each other. Moreover, pseudo-commutativity (property 9) makes properties P.11.1 and P11.2 equivalent. In the same way, P.11.3 and P11.4 are equivalent, and so are P11.6 and P11.7, P11.5 and P11.8. Thus, the eight equations of property 11 can be reduced to two, P.11.1 and P11.3 for example, when P.5 and P.9 hold, according to the following equivalence table, which exhibits two equivalence classes (Table 2).

As we will see in Section 5, duality (P.5) and pseudo-commutativity (P.9) are satisfied for all definitions, thus the restriction of P.11 to only two equations is valid.

Moreover, the increasingness and idempotence of closing and opening defined as a combination of erosion and dilation can be deduced from properties 6 and 7. It is obvious that the combination of two increasing

Table 2. Equivalence between the equations in property 11, showing two equivalence classes ($\times$ indicates equivalence between two properties)

	P.11.1	P.11.2	P.11.6	P.11.7	P.11.3	P.11.4	P.11.5	P.11.8
P.11.1	×	×	×	×				
P.11.2	×	×	×	×				
P.11.6	×	×	×	×				
P.11.7	×	×	×	×				
P.11.3					×	×	×	×
P.11.4					×	×	×	×
P.11.5					×	×	×	×
P.11.8					×	×	×	×

operations is increasing. Thus property 6 for Ψ being erosion and dilation is sufficient to ensure the increasingness of closing and opening. Let us show that the idempotence of opening can be deduced from properties 6 and 7 (the same reasoning holds for closing). As opening is anti-extensive (P.7), we have:

$$O_\nu[O_\nu(\mu)] \subset O_\nu(\mu).$$

From the definition of opening, we have:

$$O_\nu[O_\nu(\mu)] = D_\nu[E_\nu(D_\nu(E_\nu(\mu)))] = D_\nu[C_\nu(E_\nu(\mu))].$$

As closing is extensive, $E_\nu(\mu) \subset C_\nu(E_\nu(\mu))$. As dilation is increasing, $D_\nu[E_\nu(\mu)] \subset D_\nu[C_\nu(E_\nu(\mu))]$, which expresses that:

$$O_\nu(\mu) \subset O_\nu[O_\nu(\mu)].$$

In a similar way, increasingness of erosion is deduced from increasingness of dilation using duality (P.5). The same result holds for extensivity (respectively, anti-extensivity). P.5 also implies the equivalence of P.12.1 and P.12.2. Duality between closing and opening, which can be deduced from duality between erosion and dilation, allows us to consider the properties of only one of these two operations. Decreasingness of erosion with respect to ν is deduced from P.5 and from increasingness of dilation. In summary, as long as properties 5, 6, 7, 8 and 12 are concerned, we need:

- P.5 for erosion and dilation only;
- P.6 for one operation only (dilation, for example);
- P.7 for two operations (dilation and closing, for example);
- P.8 is not needed (neither for closing nor for opening) if P.6 and P.7 are satisfied;
- P.12 reduced to P.12.1 and P.12.3.

Reducing the properties related to the four basic properties of mathematical morphology (P.1, P.2, P.3 and P.4) can be achieved in a similar way. Using the duality property, invariance with respect to translation for erosion can be deduced from invariance of dilation (or conversely), and thus for opening and closing. Compatibility with homotheties for dilation does not necessarily imply the same property for erosion. However, it is verified for definitions, as it will be seen later. If the property is satisfied for both erosion and dilation, it is also satisfied for opening and closing. The property of local knowledge for dilation is sufficient to deduce the property for erosion, closing and opening. The same result applies for semi-continuity (note that for the convergence of a series of membership functions, we consider here only uniform convergence, which implies simple convergence). In summary, as far as properties 1–4 are concerned, we need:

- P.1 for one operation (dilation, for example);
- P.2 for dilation and erosion;
- P.3 for one operation;
- P.4 for one operation

to guarantee the four properties for the four basic operations.

In property P.9, the relation for erosion can be deduced from that for dilation (expressing its pseudo-commutativity) and from P.5 (duality).

The property P.13 (compatibility with BMM and GMM) need only be satisfied for dilation and then can be deduced for the other operations.

Using duality, if we have a fitting characterization (P.10) for erosion and opening, a similar characterization can be obtained for dilation and closing.

As far as P.14 is concerned, the strongest relationship we could impose on the cuts is an equality between the cuts of the fuzzy set resulting of a ΦMM operator and the result of the corresponding BMM operator applied to the cuts of the fuzzy sets, for instance, for dilation $[D_\nu(\mu)]_\alpha = D_{\nu_\alpha}(\mu_\alpha)$. At this point, it should be noted that an α-cut is usually interpreted as a decision threshold. Thus, the previous equality is equivalent to commutativity of the ΦMM operators with decision thresholding. Clearly, such a commutativity is not always desirable. Indeed, it would mean that the fuzzy model is of no use, since all operations could be performed on binary sets after the decision. On the contrary, in many applications, the interest of a fuzzy model relies on the possibility for the decision step to be rejected at the end of the reasoning process. Therefore, we will not impose *a priori* some particular relationship with the α-cuts. Table 3 summarizes these results.

3.3.2. *Relaxing some requirements.* As mentioned previously, not only may we eliminate redundancy between requirements in order to exhibit a reduced set of active constraints to create a fuzzy mathematical

Table 3. The right column lists the operations for which the properties on the left column are required in order that ΦMM inherits these properties as in GMM and FMM

Property	Needed only for
P.1: translations	Dilation
P.2: homotheties	Dilation and erosion
P.3: local knowledge	Dilation
P.4: continuity	Dilation
P.5: duality	Between erosion and dilation
P.6: increasingnesss	Dilation
P.7: extensivity	Dilation and closing
P.8: idempotence	None
P.9: pseudo-commutativity	Dilation
P.10: fitting characterization	Erosion and opening
P.11: compatibility with $\cup$ and $\cap$	Dilation (P.11.1 and P.11.3)
P.12: iteration, combination	P.12.1 and P.12.3
P.13: compatibility with classical mathematical morphology	Dilation
P.14: relationship with cuts	None

morphology, but also we may accept that some of them be less stringent. We have several arguments to do so:

- the application domain is not necessarily the most general one: for instance, image processing is dealing with discrete finite spaces instead of continuous infinite spaces;
- fuzzy set theory may induce specific properties;
- the extension of mathematical morphology to a new domain will certainly bring about compromises and loss on some properties.

Generally speaking, the raised question is the following: which are the weakest properties we need, and what for? It can be expected that if we impose only weak requirements, the deduced ΦMM operators will also have weaker properties.

Since we intend to apply ΦMM to digital pictures, which are inherently discrete, the continuity requirement (P.4) can be forgotten, as it does not make sense for finite discrete spaces.

For fuzzy sets, compatibility with homotheties seems not of prime importance, especially because only homotheties with respect to membership values can be considered. Thus, property 2 does not need to be strongly imposed. We will see in the following that it is satisfied for some definitions and not satisfied for others.

The fitting characterization (P.10) is commonly used in BMM and GMM, in particular for erosion and opening, and involves mainly set inclusion. This property is strongly used in the construction of Sinha *et al.*[20] (see See Section 6.1.4) as ΦMM is seen in that work mainly from a mathematical morphology point of view. In the other constructions (definitions 1, 2, 3 and 6), as set operators such as inclusion are generalized into their fuzzy counterparts, the derived ΦMM implicitly contains concepts based on inclusion, and thus fitting characterizations. As even in the work by Sinha *et al.*[20] the explicit formula for fitting is quite complicated, we will not impose P.10 as a strong requirement for ΦMM.

It has already been stressed in the previous section that P.14 (relationship with cuts) is not always desirable if we do not need commutativity with decision thresholding. Thus, this requirement may be relaxed. We will see in Section 5 that particular definitions of fuzzy dilation may lead to weak or strong relationships between the α-cuts of the dilated fuzzy set and the dilated α-cuts.

Finally, as long as the set of properties P.11 is concerned, we may accept that only inequalities hold, resulting in weaker implications for the derived operations. As another example of such a relaxation of our demand, we will see that relaxing extensivity (P.7) and idempotence (P.8) will result in weaker algebraic properties for opening and closing, but will not sentence the intuitive filtering effects that were expected from these operations.

3.3.3. *Testing requirements and their reduction.* To see how the reduction of requirements works, let us take the example of Sinha *et al.*,[20,28] where an inclusion indicator $I(v, \mu)$ for two fuzzy sets is defined, from which a fuzzy erosion is constructed as follows:

$$E_v(\mu)(x) = I(v + x, \mu)$$

[conversely, the inclusion indicator is related to erosion by $I(v, \mu) = E_v(\mu)(0)$].

Nine axioms are given to be satisfied by the inclusion indicator. Let us examine how they are related to the required properties of ΦMM. The question is to decide whether they are necessary and/or sufficient for assuring good properties of erosion and dilation. Let us first recall these axioms, for any inclusion indicator $I(v, \mu)$:

- A1. $I(v, \mu) = 1 \Leftrightarrow v \subset \mu^*$
- A2. $I(v, \mu) = 0 \Leftrightarrow \{x / v(x) = 1\} \cap \{x / \mu(x) = 0\} \neq \varnothing$.

* Note that $\cup$, $\cap$ and $\subset$ are here again taken in the sense of Zadeh (i.e. the union of two fuzzy sets is computed as the maximum of their membership functions, the fuzzy intersections as their minimum, $\subset$ means: $\forall x \in S$, $v_\alpha(x) \leq \mu(x)$).

Table 4. Equivalence between axioms A1–A9 and properties P.1–P.14 (only implications hold for the second and fourth items)

A1 for binary v and μ	P.13
A1 for a binary v	Implied by P.13
A2 for binary v and μ	P.13
A2 for a binary v	Implied by P.13
A3	P.6 (increasingness of erosion with respect to μ)
A4	P.6 (decreasingness of erosion with respect to v_z)
A5	P.1
A6	P.9
A7	P.11.6
A8	P.11.7
A9	P.11.5

- A3. $\mu \subset \mu' \Rightarrow I(v,\mu) \leq I(v,\mu')$.
- A4. $v \subset v' \Rightarrow I(v,\mu) \geq I(v',\mu)$.
- A5. $I(v,\mu) = I(v+t,\mu+t)$ for any translation t.
- A6. $I(v,\mu) = I(\mu^C, v^C)$.
- A7. $I(v \cup v',\mu) = \min[I(v,\mu), I(v',\mu)]$.
- A8. $I(v,\mu \cap \mu') = \min[I(v,\mu), I(v,\mu')]$.
- A9. $I(v,\mu \cup \mu') \geq \max[I(v,\mu), I(v,\mu')]$.

Equivalences between axioms A1–A9 and properties P.1–P.14 are given in Table 4 (proofs can be found in[29].

It is clear from Table 4 that the nine axioms A1–A9 are weaker than the complete set of properties in Section 3.1. For example, properties such as duality (P.5) and algebraic properties such as idempotence (P.8) do not find any equivalent among the nine axioms. Also, these axioms are not independent, since A4 can be deduced from A3 and A6, and A7 can be deduced from A6 and A8. Thus, if a minimal set of requirements were to be stated, some of the nine axioms should be discarded.

In Sinha *et al.*[18] the most important properties to be satisfied by ΦMM are: P.1, P.6, P.7, P.8 and P.10. However, the nine axioms proposed by the authors only provide weaker properties than the five which were listed.

4. TWO CONSTRUCTION PRINCIPLES

Among the above requirements, our choice is to privilege two of them and take them as construction principles: duality (property 5) and compatibility with BMM and GMM (property 13).

Duality allows us to define only one operation (dilation or erosion) and to deduce the other ones from it, as well as a set of morphological operators deduced from erosion and dilation by combination and/or iteration.

It has been stated in the Introduction that classical morphology can be applied to fuzzy sets with binary structuring elements, thus providing adequate solutions. In order to guarantee coherence with existing literature, we suggest taking this compatibility as a stringent requirement for the constructed fuzzy operations. Moreover, compatibility with classical BMM and GMM constitutes a guide for generalizing the transformations to fuzzy ones. We present two ways for achieving this generalization.*

4.1. *Duality*

At first, the principle of duality with respect to complementation establishes a strong relationship between dilation and erosion, since one operation can thus be deduced from the other. This allows us to construct only one operation. Duality between operations Φ and Ψ on fuzzy sets with respect to complementation is defined, according to Serra,[27] as:

$$\forall \mu \in M, \quad \Phi(\mu) = [\Psi(\mu^C)]^C.$$

This is the definition used in P.5. If complementation is defined as usual by:

$$\forall x \in S, \quad \mu^C(x) = 1 - \mu(x),$$

duality with respect to complementation between erosion and dilation with a structuring element v is expressed as (definition 7):

$$\forall x \in S, \quad D_v(\mu)(x) = 1 - E_v(1-\mu)(x). \quad \text{(def.7)}$$

An analogous definition between opening and closing is expressed as:

$$\forall x \in S, \quad O_v(\mu)(x) = 1 - F_v(1-\mu)(x),$$

for opening and closing defined as usual by:

$$O_v(\mu) = D_v[E_v(\mu)],$$
$$C_v(\mu) = E_v[D_v(\mu)].$$

For a general complementation function c, we have a similar definition:

$$\forall x \in S, \quad D_v(\mu)(x) = c[E_v(c(\mu))(x)],$$
$$\forall x \in S, \quad O_v(\mu)(x) = c[F_v(c(\mu))(x)].$$

The expression given in definition 7 has been used to define erosion in definitions 1–3. The general form for any complementation c has been used for definition 6.†

* As mentioned in the Introduction, we cannot expect these generalizations to be also coherent with classical FMM.

† However, a slightly different definition has been used by Sinha *et al.*[18,20] for duality (definition 8):

$$\forall x \in M, D_v(\mu)(x) = 1 - E_v(1-\mu)(x).$$

If $E_v(\mu)$ is a generalization of the classical erosion, it results from definition 8 that $D_v(\mu)$ is a generalization of the Minkowski addition, and not of the dilation in the sense of Serra:[27] for definition 4 it is expressed as $D4_v(\mu)(x) = \sup_{v \in S} \max[0, \mu(y) + v(x-y) - 1]$, and for definition 5 as $D5_{v_z}(\mu)(x) = \sup_{y \in S} \max[0, 1 - \lambda(\mu(y)) - \lambda(v_z(x-y))]$ [these equations are the original ones given by Sinha *et al.*[18,20]]. Surprisingly, opening and closing as defined in this work satisfy the principle of duality according to definition 7:

$$O_v(\mu) = 1 - C_v(1-\mu),$$

In the following, we only consider definition 7 for duality, as it provides similar relationships between dilation and erosion and between opening and closing.

4.2. *Compatibility with classical mathematical morphology for binary structuring elements*

The compatibility with BMM and GMM can be achieved in two ways. The first relies on the interpretation of a fuzzy set as a stack of binary sets. Then, fuzzy operations are obtained by "stacking" binary operations in the same way. The second consists of considering fuzzy sets as functions. Then binary operators are generalized using their functional counterpart.

4.2.1. *Fuzzification using α-cuts.* A fuzzy set can be considered as a stack of binary sets by means of its α-cuts, and reconstructed from them. Two ways are commonly used for this task:

$$\mu(x) = \int_0^1 \mu_\alpha(x)\,\mathrm{d}\alpha,$$

$$\mu(x) = \sup_{\alpha\in]0,1]} [\alpha\mu_\alpha(x)].$$

In a way similar to the above reconstruction of a fuzzy set from its own cuts, the fuzzification Φ of a binary function $\tilde{\Phi}$ can be obtained by one of the following equations:

$$\Phi(\mu) = \int_0^1 \tilde{\Phi}(\mu_\alpha)\,\mathrm{d}\alpha,$$

$$\Phi(\mu) = \sup_{\alpha\in]0,1]} [\alpha\tilde{\Phi}(\mu_\alpha)].$$

This fuzzification principle indicates the first way to construct fuzzy dilation and erosion from the binary definitions.

Let us consider the first fuzzification equation. For two fuzzy sets μ and ν, the dilation of μ by ν is obtained by fuzzification over μ then over ν, or, equivalently, by the converse:

$$D_\nu(\mu)(x) = \int_0^1 d_{\nu_\beta}(\mu_\beta)(x)\,\mathrm{d}\beta = \int_0^1\int_0^1 d_{\nu_\alpha}(\mu_\beta)(x)\,\mathrm{d}\alpha\,\mathrm{d}\beta$$
$$= \int_0^1 d_{\nu_\alpha}(\mu)(x)\,\mathrm{d}\alpha = \int_0^1\int_0^1 d_{\nu_\alpha}(\mu_\beta)(x)\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

A straightforward derivation provides:

$$D_\nu(\mu)(x) = \int_0^1 \sup_{y\in(\nu_\alpha)_x} \mu(y)\,\mathrm{d}\alpha,$$

which is definition 1.

In the same way, the erosion of μ by ν is obtained by:

$$E_\nu(\mu)(x) = \int_0^1 \inf_{y\in(\nu_\alpha)_x} \mu(y)\,\mathrm{d}\alpha.$$

These definitions guarantee that $D_\nu(\mu)$ and $E_\nu(\mu)$ are the membership functions of fuzzy sets (i.e. taking values in [0, 1]). By construction, if ν is crisp, the definitions coincide with the classical ones.

Let us now consider the second way of representing a fuzzy set by its α-cuts. Here again, we may construct a fuzzy dilation by a double fuzzification, and obtain:

$$D_\nu(\mu)(x) = \sup_{\alpha\in]0,1]} \left[\alpha \sup_{y\in(\nu_\alpha)_x} \mu(y)\right],$$

providing:

$$D_\nu(\mu)(x) = \sup_{y\in S} [\nu(y-x)\mu(y)],$$

which coincides exactly with definition 3. An analogous result is obtained for the fuzzy erosion.

4.2.2. *Translating set relationships into functional ones.* Another way to build ΦMM is to base the initial definitions on the translation of set operations (inclusion, intersection and union) into functional terms, i.e. to exhibit, for each operation, functions from $[0, 1] \times [0, 1]$ to [0, 1] which satisfy some given limit conditions expressing the compatibility with binary set operations. As many such extensions can be proposed, several definitions exist (definitions 2–5). However, we will show that they may receive a unified presentation by means of T-norms (and associated T-conorms) as they can be interpreted, respectively, as fuzzy intersection and fuzzy union (from which a fuzzy inclusion can be derived; definition 6).

The definitions and main properties of T-norms and T-conorms are provided in Appendix 2, and Fig. 3 illustrates the most used T-norms and T-conorms.

From any T-norm i taken as fuzzy intersection and the associated T-conorm u taken as fuzzy union (with respect to a complementation c), it is possible to construct a ΦMM. From the following set equivalence [where $E_B(X)$ denotes the erosion of the set X by B, and S the considered space]:

$$x\in E_B(X) \Leftrightarrow B_x \subset X \Leftrightarrow X \cup B_x^C = S \Leftrightarrow \forall y\in S, y\in X\cup B_x^C,$$

a natural way to define the erosion of a fuzzy set μ by a fuzzy structuring element ν is:

$$E_\nu(\mu)(x) = \inf_{y\in S} u[\mu(y), c(\nu(y-x))].$$

In this equation "$\cup$" has been translated in terms of a T-conorm u and "$\forall$" by an infimum.

By duality with respect to the complementation c,

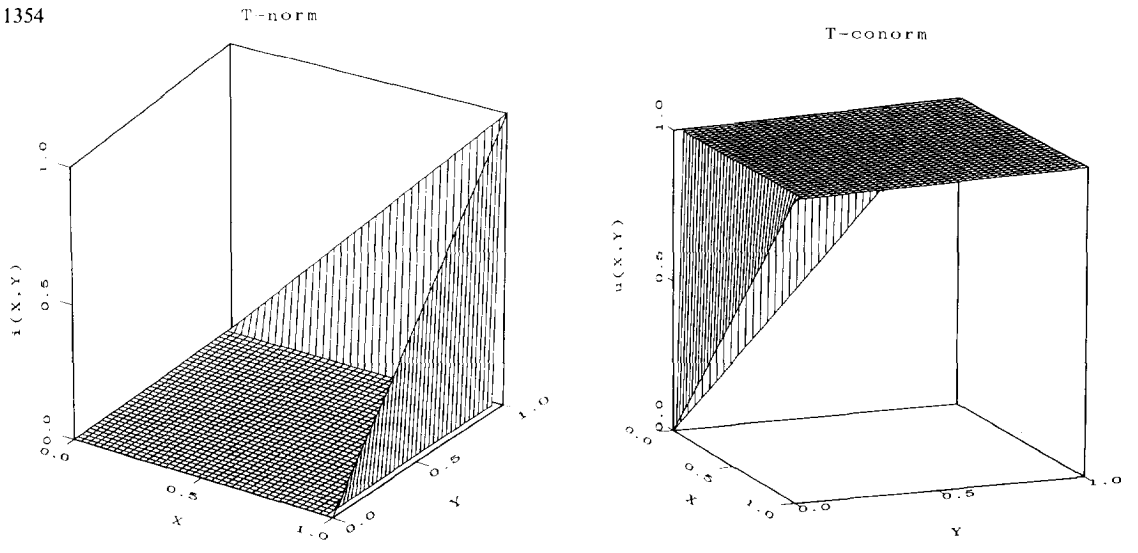
Footnote (*Continued*)

resulting from a definition of these operations different from the classical definition:

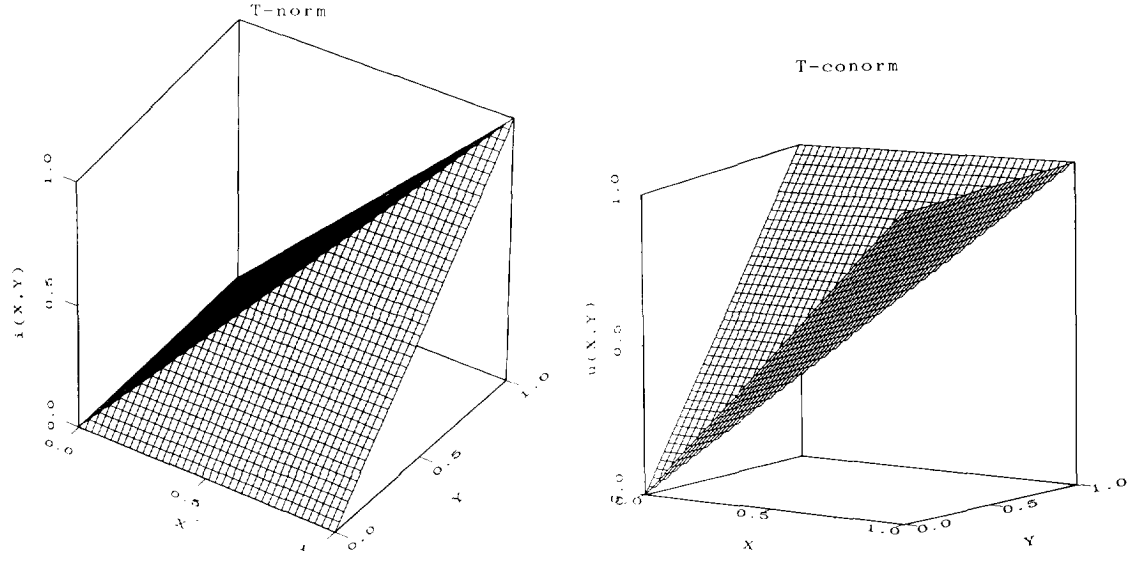
$$O_\nu(\mu) = D_\nu[E_\nu(\mu)],$$

$$C_\nu(\mu) = E_\nu[D_\nu(\mu)].$$

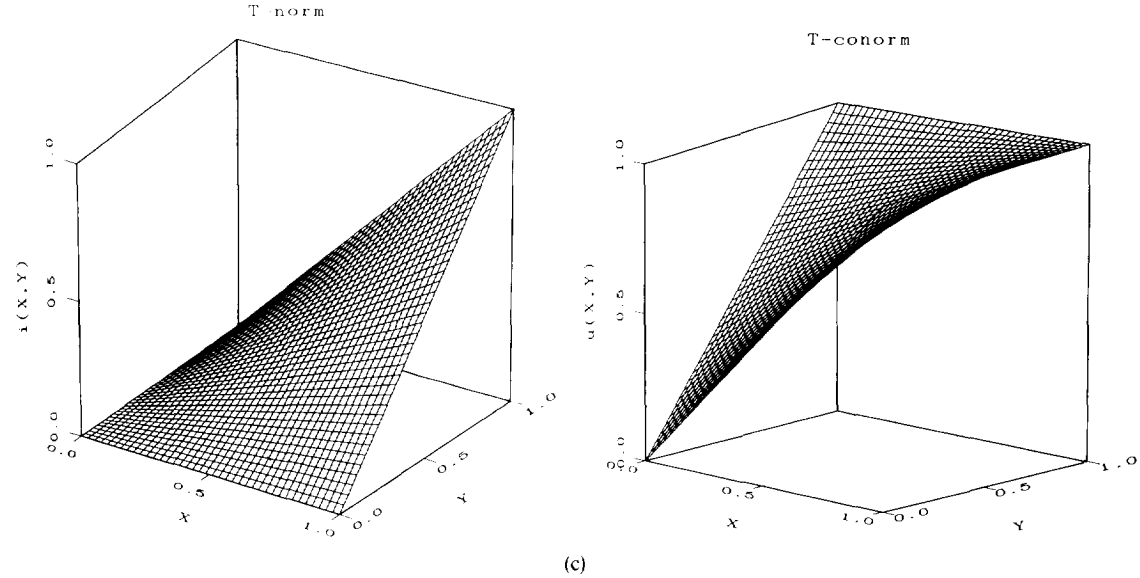
In order to treat the operations defined by definitions 4 and 5 in the same way as the others, we use the definitions 4 and 5 presented in Section 2 instead of the original equations of references (18 and 20). In this way, *D*4 and *D*5 actually generalized classical dilation and no more Minkowski addition.



(a)

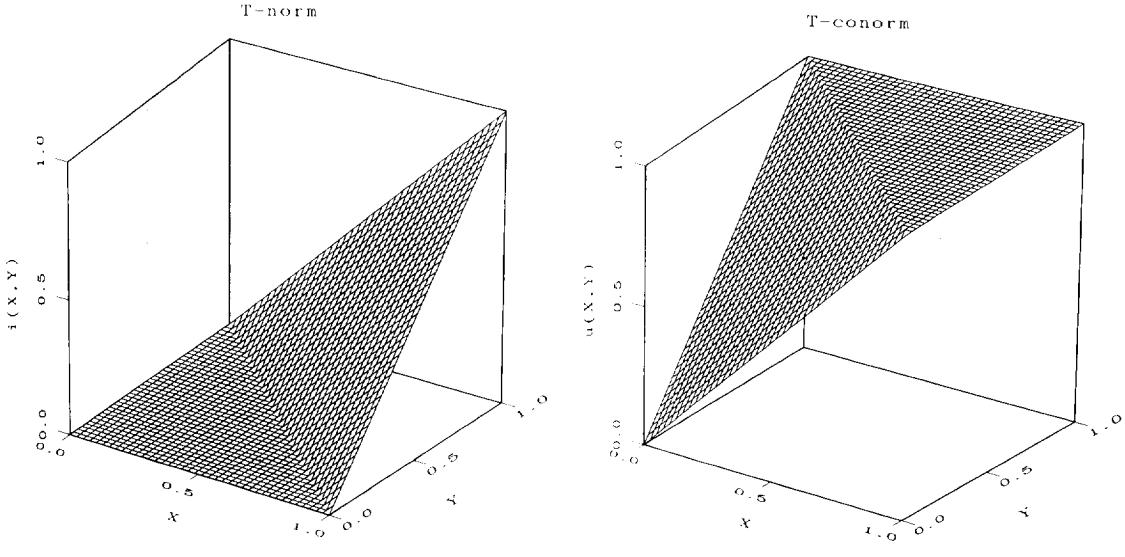


(b)



(c)

Fig. 3. Example of T-norms and T-conorms: (a) i_0 and u_0; (b) $i(x, y) = \min(x, y)$ and $u(x, y) = \max(x, y)$; (c) $i(x, y) = xy$ and $u(x, y) = x + y - xy$; (d) $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$.



(d)

Fig. 3. (*Continued*)

Table 5. Different definitions for fuzzy union and intersection, and corresponding definitions for fuzzy erosion and dilation

Def.	T-conorm	T-norm	Erosion	Dilation
2	$\max(x, y)$	$\min(x, y)$	$\inf_{y\in S} \max[\mu(y), 1 - v(y - x)]$	$\sup_{y\in S} \min[\mu(y), v(y - x)]$
3	$x + y - xy$	xy	$\inf_{y\in S} [\mu(y)v(y - x) + 1 - v(y - x)]$	$\sup_{y\in S} [\mu(y)v(y - x)]$
4	$\min(1, x + y)$	$\max(0, x + y - 1)$	$\inf_{y\in S} \min[1, 1 + \mu(y) - v(y - x)]$	$\sup_{y\in S} \max[0, \mu(y) + v_\alpha(y - x) - 1]$
5	$\min[1, \lambda(1 - x) + \lambda(1 - y)]$	$\max[0, 1 - \lambda(x) - \lambda(y)]$	$\inf_{y\in S} \min[1, \lambda(1 - \mu(y)) + \lambda(v(y - x))]$	$\sup_{y\in S} \max[0, 1 - \lambda(\mu(y)) - \lambda(v_\alpha(y - x))]$
	(weak T-conorm)	(weak T-conorm)		
6	T-conorm $u(x, y)$	T-norm $i(x, y)$	$\inf_{y\in S} u[c(v(y - x)), \mu(y)]$	$\sup_{y\in S} i[v(y - x), \mu(y)]$

fuzzy dilation is then defined by:

$$D_v(\mu)(x) = \sup_{y\in S} i[\mu(y), v(y - x)],$$

which corresponds to the following set equivalence:

$$x \in D_B(x) \Leftrightarrow \bar{B}_x \cap X \neq \varnothing \Leftrightarrow \exists y \in S, y \in \bar{B}_x \cap X.$$

Here, "$\cap$" has been translated in terms of a T-norm i and "$\exists$" by a supremum.

Table 5 shows that all definitions given in Section 2 except the first can be stated in this framework, i.e. they all correspond to a particular T-norm and the associated T-conorm. It should be noted that definition 5[20] corresponds in the general case only to weak T-norm and T-conorm, as they are not associative and do not admit 1 (respectively, 0) as the unit element if $\lambda \neq \lambda_0$. Figure 4 gives an example of such weak T-norm and T-conorm. We will show in Section 5 that this will have some consequences on the properties derived from this definition.

Let us now consider the other definitions given in the literature (see the Introduction) and not mentioned in Section 2 because they are less general. The definition of Rosenfeld[3] is given for binary structuring elements and corresponds exactly to the classical definition of grey-level operators. As the operations derived from any T-norm are compatible with GMM, they include the definition of Rosenfeld[3] as a particular case. We have the same result for the definition given by Goetcherian,[4] which also considers only binary structuring elements. From the Minkowski addition defined by Kaufmann[5] through the α-cuts by $(\mu \oplus v)_\alpha = \mu_\alpha \oplus v_\alpha$,

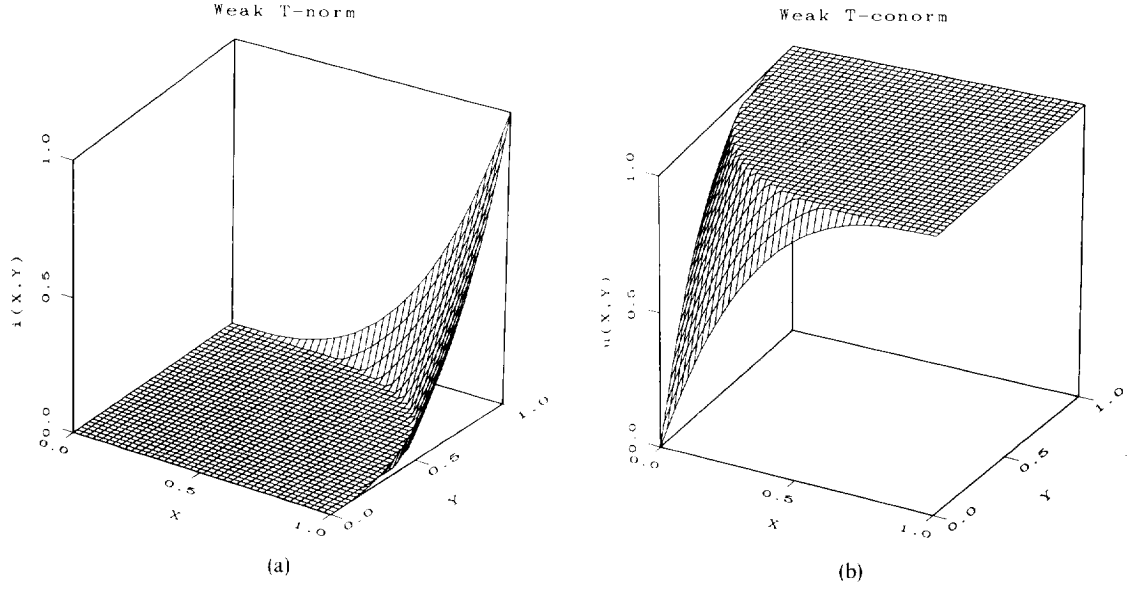


Fig. 4. Example of a weak T-norm derived from a λ function and the associated weak T-conorm [$\lambda(x) = 1 - x^3$]: (a) $i(x, y) = \max[0, 1 - \lambda(x) - \lambda(y)]$; (b) $u(x, y) = \min[1, \lambda(1 - x) + \lambda(1 - y)]$).

the α-cuts of the dilation are obtained by:

$$[D_v(\mu)]_\alpha = D_{v_\alpha}(\mu_\alpha),$$

and the erosion is obtained by duality. It can then be shown[21] that, according to this equation, the derived expression for dilation is:

$$D_v(\mu)(x) = \sup_{y \in S} \min[v(y - x), \mu(y)],$$

which corresponds to the definition obtained from the T-norm "min". These remarks show that the definitions proposed in previous works fall within definition 6. Note that the definition given by Giardina[7] for fuzzy dilation takes the form:

$$\sup_{y \in S} \min[1, \mu(y) + v(y - x)],$$

and thus involves a T-conorm instead of a T-norm. Some important consequences are that this definition is not compatible with BMM and GMM, and, as soon as there exist x in S such that $v(x) = 1$ or $\mu(x) = 1$, the result of dilation is completely saturated (i.e. = 1).

In the same way, the definitions proposed by di Gesu *et al.*,[9,10,11] based on symmetrical difference or average operators, are not compatible with BMM.

This principle of translating set operations into functional ones has also been used by Sinha and Dougherty.[18] Instead of translating directly relationships involving union and intersection, they start from the fitting characterization of erosion which is considered as a marker in a morphological sense.[19,30] Thus, they define an inclusion indicator for fuzzy sets $I(v, \mu)$, from which a fuzzy erosion is derived. The two approaches are strongly related to each other, as inclusion can be defined from union and complementation. In the binary case, we have:

$$A \subset B \Leftrightarrow A^C \cup B = S.$$

In the fuzzy case, a similar equation provides:

$$I(v, \mu) = \inf_{x \in S} (v^C \cup \mu)(x),$$

for given fuzzy union and fuzzy complementation. $I(v, \mu)$ represents the degree to which the fuzzy set v is included in the fuzzy set μ. Definitions 4 and 5[18,20] correspond, respectively, to:

$$I(v, \mu) = \inf_{x \in S} \min[1, 1 - v(x) + \mu(y)],$$

$$I(v, \mu) = \inf_{x \in S} \min[1, \lambda(v(x)) + \lambda(1 - \mu(y))].$$

These definitions correspond to particular T-norm and T-conorm (eventually weak ones), as we have seen before.

Other inclusion indicators have been proposed in the literature, but they are generally not convenient for ΦMM as they do not lead to the required morphological properties. For example, let us consider the definitions of Kosko[31] (definition 9) and Ishizuka[32] (definition 10) for a finite space S:

$$I_K(v, \mu) = 1 - \frac{\sum_{x \in S} \max[0, v(x) - \mu(x)]}{\sum_{x \in S} v(x)}$$

(definition 9)

$$I_I(v, \mu) = \frac{\min_{x \in S} \min[1, 1 - v(x) + \mu(x)]}{\max_{x \in S} v(x)}$$

(definition 10)

These two expressions are similar to those obtained with the bounded sum as T-conorm followed by a

normalization. The poor properties obtained for fuzzy morphological operations derived with definitions 9 and 10 are due to this normalization. Fuzzy erosion derived from definition 9 is not internal in [0, 1] in the general case, and thus leads to a result which is not a fuzzy set. Definition 10 provides operations which are not compatible with the binary case. For example, for a binary structuring element, the dilation derived from definition 10 is exactly a mean operator, in complete contradiction with the basically non-linear nature of mathematical morphology. For these reasons, these two inclusion indicators will not be considered in the following.

5. COMPARISON OF PROPERTIES BETWEEN THE DIFFERENT DEFINITIONS

In this section, we establish some properties of the fuzzy mathematical operators constructed according to the principles described in Section 4. We first look at the required properties defined in Section 3 and examine each definition of Section 2 with respect to these properties. The main results for definition 4 and some for definition 5 have already been given in references (18, 20).

Some additional relationships between these six definitions are finally stated.

As duality (P.5) and compatibility with BMM and GMM (P.13) are taken as construction principles, these properties are satisfied for definition 1 and for all definitions derived from the general form (definition 6). However, for definition 5 which corresponds only to a weak T-norm, P.5 holds for any function λ but P.13 is satisfied if and only if $\lambda = \lambda_0$ (see definition 5, in part 2).

P.2 (translation invariance) holds for all definitions if S is translation invariant (i.e. S is infinite). Note that the translation invariance condition for S is the same as for classical morphology and is not an additional restriction for fuzzy morphology.

The fuzzy dilation is compatible with homotheties for definition 1 and for definition 6 if and only if i is compatible with homotheties (thus P.2 holds for definition 3, but not for definitions 2, 4 and 5). For example, for dilation defined from the product T-norm (definition 3), we have:

$$\forall x \in S, \quad \forall \lambda \in [0,1], \quad D_v(\lambda\mu)(x) = \sup_{y \in S} [\lambda\mu(y)v(y-x)] = \lambda \sup_{y \in S} [\mu(y)v(y-x)] = \lambda D_v(\mu)(x).$$

The computation of the fuzzy dilation (for any definition) of a fuzzy set μ in a binary mask Z is sufficient to know the result of the operation in the eroded set Z by the support $supp(v)$ of the structuring element:

$$[D_v(\mu \cap Z)] \cap (E_{supp(v)}(Z)) = D_v(\mu) \cap (E_{supp(v)}(Z)).$$

Thus, P.3 (local knowledge property) holds for all definitions.

Fuzzy dilation is continuous for definition 1 and for definition 6 iff i is continuous. This is the case for definitions 2–4 and for definition 5 if and only if λ is continuous.

P.6 (increasingness of dilation with respect to μ and to v_x) holds for all definitions. Similar properties hold for erosion, opening and closing.

Fuzzy dilation is extensive (P.7) [i.e. $D_v(\mu) \geq \mu$] iff $v(0) = 1$, for definitions 1, 2, 3, 4 and 6 (0 being the origin of the space S). The condition $v(0) = 1$ corresponds to "$0 \in B$" in the binary case. For definition 5, it holds iff $v(0) = 1$ and $\lambda = \lambda_0$.

Increasingness of closing and opening (P.6) are deduced from increasingness of dilation and erosion, but the other algebraic properties (extensivity or anti-extensivity, idempotence) of opening and closing are usually not satisfied. They are not satisfied for definition 1. For definition 6, they hold iff the following relationship holds between i and u:

$$\forall (a,b) \in [0,1]^2, \quad i[b, u(c(b)), a)] \leq a.$$

In particular, this condition implies the non-contradiction principle and is not satisfied by min and max, nor by the product and algebraic sum. Thus, opening and closing derived from definitions 2 and 3 do not satisfy P.7 and P.8. The only definition (among definitions 2 to 5 of Section 2) which leads to algebraic opening and closing is definition 4. For definition 5, the properties holds iff $\lambda = \lambda_0$ (i.e. when definition 5 coincides with definition 4).

Fuzzy dilation is pseudo-commutative (P.9) for all definitions. Note that in the original definitions of Sinha *et al.*,[18,20] the dilation is commutative, since it corresponds to Minkowski addition.

For definitions 2–6, the fuzzy dilation is compatible with union (P.11.1):

$$D_v[\max(\mu, \mu')] = \max[D_v(\mu), D_v(\mu')].$$

Definition 1 verifies a weaker property:

$$D1_v[\max(\mu, \mu')] \geq \max[D1_v(\mu), D1_v(\mu')].$$

P.11.3 (relationship to intersection) holds for all definitions.

The iteration relation (property 12.1) does not hold for definition 1. It holds for definitions 2, 3, 4 and 6. For definition 5, the property is satisfied iff $\lambda = \lambda_0$, since a weak T-norm is generally not associative, a property strongly involved in the demonstration of P.12.1 for T-norms and T-conorms. The only function λ for which the weak operators $i(x, y) = \max[0, 1 - \lambda(x) - \lambda(y)]$ and $u(x, y) = \min[1, \lambda(1 - x) + \lambda(1 - y)]$ are associative is λ_0.

The combination property (P.12.3) is not satisfied for definition 1. For definition 6, it holds iff the following relationship between i and u is satisfied:

$$\forall (a,b,c) \in [0,1]^3, \quad i[a, u(b,c)] \leq u[b, i(a,c)].$$

This condition is satisfied, for example, for $i = \min$ and $u = \max$, for $i(x, y) = xy$ and $u(x, y) = x + y - xy$, for $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$,

Table 6. Comparison between the properties of definitions 1–6

	$\int_0^1 \sup_{y\in(v_\alpha)_\lambda} \mu(y)\,d\alpha$ (definition 1)	$\sup_{y\in S} i[\mu(y), v(y-x)]$ (definitions 2, 3, 4 and 6); (i; T-norm)	$\sup_{y\in S} \max[0, 1-\lambda(\mu(y))-\lambda(v(y-x))]$ (definition 5) (weak T-norm)
Duality (P.5)	Yes	Yes	Yes
Compatibility with $\mu \oplus v$ if v is binary (P.13)	Yes	Yes	Iff $\lambda = \lambda_0$
Compatibility with translations (P.1)	Yes	Yes	Yes
Compatibility with homotheties (P.2)	Yes	Iff i is compatible with homotheties	No
Local knowledge (P.3)	Yes	Yes	Yes
Continuity (P.4)	Yes	Iff i is continuous	Iff λ is continuous
Increasingness (P.6)	Yes	Yes	Yes
Extensivity of dilation (P.7)	Iff $v(0) = 1$	Iff $v(0) = 1$	Iff $v(0) = 1$ and $\lambda = \lambda_0$
Extensivity of closing (P.7)	No	Iff $i[b, u(c(b), a)] \leq a$	Iff $\lambda = \lambda_0$
Idempotence (P.8)	No	Iff $i[b, u(c(b), a)] \leq a$	Iff $\lambda = \lambda_0$
Pseudo-commutativity (P.9)	Yes	Yes	Yes
Union (P.11.1)	$\geq$	$=$	$=$
Intersection (P.11.3)	$\leq$	$\leq$	$\leq$
Iteration (P.12.1)	No	Yes	Iff $\lambda = \lambda_0$
Combination (P.12.3)	No	Iff $i[a, u(b, c)] \leq u[b, i(a, c)]$	Iff $\lambda = \lambda_0$
Cuts (P.14)		$[D2_v(\mu)]_\alpha = D2_{v_\alpha} = D2_{v_\alpha}(\mu_\alpha)$ $[D6_v(\mu)]_\alpha \subset D6_{v_\alpha}(\mu_\alpha)$	

but not for i_0 and u_0 (defined in Appendix 2). Thus, we have for definitions 2, 3 and 4:

$$D_v[E_v(\mu)] \leq E_v[D_v(\mu)].$$

For definition 5, it is generally not satisfied, except iff $\lambda = \lambda_0$.

We have no general equation relating the cuts of the dilated fuzzy set to the dilation of the cuts of the fuzzy set (P.14), but for definition 2 an interesting equality holds:

$$[D_{2_v}(\mu)]_\alpha = D_{2_{v_\alpha}}(\mu_\alpha).$$

and for the general definition 6, only an inclusion holds:

$$[D6_v(\mu)]_\alpha \subset D6_{v_\alpha}(\mu_\alpha).$$

The comparison of the properties is summarized for definitions 1–6 in Table 6. Complete proofs can be found in references (16, 21, 29).

Moreover, we have the following result between fuzzy dilations:

The dilation obtained from "min" (definition 2) is the largest fuzzy dilation which can be built from definition 6, i.e. for any T-norm i, the associated fuzzy dilation D satisfies:

$$\forall x \in S, \quad D_v(\mu)(x) \leq \sup_{y\in S} \min[\mu(y), v(y-x)] = D2_v(\mu)(x).$$

In the same way, for any T-conorm u, and taking for the complementation $c(x) = 1 - x$, we have:

$$\forall x \in S, \quad E_v(\mu)(x) \geq \inf_{y\in S} \max[\mu(y), 1 - v(y-x)]$$

$$= E2_v(\mu)(x),$$

and thus the fuzzy erosion obtained from "max" is the smallest erosion that can be built for $c(x) = 1 - x$ following definition 6.

This result shows that the operations obtained from "min" and "max" are those which have "the largest effect" on the initial fuzzy sets (see Fig. 2).

In the same way, the fuzzy dilation D_0 and the fuzzy erosion E_0 obtained from i_0 and u_0 are, respectively, the smallest dilation and the largest erosion built according to definition 6, and thus have the less effect on the initial fuzzy sets. This result is, however, not satisfied if we consider only a weak T-norm and the associated weak T-conorm, such as in definition 5.

The following inequalities hold between the definitions given in Section 2 [with $\lambda(x) = 1 - x^n$ in definition 5 and more generally for any function λ such that $\lambda(x) \geq 1 - x$]:

$$D_5 \leq D_4 \leq D_3 \leq D_2,$$

$$E_5 \geq E_4 \geq E_3 \geq E_2.$$

We also have:

$$D_0 \leq D_4 \leq D_3 \leq D_2,$$

$$E_0 \geq E_4 \geq E_3 \geq E_2.$$

but no general relationship between D_0 and D_5, nor between E_0 and E_5.

These properties are of prime importance for the choice of a morphology for a given application.

6. DISCUSSION: COMPARED INTEREST OF THE DIFFERENT DEFINITIONS

In this section, we compare under different aspects the different fuzzy mathematical morphologies presented in the previous sections. Section 6.1 deals with morphological aspects, Section 6.2 with aspects related to fuzzy sets and fuzzy measures. At last, Section 6.3 provides a discussion about the interest of ΦMM for decision problems in image processing.

6.1. *Comparison with respect to mathematical morphology*

Table 6 shows that most of the definitions share almost all properties of classical morphology. Definitions 1 and 5 (in the general case where $\lambda \neq \lambda_0$) lead to the weakest properties. Definitions 2–6, i.e. derived from a T-norm or from a particular class of weak T-norms (associated with a λ function), may have all properties of classical morphology, at least for an adequate choice of the T-norm and associated T-conorm or of the function λ. Let us first consider definition 6 (and definitions 2–4 as special cases of definition 6).

6.1.1. *Morphological aspects for definition 6. Compatibility of dilation with homotheties* is satisfied iff i is compatible with homotheties, i.e. $i(\lambda x, y) = \lambda i(x, y)$ for $\lambda \in [0, 1]$. This property is satisfied for $i(x, y) = xy$ but neither for $i(x, y) = \min(x, y)$, nor for $i(x, y) = \max(0, x + y - 1)$. However, the importance of the membership scale invariance is questionable for ΦMM. For most applications we had, we found it not useful, but contrary opinions may exist.

As far as *semi-continuity* is concerned, most of the used T-norms are continuous. For example, definitions 2, 3 and 4 lead to continuous operators. As a counter-example, the dilation derived from i_0 is not continuous. Note that surprisingly, the proofs do not involve properties on μ and/or ν, but the quite strong requirement of uniform convergence (which could perhaps be relaxed). Despite the fact that this property guarantees the robustness of the operators with respect to small changes in the fuzzy sets, it is seldom used when working on discrete finite spaces (e.g. in image processing).

The condition $\nu(0) = 1$ is necessary and sufficient to guarantee the *extensivity of dilation*, and thus anti-extensivity of erosion. This condition is not an additional restriction with respect to classical morphology and corresponds to the classical condition $0 \in B$. If this condition is not satisfied, we may for instance have more imprecision or uncertainty in an eroded fuzzy set than in the initial one: 0 valued points in the initial fuzzy set (complementary of the support) are no more 0 valued (see Fig. 5). However, this effect is limited to a neighbourhood of the support if there exists a point x_0 with membership value 1 in the fuzzy structuring element (see Fig. 6). This is due to the translation invariance property, which assures that the result is simply the translation of the result we would obtain by translating (spatially) the structuring element by x_0 (so that the space origin 0 has membership value 1). This effect is similar to that observed in classical mathematical morphology. It can be exploited for directional transformations, which are useful for granulometry, covariograms, etc.

The properties of *extensivity* (respectively, *anti-extensivity*) *and idempotence of closing* (respectively, *opening*) are only satisfied for particular T-norms and T-conorms. The required condition is very strong: among the T-norms mentioned in this paper, only the

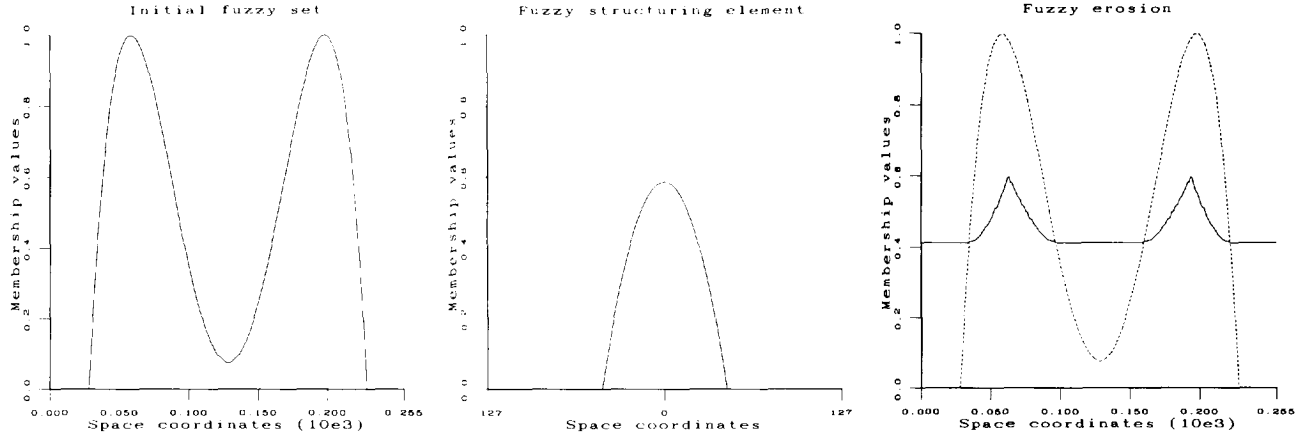


Fig. 5. Fuzzy erosion with $\nu(0) < 1$: erosion is no more anti-extensive (0 valued points in the initial fuzzy set are no more 0 valued). Definition 2 has been used in this figure, but similar behaviour can be observed with all definitions.

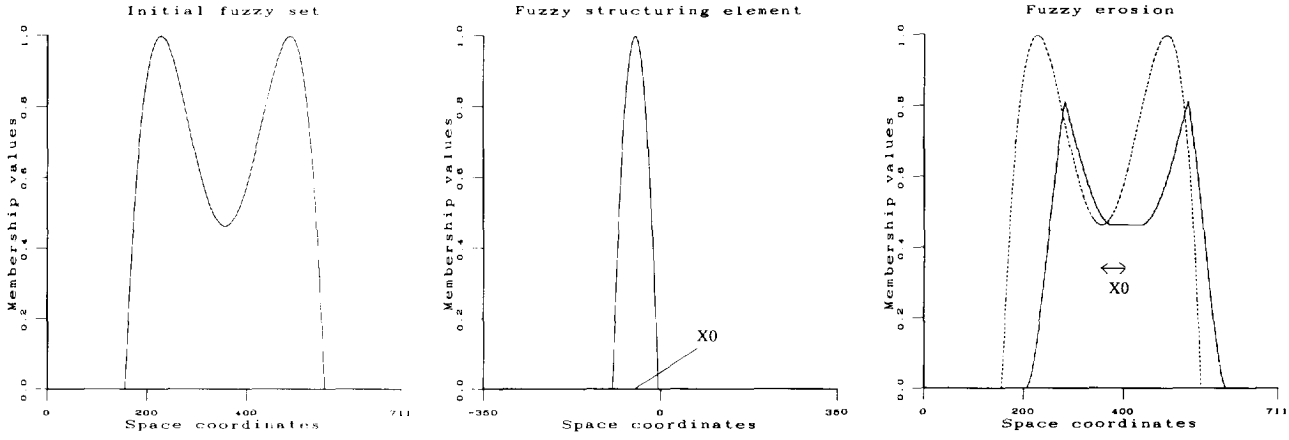


Fig. 6. Fuzzy erosion with a fuzzy structuring element v such that $v(0) < 1$ but $v(x_0) = 1$ for some point x_0 of S. The effect on 0 valued points shown in Fig. 5 is limited to a neighbourhood of the support of μ. The result is the translation by x_0 of the result we would obtain with a translated structuring element v_{-x_0}.

T-norm $i(x, y) = \max(0, x + y - 1)$ and the associated T-conorm $u(x, y) = \min(1, x + y)$ used in definition 4 satisfy this condition (see Fig. 7). If we absolutely need these algebraic properties, then the choice of a particular mathematical morphology is clearly dictated. Unfortunately, definition 4 leads to effects on the initial fuzzy sets which may be judged too weak. Moreover, it is strongly related to FMM, as the results are exactly the same as those obtained with classical morphology and then by shifting and truncating the obtained function (see Fig. 1). Actually, it is not surprising that the definition which is the closest to the classical definition on functions also leads to the closest properties to the classical ones.

The *combination property* is also satisfied only for particular T-norms. However, the required condition is here not too stringent. It is satisfied for the "min", the product, the bounded sum and thus the combination relation holds in particular for definitions 2, 3 and 4..

6.1.2. *Morphological aspects for definition* 5. Let us now consider definition 5, i.e. obtained for a weak T-norm derived from a λ function. Its properties are weaker than those obtained from a T-norm, as expected. As opposed to all other definitions, definition 5 is *not always compatible with GMM* if the structuring element is binary. Actually this compatibility is satisfied if and only if $\lambda = \lambda_0$, i.e. in the case where definition 5 reduces to definition 4 (see Fig. 8). In the general case, however, definition 5 is compatible with BMM.

As weak T-norms deriving from a λ function are not compatible with homotheties, definition 5 leads to transformations which are *not compatible with homotheties*.

For the *continuity property*, the condition "λ continuous" is necessary and sufficient and is similar to the condition stated for definition 6.

A strong limitation of definition 5 concerns *algebraic properties* of the operators: extensivity of dilation (and thus anti-extensivity of erosion), extensivity of closing (anti-extensivity of opening), idempotence of opening and closing, iteration relation and combination relation are all satisfied iff $\lambda = \lambda_0$ (see Fig. 8).

Our feeling is that definition 5 does not constitute a real generalization of definition 4 as far as morphological aspects are concerned, because too much algebraic properties are lost as soon as definition 5 differs from definition 4 (i.e. $\lambda \neq \lambda_0$). However, we will see in the next subsection that the use of a λ function can be useful from a fuzzy set point of view.

6.1.3. *Morphological aspects for definition* 1. Let us now consider the first definition. The underlying principle is completely different from that of the other definitions. For definitions 2–6, the key point is mainly a generalization of binary set operators into fuzzy ones. The good properties derived for these definitions are due to the properties imposed on the fuzzy set operators, which have to match the binary set properties used for mathematical morphology. For definition 1, the construction relies mainly on a generalization of binary sets into fuzzy ones by means of stacking crisp sets (α-cuts). Then morphological operators are constructed in the same way by stacking operations on the α-cuts. In this construction, no property related to morphological properties appears explicitly, as opposed to the other definitions. So, it is not surprising that the morphological properties obtained for the first definition are weaker than for the other ones. Figure 9 shows examples where, respectively, compatibility with union, iteration relation, combination relation, anti-extensivity and idempotence of opening are not satisfied. However, this definition is not without interest. The construction principle is interesting for itself and has been used for generalizing other operations on fuzzy sets, like the degree of connectivity for instance.[3,33] Another interest lies in the regularization effect of the operators, due to the integral in the formula (see Fig. 2). Even operators with weak properties may have effects useful for the applications: for instance, erosion followed

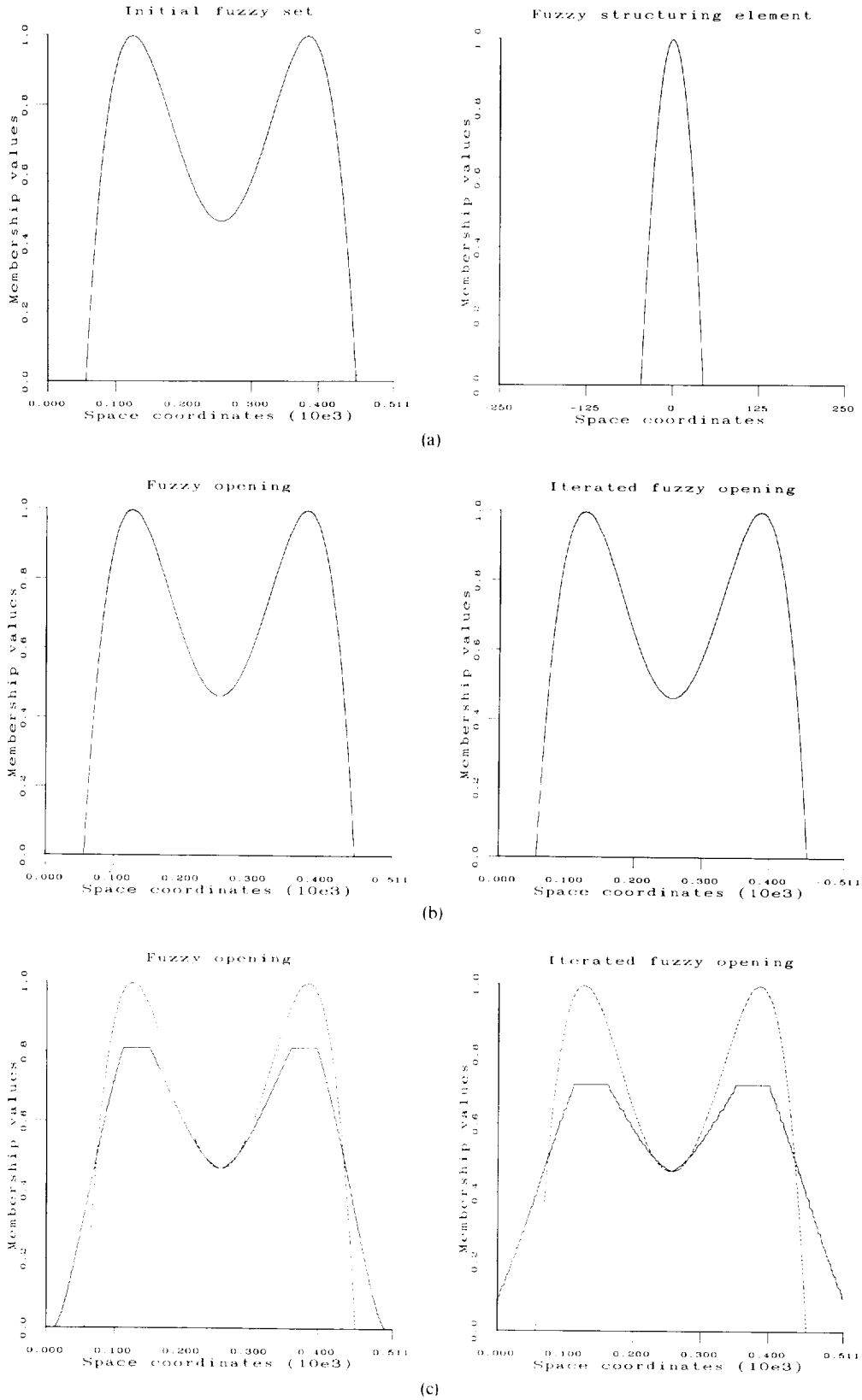
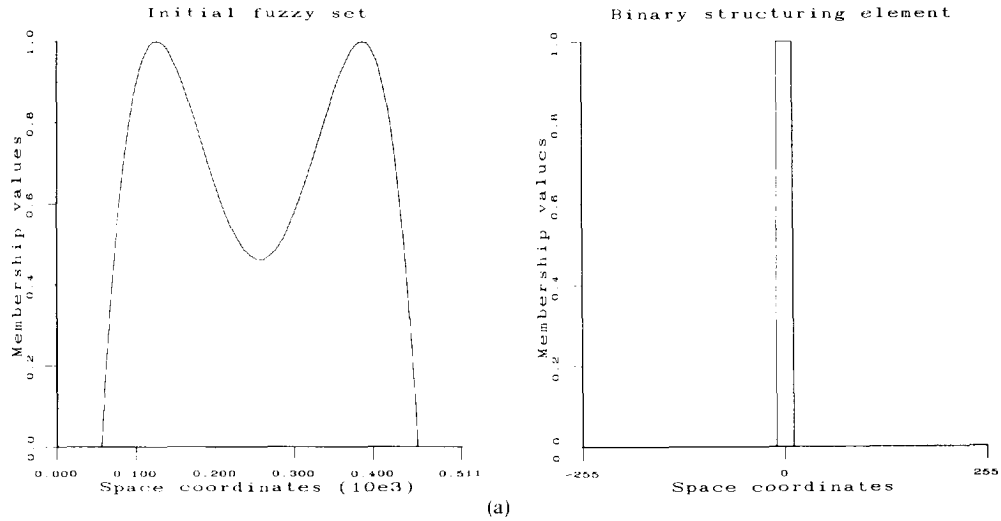
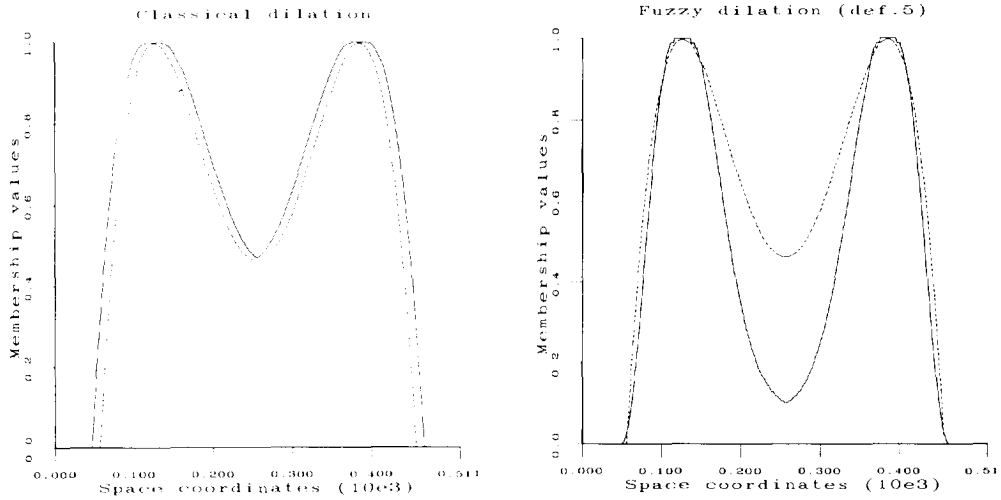


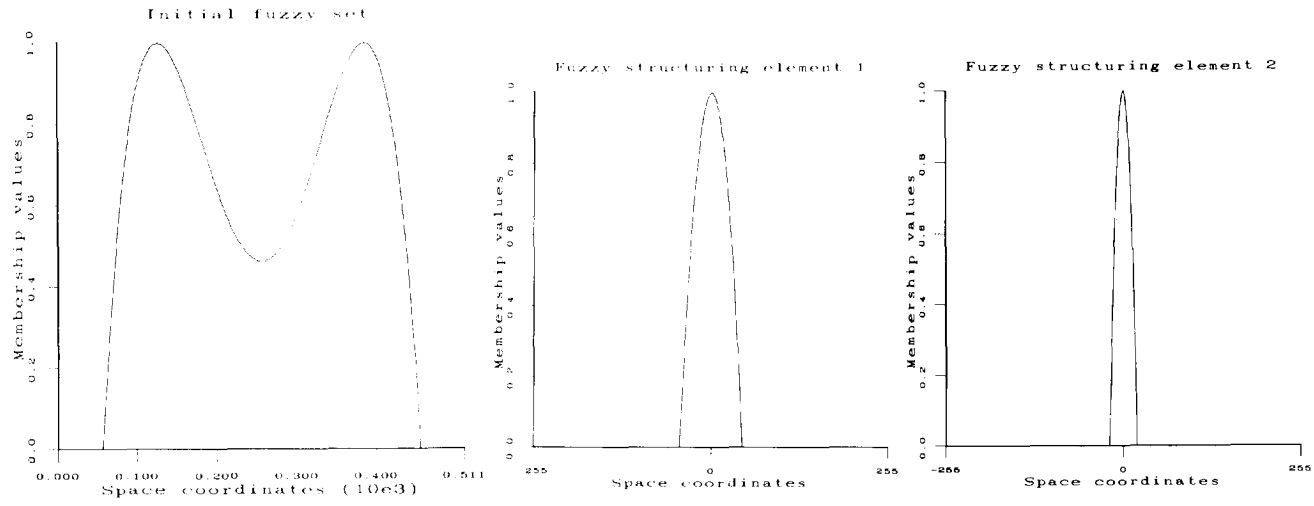
Fig. 7. Anti-extensivity and idempotence of fuzzy opening: (a) initial fuzzy set μ and fuzzy structuring element v; (b) $O_v(\mu)$ and $O_v[O_v(\mu)]$ for $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$: this opening is anti-extensive and idempotent; (c) $O_v(\mu)$ and $O_v[O_v(\mu)]$ for $i(x, y) = \min(x, y)$ and $u(x, y) = \max(x, y)$: this opening is not anti-extensive and not idempotent.



(a)

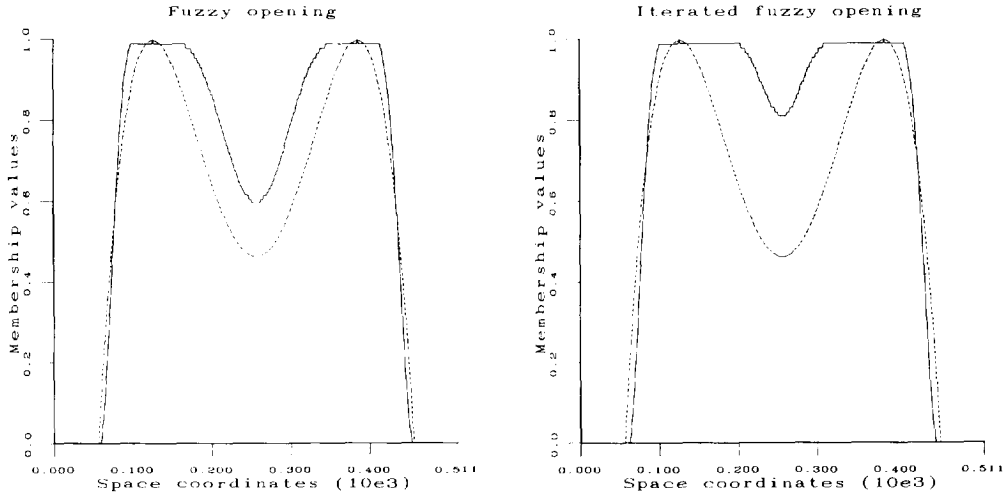


(b)



(c)

Fig. 8. Weak properties of definition 5 for $\lambda \neq \lambda_0$ [here, $\lambda(x) = 1 - x^3$ has been used] (a) fuzzy set and binary structuring element; (b) classical dilation (left) and fuzzy dilation obtained by definition 5 (right) illustrating the non-compatibility of definition 5 with GMM; (c) initial fuzzy set μ and fuzzy structuring v_1 and v_2; (d) $O_v(\mu)$ and $O_v[O_v(\mu)]$: this opening is not anti-extensive and not idempotent; (e) $D_{v_2}[E_{v_1}(\mu)]$ and $E_{v_1}[D_{v_2}(\mu)]$: the combination equation is not satisfied; (f) $D_{v_2}[D_{v_1}[(\mu)]$ and $D_{v_2(v_1)}(\mu)$: the iteration equation is not satisfied.



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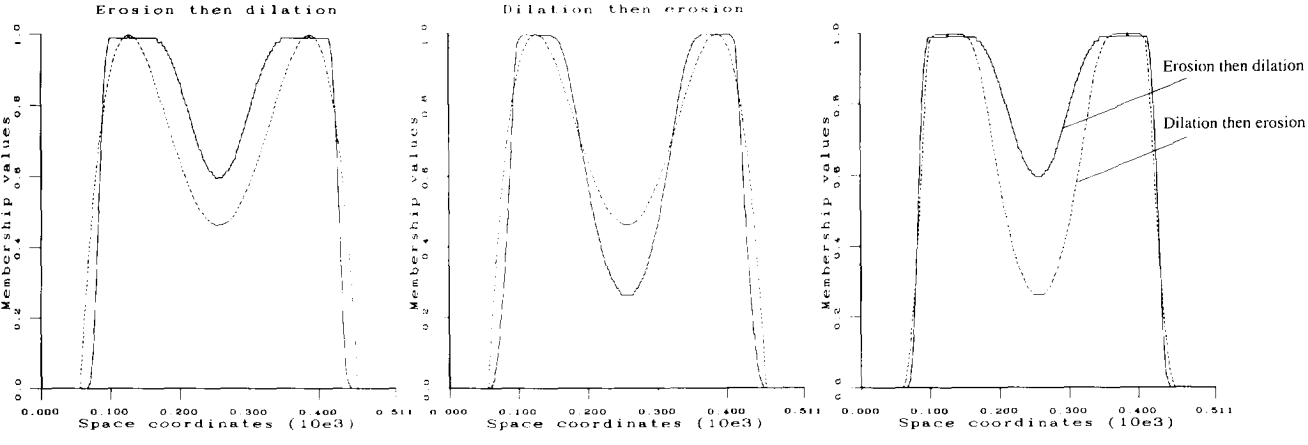

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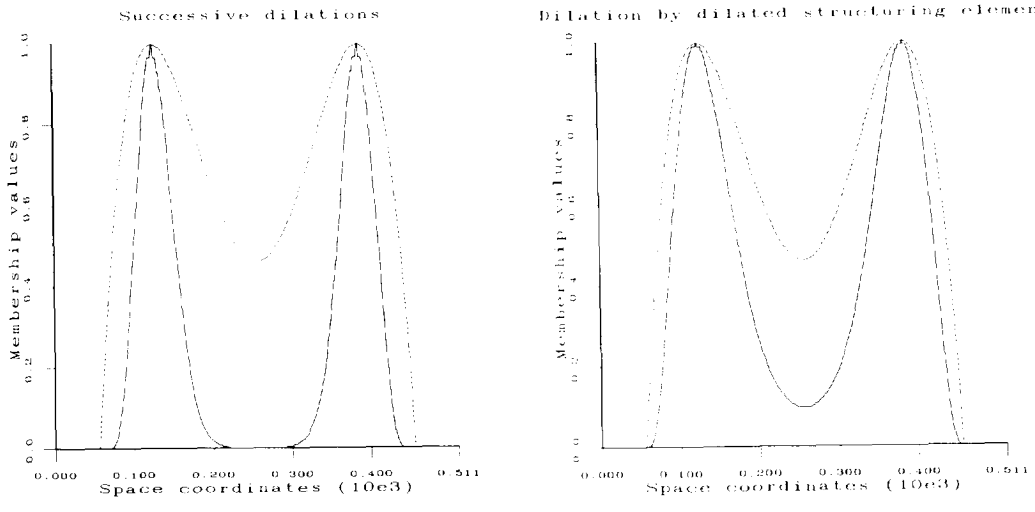

(f)

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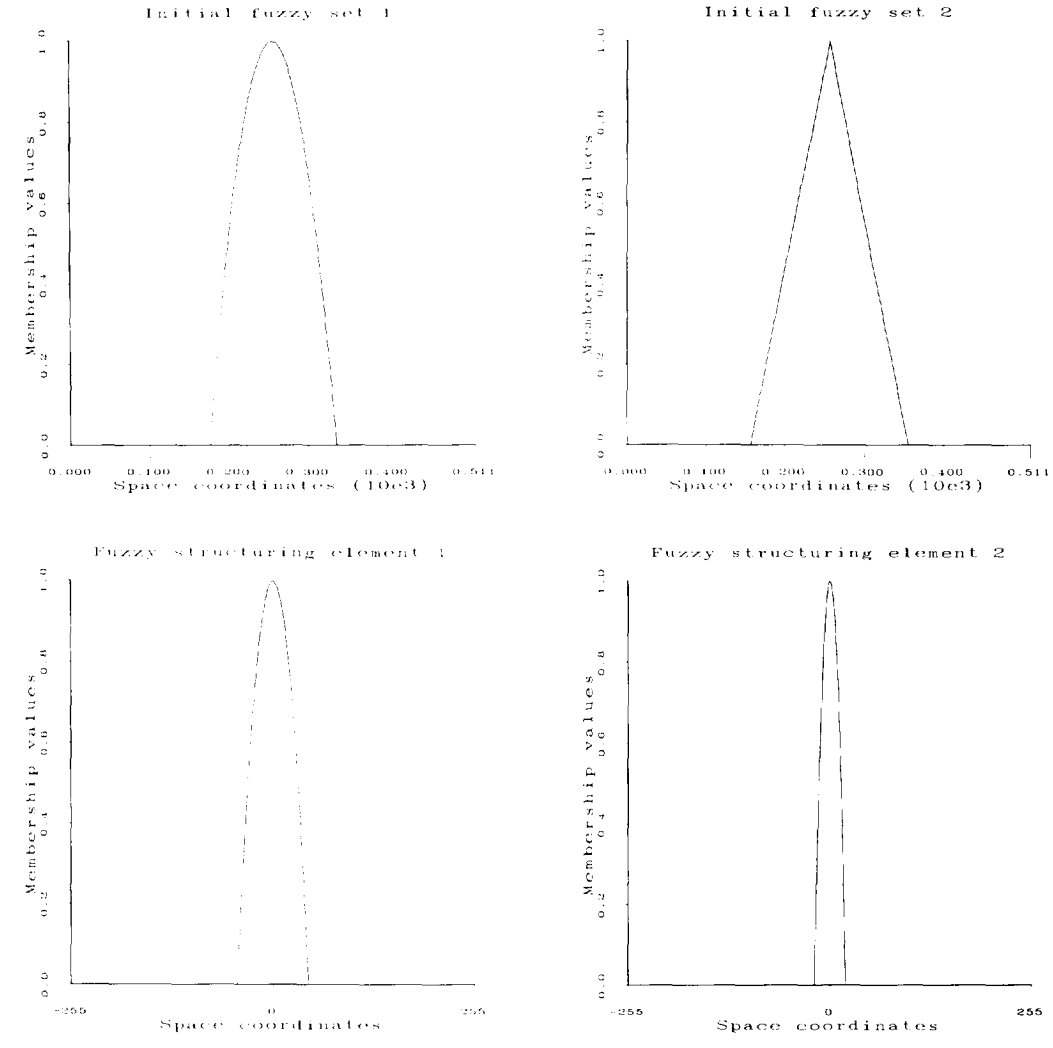


(a)

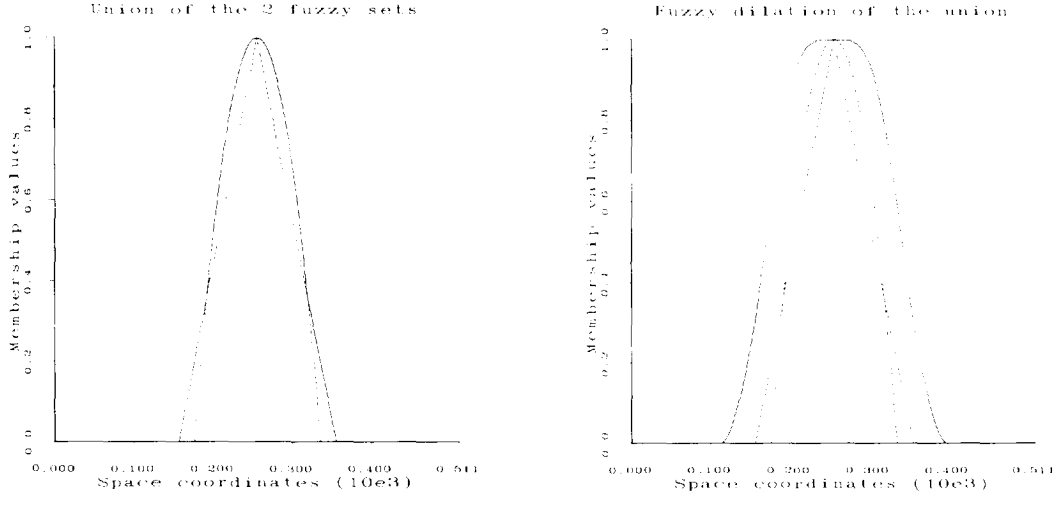
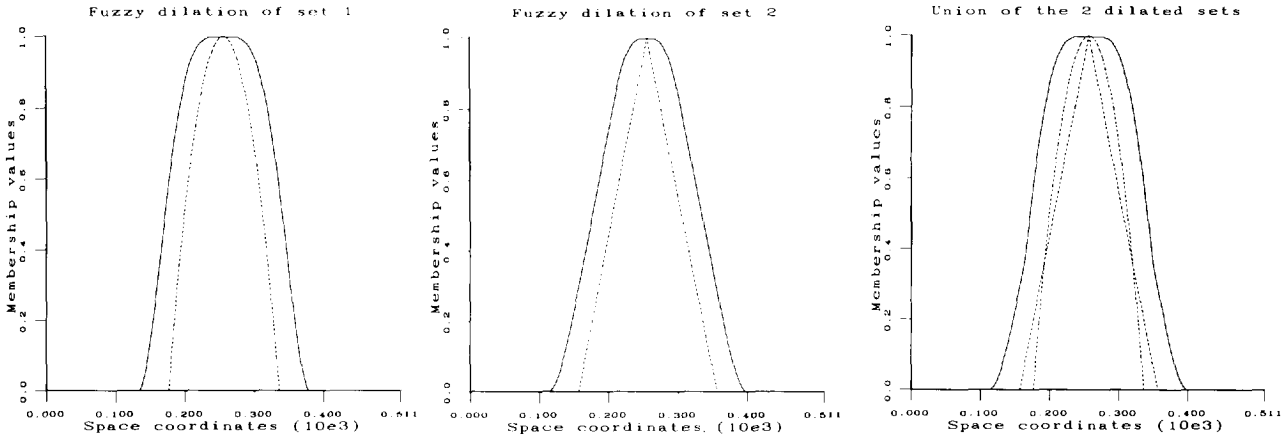
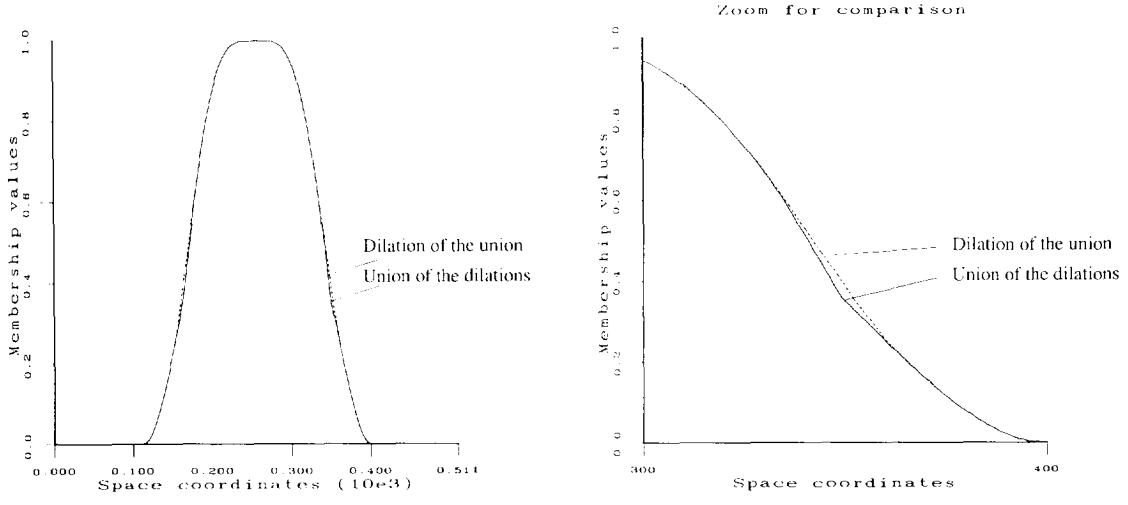


Fig. 9. Weak properties of definition 1: (a) initial fuzzy sets μ and μ' and fuzzy structuring elements v_1 and v_2; (b) $D_v(\mu \cup \mu')$ and $D_v(\mu) \cup D_v(\mu')$: P.11.1 (compatibility of dilation with union) is not satisfied; (c) $F_v(\mu)$ and $F_v[F_v(\mu)]$: this closing is not anti-extensive and not idempotent; (d) $O_v(\mu)$ and $O_v[O_v(\mu)]$: this opening is not anti-extensive and not idempotent; (e) $D_{v_2}[E_{v_1}(\mu)]$ and $E_{v_1}[D_{v_2}(\mu)]$: the combination equation is not satisfied; (f) $D_{v_2}[D_{v_1}(\mu)]$ and $D_{D_{v_2}(v_1)}(\mu)$: the iteration equation is not satisfied.





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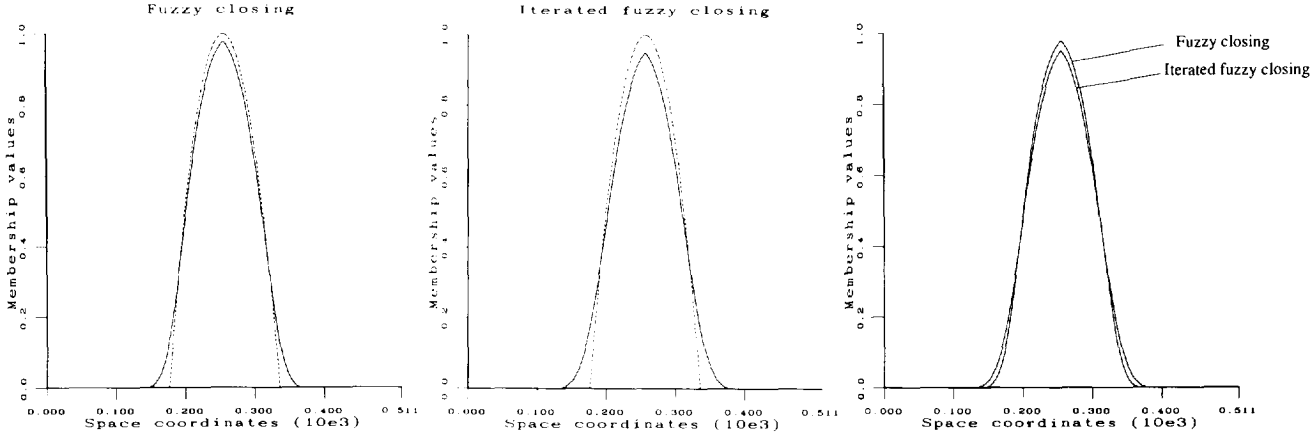

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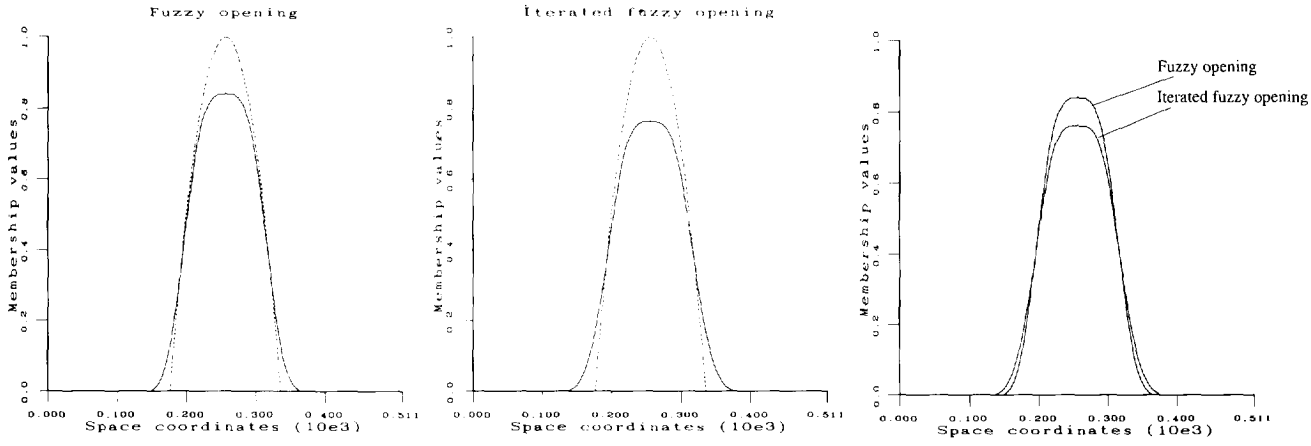

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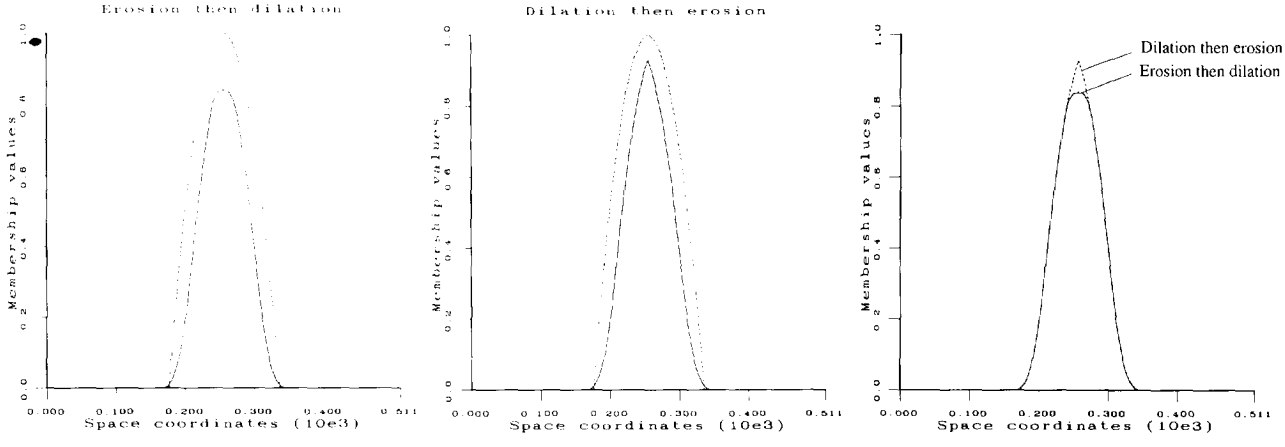

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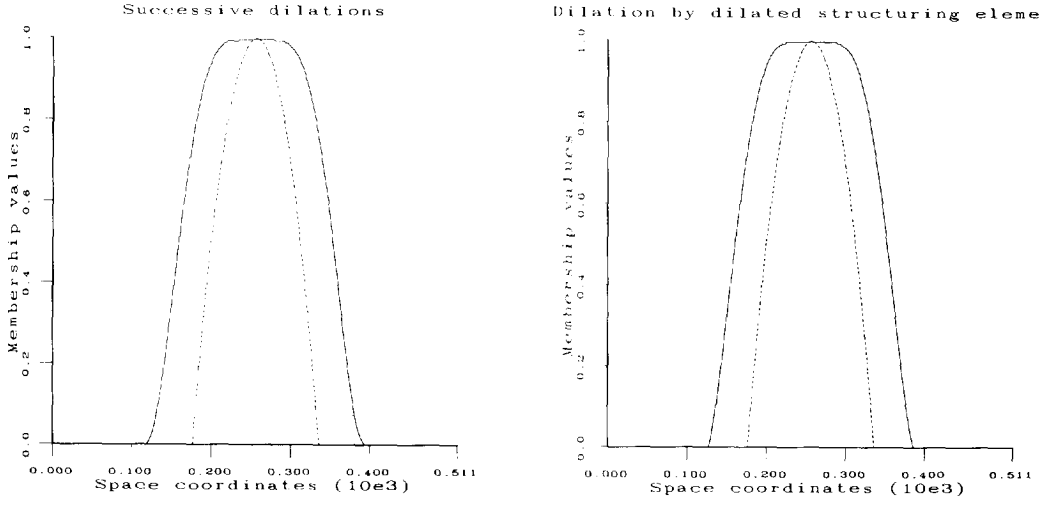

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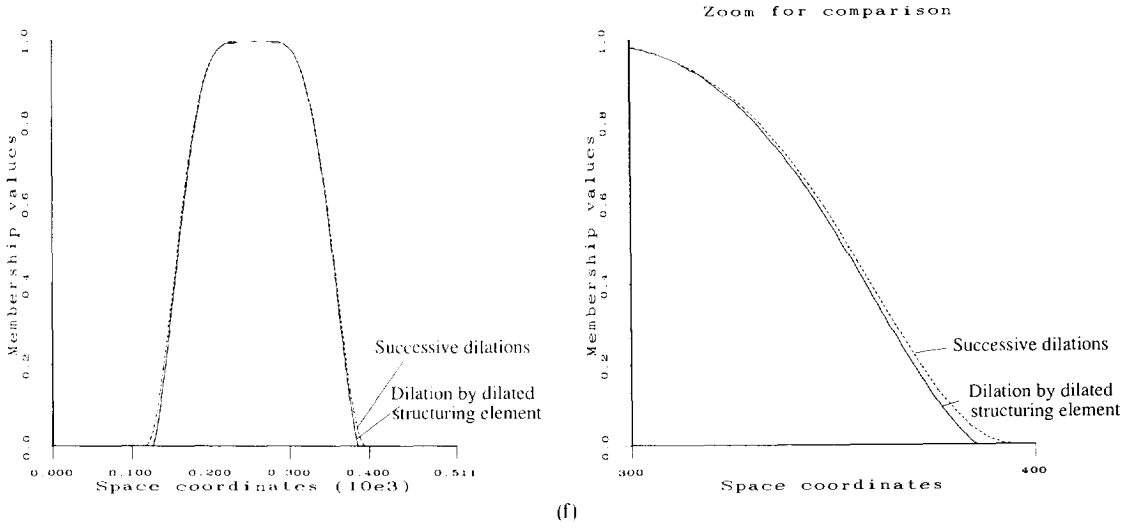
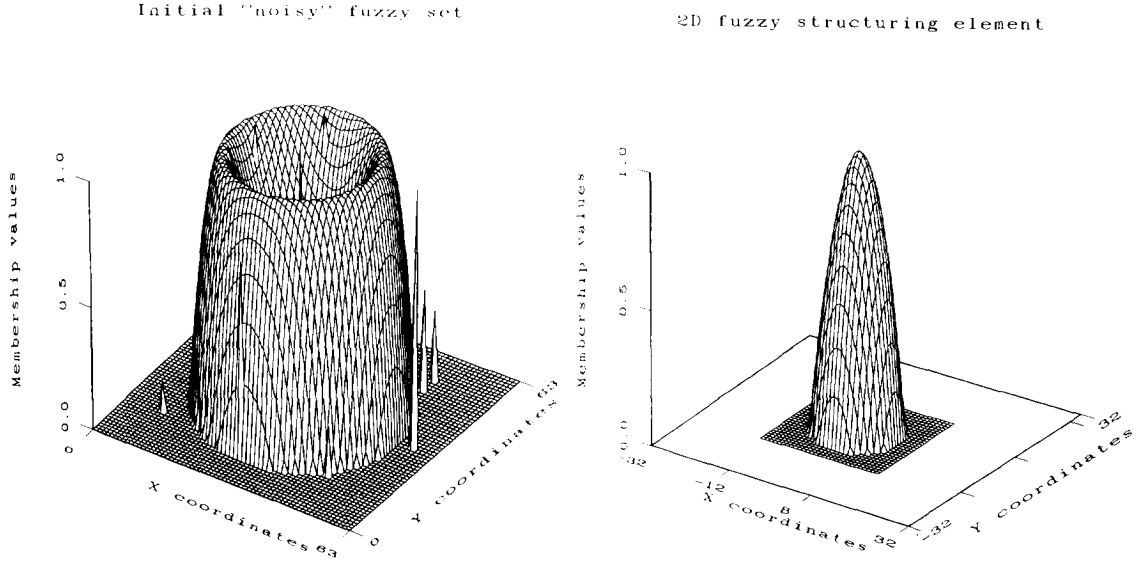




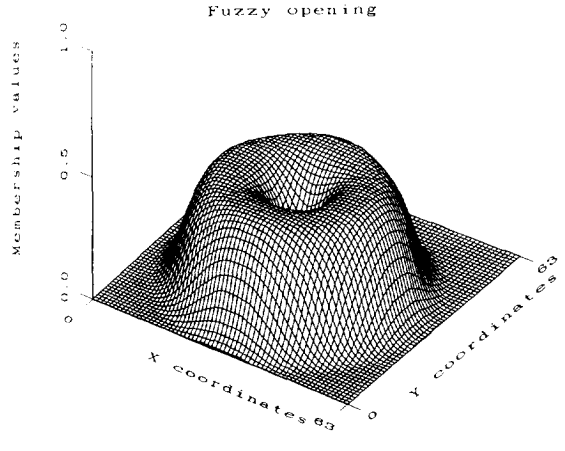


Fig. 10. (a) "Noisy" fuzzy set μ and structuring element ν; (b) $O_\nu(\mu)$ for definition 1: noise has been eliminated.

by dilation is not an algebraic opening for this definition but is useful for suppressing noise, as can be expected from any opening (see Fig. 10). Note also that the satisfied properties, although weak, are not weaker than those of definition 5 for $\lambda \neq \lambda_0$.

A last remark about this first definition concerns inclusion indicator: it can be defined from the erosion according to $I(\nu, \mu) = E1_\nu(\mu)(0)$, and thus as:

$$I(\nu, \mu) = \int_0^1 \inf_{y \in \nu_\alpha} \mu(y)\, d\alpha.$$

This inclusion indicator derived from the first definition satisfies A2, A3, A4, A5, A6 and A9, axiom A1 in a weaker form, and axioms A7 and A8 with inequalities, due to the weaker form of P.11.1 for definition 1.

6.1.4. *Further properties and operators.* As a general rule, the loss of a property for one of the definitions has consequences at different levels: at theoretical and algorithmical levels, and at a qualitative level for the results of fuzzy morphological transformations.

One theoretical consequence is the derivation of further properties. Let us consider, for example, the generalization of the Matheron's representation theorems (see e.g. Serra[34] and Heijmans[35]). It makes sense to consider the possibility of such a generalization for definitions having good algebraic properties for openings, that is, mainly for definition 4. This problem has been addressed by Sinha *et al.*[36] The authors consider spatially translation-invariant openings and show that an F-opening (i.e. an opening acting on membership functions) has a representation in terms of elementary H-openings (acting on general real functions), but not in terms of union of fuzzy openings. For the representation by erosions, they also obtain a weaker result than for classical mathematical morphology: it cannot be shown that any increasing spatially translation invariant operation can be represented by a union of fuzzy erosions.

Another theoretical consequence concerns filtering. A morphological filter is an increasing and idempotent mapping (see e.g. Serra[34] for a general theory of morphological filtering). An anti-extensive filter is called algebraic opening, and an extensive filter is called algebraic closing. Of course, morphological openings and closings derived as the compound of an erosion and a dilation are algebraic openings. In the fuzzy case, as morphological openings and closings have weaker properties than GMM ones, they are not morphological filters. Thus, all theoretical results related to morphological filtering theory are not valid in general, except for particular definitions. However, algebraic openings can be constructed in another way. For example, the iterative construction proposed in Serra[34] is also applicable in the fuzzy case: given an increasing mapping ψ, a decreasing series is constructed as $[\min(Id, \psi)]^n$, where Id denote the identical mapping. For any value of n, $[\min(Id, \psi)]^n$ is increasing and anti-extensive by construction. Let us now consider the case of a finite space. The decreasing series $[\min(Id, \psi)]^n$ admits a limit $[\min(Id, \psi)]^{n_0}$. As this limit is such that $[\min(Id, \psi)]^{n_0} = [\min(Id, \psi)][\min(Id, \psi)]^{n_0}$, it is idempotent and thus is an algebraic opening (and an anti-extensive filter). This construction can be generalized to the fuzzy case if we consider a finite space S and a finite number of possible values for the membership functions. So there exist algebraic openings for all definitions. This example shows that it is possible to derive fuzzy morphological filters having the same properties as in classical morphology, whatever the chosen basic definition.

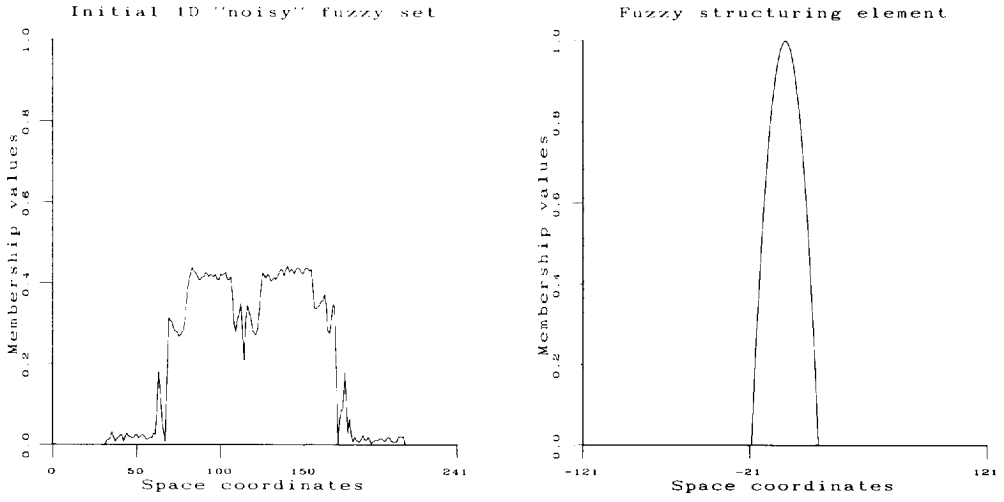
An example of fuzzy filtering is given by Dougherty[37] for an opening derived from definition 4. The comparison with a mean operator shows better results for the fuzzy opening. A deeper comparison should be made with classical mathematical morphology with binary and functional structuring elements. Similar results can be obtained with the other definitions, even if the algebraic properties are not satisfied (see Fig. 11). An advantage of fuzzy filtering is that the great variety of fuzzy morphologies and of possible fuzzy structuring elements provides more flexibility. Composed filters like alternate sequential filters[34] can also be built for ΦMM in a similar way as for classical morphology: only an increasing sequence of structuring elements is needed. Its effects to filter out pepper-and-salt-like noise are similar (see Fig. 12). Thus, the consequences of weaker properties are not as crucial at a qualitative level as at a theoretical one.

The loss of algebraic properties for fuzzy morphological operators has also algorithmical consequences. Let us consider, for example, the iteration relation. If it is satisfied, then it guarantees that it is equivalent to perform, for instance, a dilation by ν and then by ν' or directly a dilation by $D_\nu(\bar{\nu}')$. This equivalence may lead to more efficient algorithms. If it is not satisfied (like for definition 1 and definition 5 for $\lambda \neq \lambda_0$), an alternative way to implement the algorithm is lost and a potential improvement too. More generally, morphological algorithms are based on chaining operators. One advantage of mathematical morphology lies in the strong properties of the operators which assure, for example, that it is not worth performing the same opening twice, etc. For ΦMM, the design of an algorithm has to be made more carefully, as the properties not only depend on the chosen operation but also on the chosen morphology.

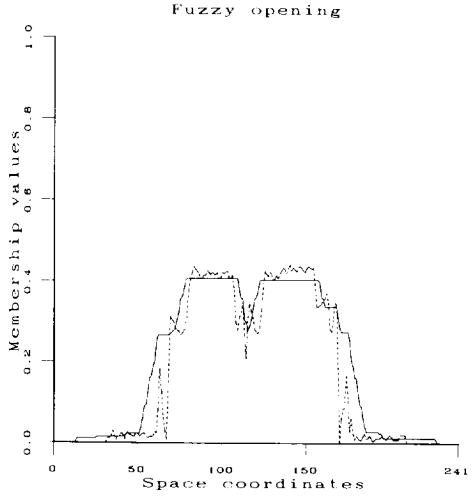
Examples of algorithms derived from the four basic operators are presented by Dougherty:[37] top-hat transform, hit or mass transform, morphological gradient. The approach presented in this report is mainly based on a fitting characterization of the morphological operators: erosion serves as a fuzzy marker and characterizes to which degree the structuring element fits in the shape. They give fitting characterization for erosion, dilation and opening.[36] For example, for definition 4, the following equation holds:

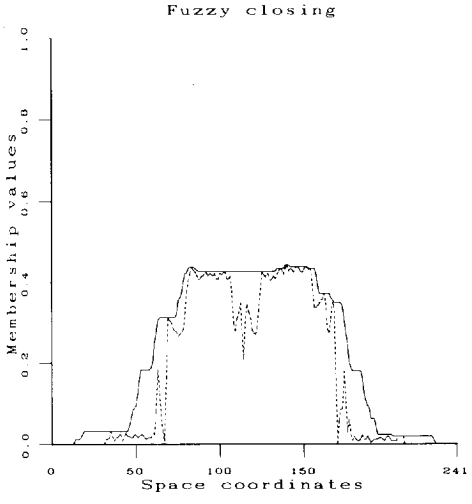
$$E4_\nu(\mu)(x) = \sup\{(1-\alpha) \in [0,1], \nu_x \diamond (-\alpha) \leq \mu\},$$

where $[\nu_x \diamond (-\alpha)](y) = \min[1, \max(0, \nu(y-x) - \alpha)]$. This equation is issued from the equivalence between



(a)





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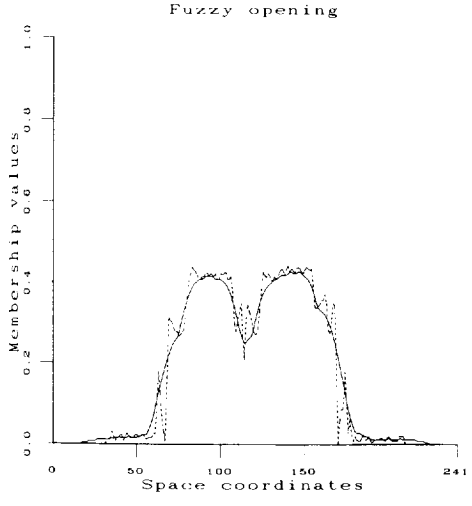
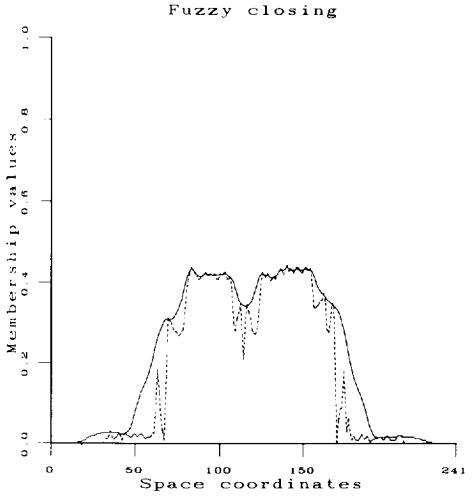
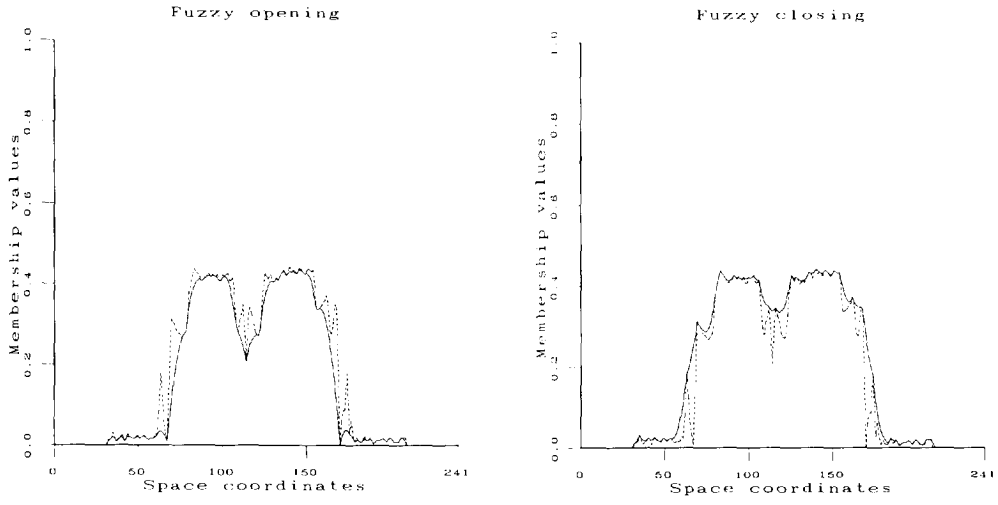




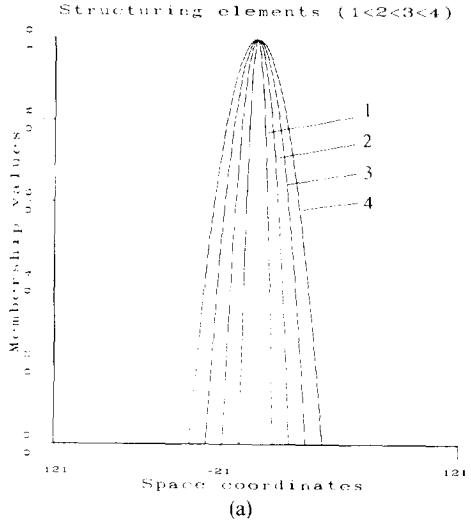
(c)

Fig. 11. Fuzzy filtering by opening and closing: (a) initial fuzzy set showing "pepper-and-salt like noise" and fuzzy structuring element; (b) opening and closing for definition 2; (c) opening and closing for definition 3; (d) opening and closing for definition 4.

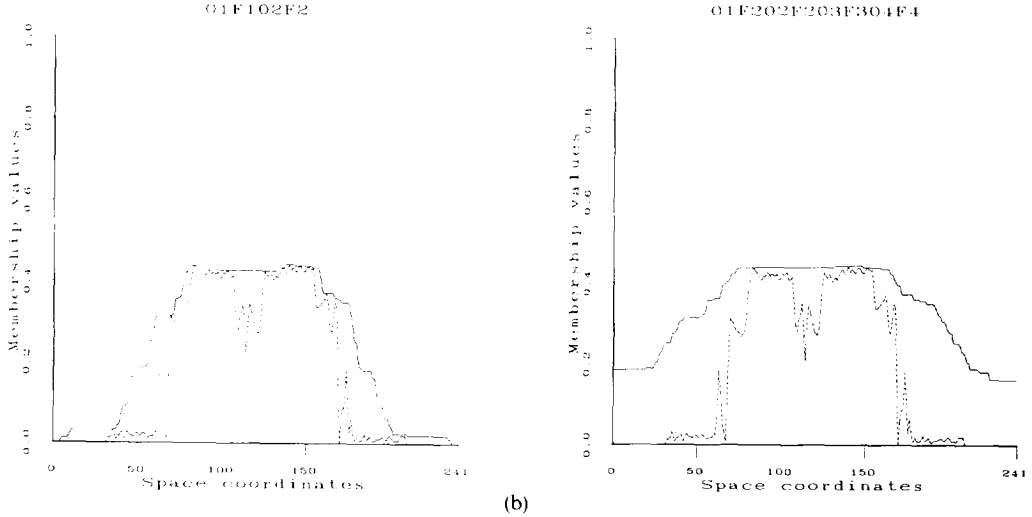


(d)

Fig. 11 (*Continued*)

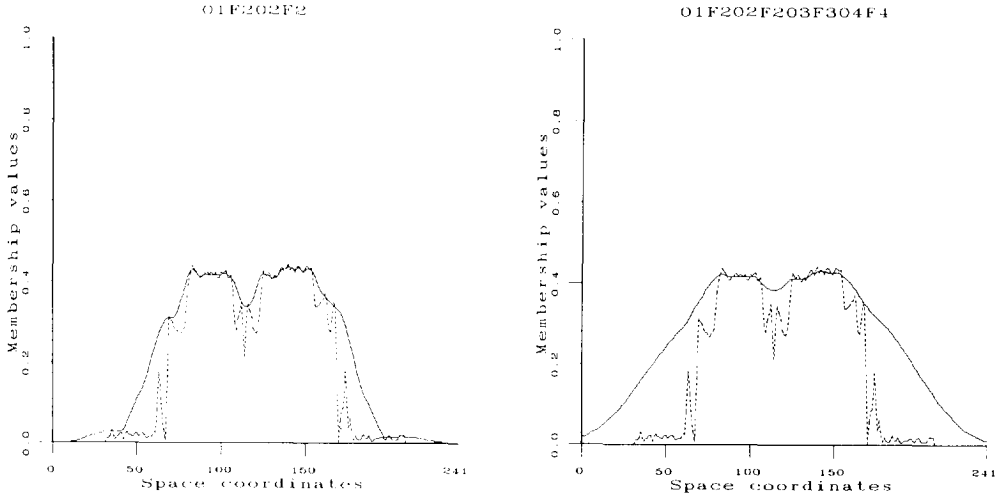


(a)

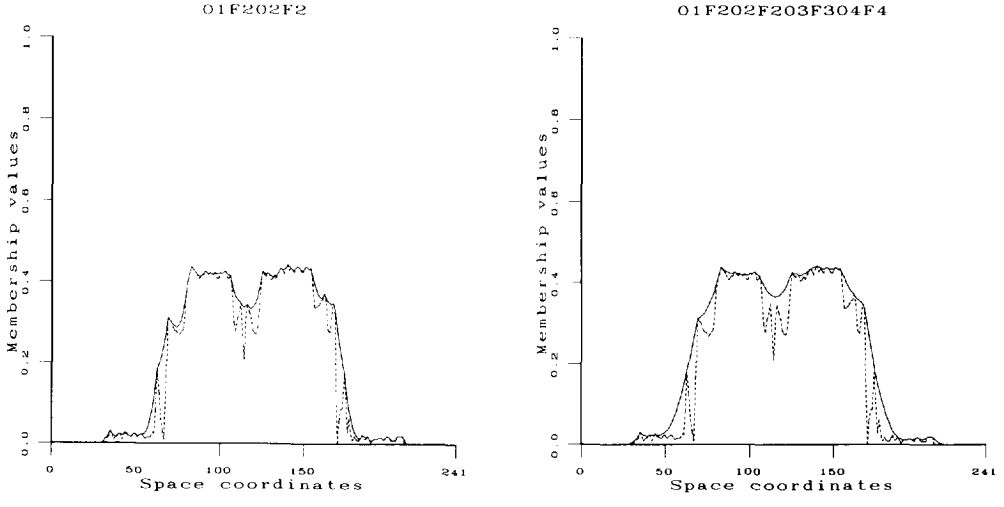


(b)

Fig. 12. Alternate sequential filter on the fuzzy set of Fig. 11: (a) increasing sequence of structuring elements; (b) result of alternate sequential filter for definition 2; (c) result of alternate sequential filter for definition 3; (d) result of alternate sequential filter for definition 4. Two iterations are shown: after opening and closing with structuring elements 1 and 2 (left) and after opening and closing with the four structuring elements (right).



(c)



(d)

Fig. 12 (*Continued*)

this definition and FMM shifted and truncated. The shift appears in the expression $(1-\alpha)$ [instead of $(-\alpha)$ in the classical fitting characterization] and the truncation in the truncated range translation $\diamond$ (instead of a simple range translation in the classical fitting characterization). As the other definitions (except the first one) rely on a generalization of union and intersection, and thus of inclusion, they contain implicitly such a characterization. However, the explicit formula may be more complicated. Going further in the interpretation, we note that, for the general definition 6, erosion is interpreted as fuzzy inclusion since we have $E_v(\mu)(x) = I(v_x, \mu)$, which represents the degree to which the fuzzy structuring element v translated at point x is included in the fuzzy set μ. Thus, the membership value to a fuzzy eroded set corresponds exactly to a degree of fitting. A dual interpretation holds for dilation, as a degree of intersection.

Other transformations can be built from the four basic ones. For example, conditional operations can be easily generalized. Let $T_v(\mu)$ be a fuzzy morphological transformation (a dilation for instance). We define the corresponding conditional operator (conditional dilation of a fuzzy marker μ' with respect to a fuzzy set μ) as:

$$CT_v(\mu, \mu') = i[T_v(\mu'), \mu],$$

where i represents a fuzzy intersection, i may be the standard Zadeh intersection (min) or any T-norm. It is not necessarily the same as that chosen for defining the ΦMM if definition 6 is used, and may thus have a different interpretation with respect to fuzzy set theory. An example of conditional dilation is shown in Fig. 13. Other possible generalizations are thinning and thickening, derived from the hit or miss transform. As for classical morphology, accurate choices of structuring

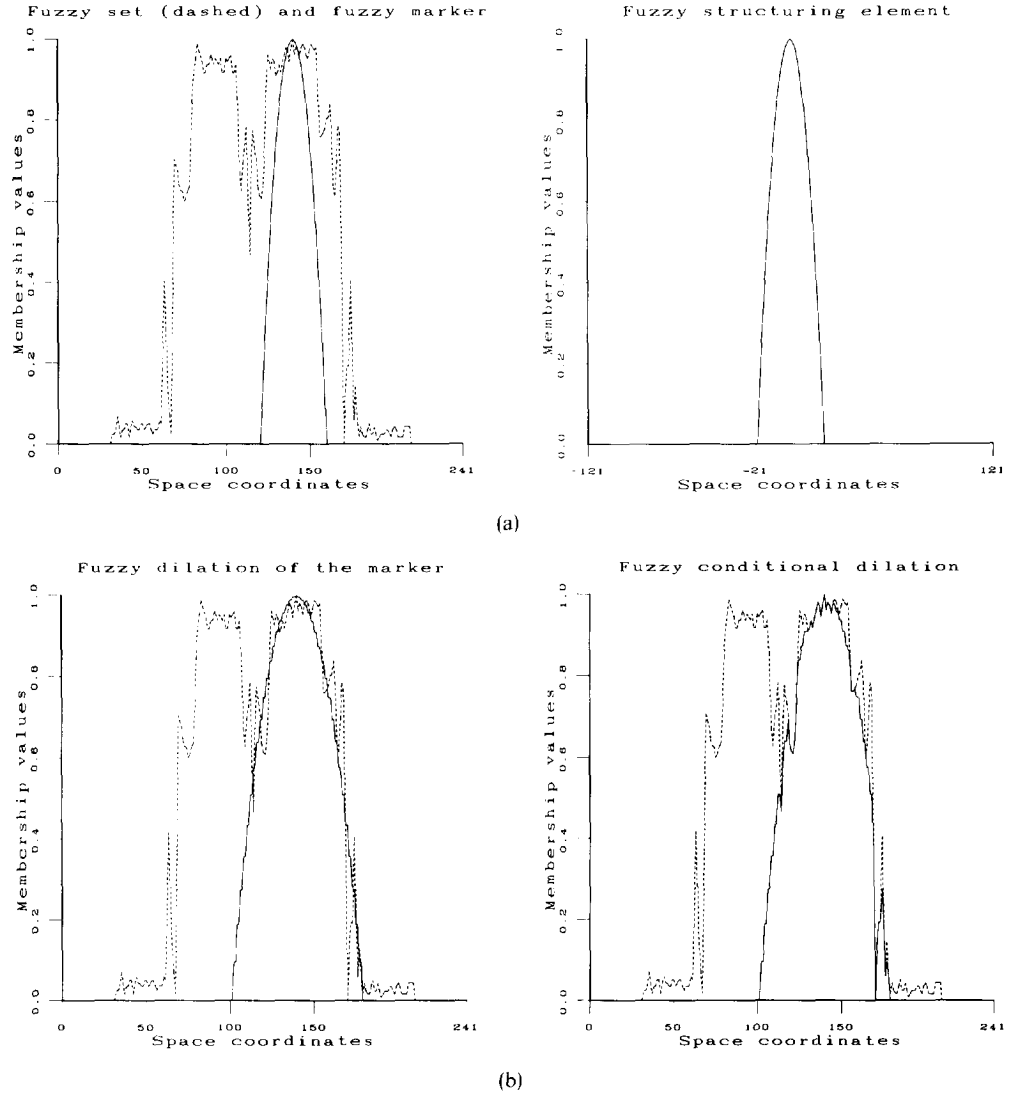


Fig. 13. Effect of conditional dilation: (a) fuzzy set μ, fuzzy marker μ' and fuzzy structuring element v; (b) fuzzy conditional dilation $i[D_v(\mu'), \mu]$ (with definition 2, and $i = \min$).

elements provide particular operations (examples for BMM are convex hull, skeleton, etc.). In the fuzzy case, a deeper study is needed to design interesting thinning and thickening.

That morphological-based pattern recognition may be of great interest for fuzzy sets has already been pointed out,[37] where a fuzzy set is considered as a noise version of a binary set, and fuzzy operations were shown to provide better results than a preliminary threshold to recover a binary set. However, we claim that this approach is too restrictive. Fuzziness cannot always be assimilated to noise. Imprecision and uncertainty may be inherent to the observed phenomenon (and not due to noise) and adequately represented by fuzzy sets. In reference (22) we presented an application in medical image processing where a fuzzy structuring element was used to represent the imprecision in the matching between two images. For pattern recognition purposes, imprecision and uncertainty can be taken into account by means of ΦMM. The fuzziness of the result may be used to define a degree of detection or recognition. We will discuss this aspect in Section 6.3.

6.1.5. *A general form for fuzzy mathematical morphology?* Let us now briefly discuss what could be the most general form for ΦMM satisfying the requirements stated in Section 3. In Sinha[28] the following form is proposed for the inclusion indicator:

$$I(\nu, \mu) = \theta[\psi[\lambda(\mu(x)), \phi(\nu(x))], x \in S],$$

with a series of conditions on the functions θ, ψ, λ and ϕ. As already mentioned above when discussing definition 5, the functions λ and ϕ act separately on the fuzzy set and the fuzzy structuring element, respectively. Thus, they are only modifications of μ and ν and do not provide any relation between μ and ν. More inte-

resting are θ and ψ, which actually perform the morphological operation between μ and ν. The function θ may be supremum, infimum, product, mean and the function ψ may be maximum, minimum, bounded sum, bounded difference. The conditions on θ, ψ (and λ, ϕ) are such that they are necessary and sufficient to guarantee that $I(\nu, \mu)$ satisfies the nine axioms A1–A9. Here again, no condition guarantees the other properties like P.7, P.8, P.12.

Let us consider definition 6, where the transformations are derived from a combination operator. Is the family of T-norms too restrictive or could a more general family also be used? Table 7 presents the properties of the T-norms and T-conorms (described in Appendix 2) involved in the demonstrations of the results given in Section 5 (morphological properties summarized in Table 6). This table shows that all properties of the T-norms are used at least once. So if we want all morphological properties, we cannot extend the family of combination operators and must remain in the T-norm family (for morphological operators expressed in the form of definition 6). However, if we accept to lose some properties, we can consider a larger family. For instance, T2 and T'2 (associativity) is used only to prove the iteration relation. If we do not need this relation for a particular application, we can consider non-associative operators. However, from a data fusion point of view, associativity is a key property which is satisfied for all commonly used combination operators, except for mean operators.[38] (However, mean operators have no unit element and thus cannot provide ΦMM operators which are compatible with BMM.) Weak T-norms and T-conorms do not satisfy T2 and T'2, nor T3 and T'3. If we accept to lose the properties involving T2 and T3 (extensivity of dilation, compatibility with GMM, iteration relation), weak T-norms may be used. However, the choice for a weak T-norm should be made more carefully if we want to have the other properties. In particular, extensivity and idempotence of closing requires a strong relation which is seldom satisfied.

More generally, we have proved that if the fuzzy dilation takes the following form:

$$D_\nu(\mu)(x) = \sup_{y \in S} f[\nu(y - x), \mu(y)],$$

then f is necessary a T-norm if we want the required properties. Some conditions must be added if we want

Table 7. Properties of i and u involved in the proofs of the morphological operators for definitions deriving from a (weak) T-norm and T-conorm (× means that no property, or no additional property, is required)

Properties of fuzzy mathematical morphology operators	Properties of i and u involved in the proofs	Additional necessary and sufficient properties
Duality	D, C1, C2	×
Compatibility with classical morphology if ν is crisp	T3, T'3, T0, T'0, T4, T'4	×
Compatibility with translations	×	×
Compatibility with homotheties	×	i and u compatible with homotheties
Local knowledge	T4, T0, T'4, T'0	×
Continuity	×	T5, T'5
Increasingness of dilation	T4	×
Decreasingness of erosion with respect to ν	T'4, C1	×
Extensivity of dilation iff $\nu(0) = 1$	T3	×
Anti-extensivity of erosion iff $\nu(0) = 1$	T'3, C0	×
Extensivity of closing anti-extensivity of opening	T4, T'4	$i[b, u(c(b), a)] \leq a$
Idempotence	T4, T'4	$i[b, u(c, (b), a)] \leq a$
Pseudo-commutativity of dilation	T1	×
Dilation and erosion of (or by) an intersection or an union	T4, T'4	×
Iteration	T4, T2, T'4, T'2	×
Combination	T4, T'4	$i[a, u(b, c)] \leq u[b, i(a, c)]$
Inclusion of the cuts of the dilation in the dilation of the cuts	T4, T3	×

compatibility with homotheties, extensivity and idempotence of closing and/or the combination relation (see Table 6).

Now let the fuzzy dilation be in the following form:

$$D_v(\mu)(x) = g\{f[v(y-x), \mu(y)], y \in S\},$$

where g is a function of eventually an infinity of variables, and let g_1 be the corresponding function of one variable only. Then we have the following result:

If g_1 is a continuous function, then g_1 is necessarily a bijection, $g_1 \bigcirc f$ must be a T-norm (again with some restriction if we want all properties), and $g = g_1 \bigcirc (\sup_{y \in S})$ if g_1 is increasing or $g = g_1 \bigcirc (\inf_{y \in S})$ if g_1 is decreasing. So this case is the same as the previous one up to a continuous bijection.

These results show that definition 6 is the most general for a dilation taking the above form if we want all properties. Proofs of these results can be found in reference (29). Note that the conditions on θ and Ψ obtained by Sinha and Dougherty[28] are weaker because they only require the nine axioms A1–A9 and not the other properties.

6.1.6. *Use of fuzzy mathematical morphology.* A last remark in this section concerns the use of ΦMM. When a particular definition has to be chosen, the choice is guided by the morphological properties we look for. If we need weak properties, the choice is large; if we need stronger properties (like idempotence for intance), then the choice is much more restricted. If definition 6 has been chosen, another guide comes from the requirements on the properties of i and u. This will be discussed in Subsection 6.3 as it concerns mainly fuzzy aspects related to combination operators. Then, as for classical morphology, particular operators and structuring elements have to be chosen. The problem is similar to the classical one and is solved by looking at the desired effects and the desired properties. In classical mathematical morphology, compact structuring elements are often used[34] as they lead to a better behaviour of the transformations and to more mathematical properties. We suggest to define a compact fuzzy set by means of one of the three following conditions (leading to different definitions):

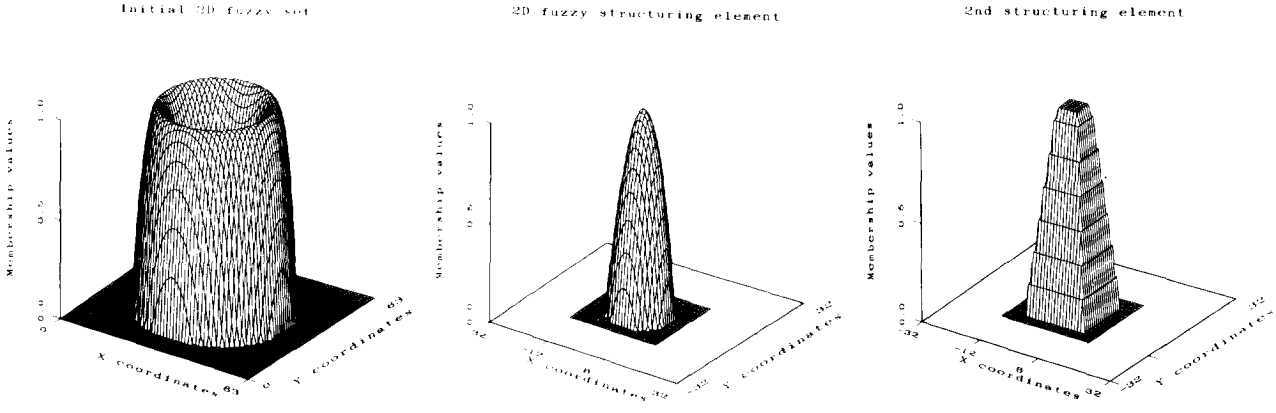
- a fuzzy set is compact iff its support is compact;
- a fuzzy set is compact iff all its α-cuts are compact (this definition is more restrictive);
- a fuzzy set is said to be compact to a degree β iff $\beta = \sup\{\alpha \in]0, 1], \mu_\alpha \text{ is } compact\}$.

From the previous lines, we see that there is a lot of elements to chose in order to design a ΦMM application. Fig. 14 shows the effect of the shape of the structuring element.

Unfortunately, no experience exists until now for real fuzzy applications similar to those we have for classical mathematical morphology. However, we hope that we can benefit from the many applications in mathematical morphology on one hand and in fuzzy set theory on the other hand to soon acquire this necessary experience.

6.2. *Comparison with respect to fuzzy sets*

6.2.1. *New operations on fuzzy sets.* As mentioned earlier, since the introduction of fuzzy sets, many fuzzy operations have been proposed. They mainly concern logical aspects ("and/or" operators, implications, entailment schemes, etc.) and set operators directly related to the logical ones. However, to process spatial information as is often done in picture processing, only some elementary topological and geometrical operators exist. In this context, ΦMM appears as a powerful theory as it provides a large class of operations, widely used in classical image processing and now also available for fuzzy image processing. Basically, from one operation (dilation, for example) four basic operations are constructed whose properties are well known, and



(a)

Fig. 14. Influence of the shape of the structuring element (definition 3 is used): (a) initial fuzzy set μ and two different shaped fuzzy structuring elements v_1 and v_2; (b) $D_{v_1}(\mu)$ and $D_{v_2}(\mu)$; (c) $E_{v_1}(\mu)$ and $E_{v_2}(\mu)$; (d) $O_{v_1}(\mu)$ and $O_{v_2}(\mu)$.

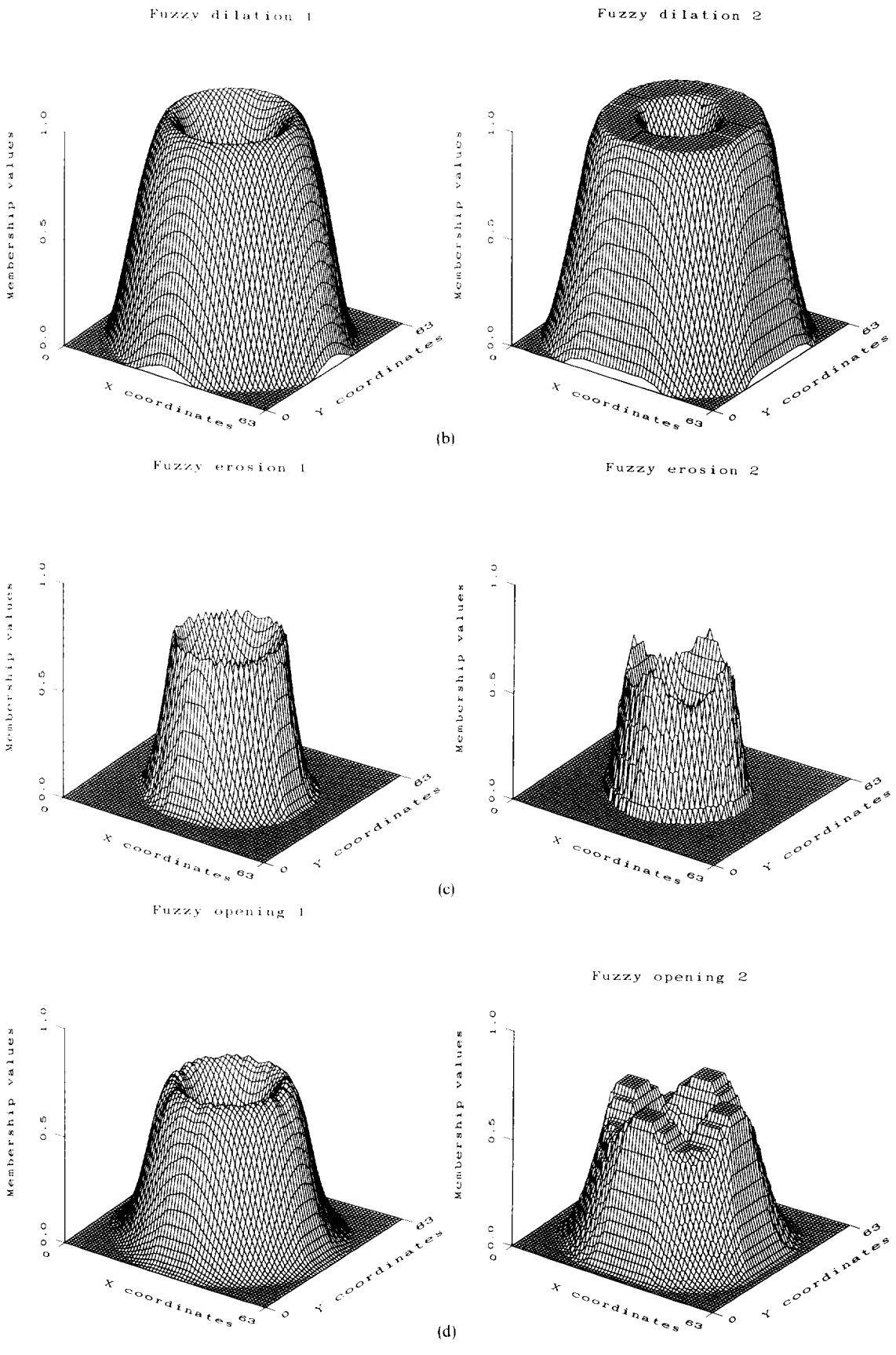
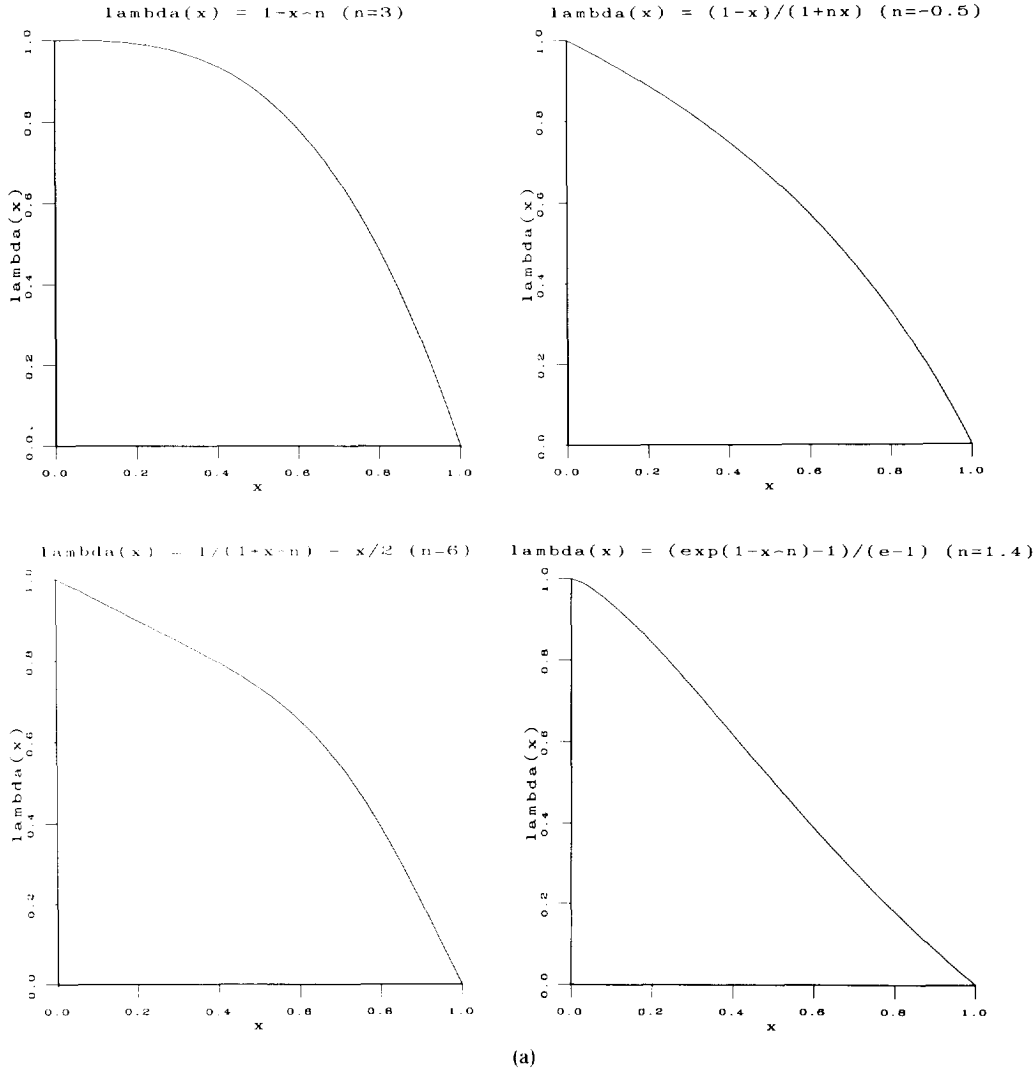


Fig. 14 (*Continued*)

then a complete set of morphological operators generalized for fuzzy sets is built. From this, the set of possible operators for manipulating fuzzy sets is greatly extended. Moreover, ΦMM provides operations whose effects are spatially controlled, for instance, dilation allows us to propagate fuzziness to a extent defined by the structuring element.

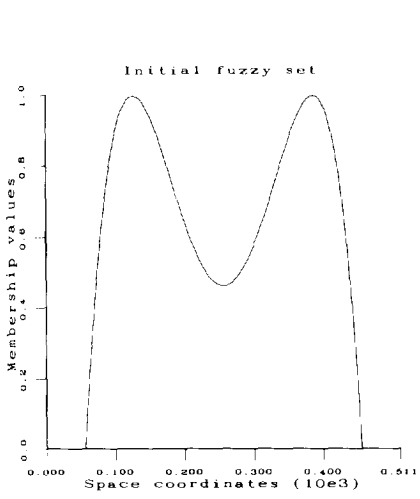
6.2.2. *Interpretation of the λ function in definition 5 in terms of fuzziness.* Another effect on the fuzziness is provided by the function λ introduced in definition 5. This function acts on the imprecision or uncertainty representation by modifying μ and ν. Definition 5 applied to a fuzzy set μ and a fuzzy structuring element ν is equivalent to applying definition 4 to $1 - \lambda(\mu)$ and $1 - \lambda(\nu)$, providing a ΦMM which is close to FMM (see Section 6.1). Examples of the effect of λ on a fuzzy set are shown in Fig. 15. For instance, the function $\lambda(x) = 1 - x^n$ decreases the fuzzy character of μ and makes the set crisper when n increases. As λ does not represent any relationship between μ and ν, it does not really help in designing ΦMM operators. Thus, we propose that it be introduced in the modelling phase where the fuzzy sets and structuring elements are defined from the problem to be solved. In this stage, from the measures made on the image, we have to derive μ and ν membership functions. As the fuzzy set and the fuzzy structuring element derive from different deductions, there is theoretically no reason that they make use of a similar λ function.

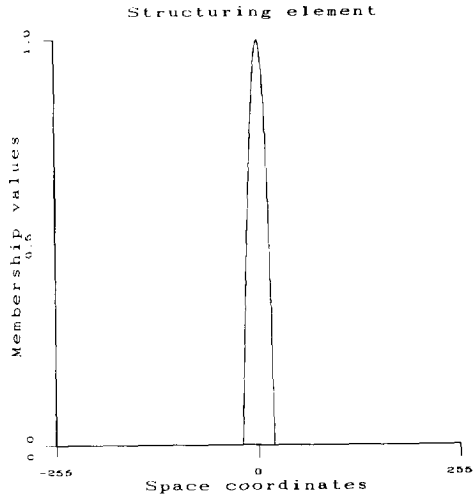
6.2.3. *Functions vs fuzzy sets.* More generally, one of the main contributions of ΦMM with respect to FMM relies in the modelling of fuzziness. Membership functions are functions into [0, 1] which cannot be considered as any function, since the underlying interpretation plays an important role: 0 and 1 are particular values representing certainty about the presence or the

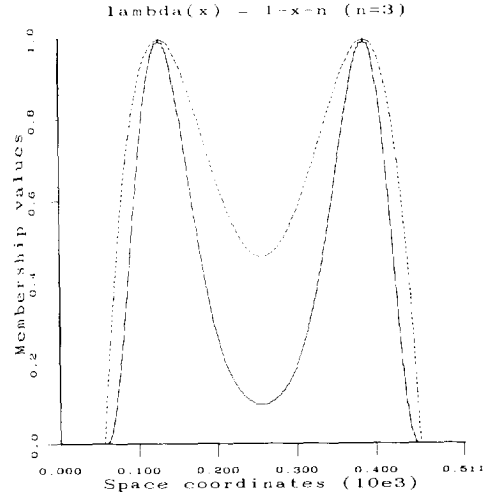


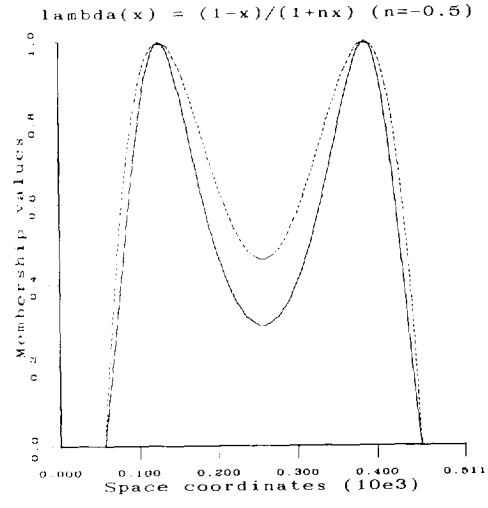
(a)

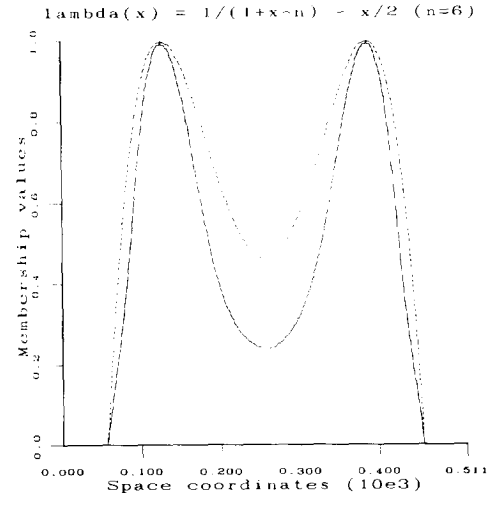
Fig. 15. Influence of the choice of the λ function: (a) different λ functions; (b) $1 - \lambda(\mu)$ for the above functions; (c) corresponding fuzzy dilation for definition 5.

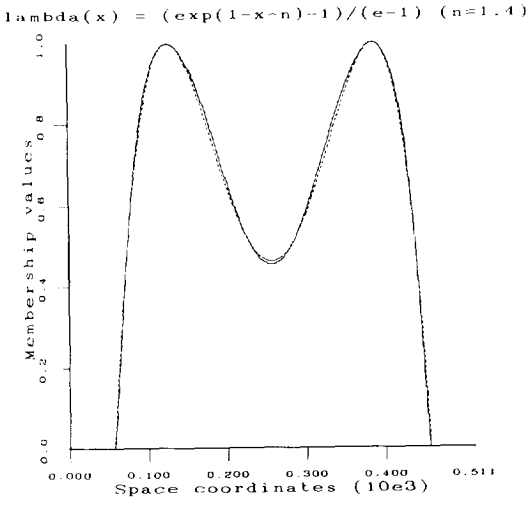






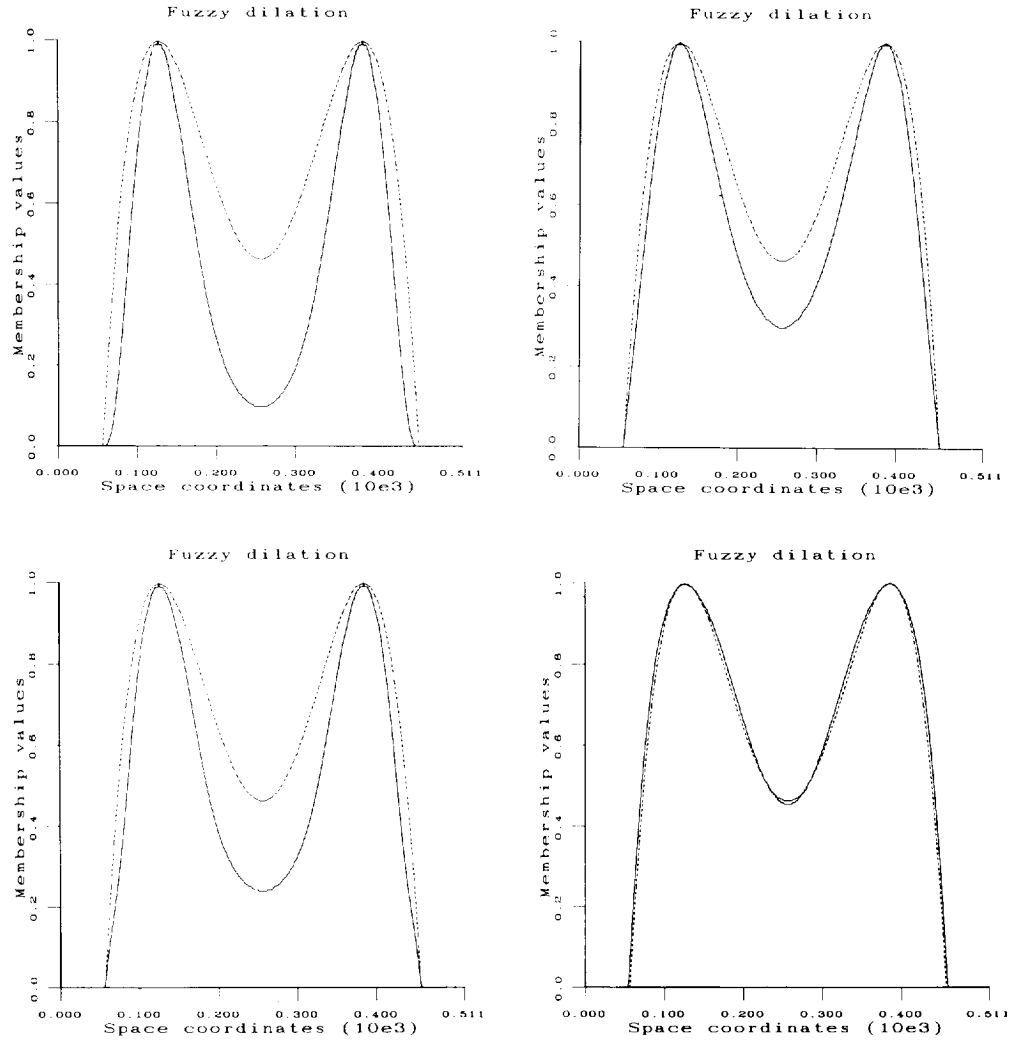






(b)

Fig. 15 (*Continued*)



(c)

Fig. 15 (*Continued*)

absence of information. The value 0.5 also often plays a privileged role, as a decision thresholding for instance. As said before, the problem when using FMM directly on membership functions is that the result is generally not a membership function, and we have seen that shifting and truncating the result does not lead to a satisfactory result. This could suggest proceeding in another way. We may transform any function with values in $[0, 1]$ in a function with values in $[0, +\infty]$ (or $[-\infty, +\infty]$) by a bijective increasing mapping f such that $f(0) = 0$ [or $f(0) = -\infty$] and $f(1) = +\infty$. Transforming a fuzzy set μ and a structuring element v according to this mapping, it is now possible to dilate them using classical mathematical morphology. The resulting function, taking values in $[0, +\infty]$ (or in $[-\infty, +\infty]$), is then transformed by the inverse mapping f^{-1} in order to recover a membership function. Thus, a fuzzy dilation can be defined as:

$$D_v(\mu)(x) = f^{-1}\left[\sup_{y \in S} [f(\mu(y)) + f(v(y-x))]\right].$$

The main drawback of this approach is to distort the fuzziness scale in a way which makes the interpretation more difficult. F instance, value 1, expressing certainty, is no longer accessible since it goes to the numerically inexistent infinite value. This problem would also exist for the value 0 if we take f in $[+\infty, -\infty]$.

Other drawbacks appear at the property level. Let us assume that there exists, z in S such that $v(z) = 1$. Then we have:

$$\forall x \in S, \quad \exists y \in S, \quad f(\mu(y)) + f(v(y-x)) = +\infty,$$

and thus:

$$\forall x \in S, \quad D_v(\mu)(x) = 1.$$

This means that the dilation is completely saturated, and does not fit the intuitive idea of a dilation which respects the shape and size of the structuring element. Moreover, this shows that compatibility with BMM and GMM cannot be achieved using this definition.

Note that if we choose f taking values in $[0, +\infty]$, then the form:

$$f^{-1}[f(a)+f(b)],$$

which is involved in the dilation, corresponds to the general form of a strictly monotonous archimedian T-conorm.[39] Thus, the dilation is obtained as a "sup" of a T-conorm instead of the "sup" of a T-norm as in definition 6, and this explains the previous behaviour, as well as the loss of properties related to the interpretation of fuzzy dilation as a degree of intersection between the shape and the fuzzy structuring element.

Note that an analogous form has been proposed by Giradina[7] with the T-conorm $\min(1, x+y)$, which leads to the same drawback.

Let us now consider fuzzy erosion. It can be built in two ways: either from dilation by duality, or following the same construction scheme as for dilation. In the first case, we obtain directly:

$$E_v(\mu)(x) = \inf_{y \in S}[1 - f^{-1}[f(1-\mu(y)) + f(v(y-x))]].$$

In the second case, we first transform the fuzzy sets μ and v according to a function ϕ from $[0,1]$ to $[-\infty, +\infty]$, use classical FMM and then apply ϕ^{-1}. This leads to:

$$E_v(\mu)(x) = \inf_{y \in S} \phi^{-1}[\phi(\mu(y)) - \phi(v(y-x))].$$

However, there is no reason to transform fuzzy sets differently depending on the operation we would like to apply. Thus, a reasonable choice is $\phi = f$. The two ways for defining fuzzy erosion are then equivalent iff:

$$\forall (a,b) \in [0,1]^2, \quad f^{-1}[f(1-a)+f(b)] + f^{-1}[f(a)-f(b)] = 1,$$

which is obviously not satisfied for any function f from $[0,1]$ to $[-\infty, +\infty]$. Thus, this is an additional drawback of this approach.

6.2.4. *Excluded middle and non-contradiction.* One important feature of fuzzy sets is their behaviour with respect to the rules of excluded middle and non-contradiction. For binary sets, the rule of excluded middle always holds and is expressed, for a subset A of a space S, as:

$$A \cup A^C = S.$$

By duality, the rule of non-contradiction for crisp sets holds and is expressed as:

$$A \cap A^C = \varnothing.$$

For fuzzy sets, these rules may be or may not be satisfied depending on the fuzzy union and intersection used. For a T-norm i and the corresponding T-conorm u with respect to a complementation c, the two rules are expressed equivalently as:

$$\begin{aligned} & i[x, c(x)] = 0, \quad \text{(non-contradiction)} \\ \Leftrightarrow\ & u[x, c(x)] = 1 \quad \text{(excluded-middle)}. \end{aligned}$$

Let us consider the T-norms given in Table 5, "min" and product do not satisfy the rule of non-contradiction. The T-norm $\max(0, x+y-1)$ and the weak T-norm $\max(0, 1-\lambda(x)-\lambda(y))$ satisfy this rule (as for any function λ the equation $\lambda(x)+\lambda(1-x) \geq 1$ holds, see definition 5, part 2). Let us see now how these two rules are related to ΦMM. Let us consider the necessary and sufficient condition to be satisfied by i and u for the extensivity and idempotence properties (needed to have algebraic openings and closings):

$$i[b, u[c(b), a]] \leq a.$$

By setting $a = 0$, the rule of non-contradiction is deduced. We conclude that non-contradiction for i and u is a necessary condition to ensure extensivity and idempotence of openings and closings, and it reduces drastically the set of eligible T-norms. Unfortunately, non-contradiction is not a sufficient condition: for instance, a weak T-norm derived from a λ function for $\lambda \neq \lambda_0$ does not lead to algebraic closings and openings, although it satisfies the non-contradiction rule.

Let us now consider the condition for the combination relation:

$$i[a, u(b,c)] \leq u[b, u(a,c)].$$

This condition is not as strong as the one for extensivity and idempotence and is satisfied for a larger family of T-norms and T-conorms. This can be easily shown by the following result:[29]

For any T-norm i and associated T-conorm u with respect to a complementation c such that the condition for extensivity and idempotence $i[b, u[c(b), a]] \leq a$ is satisfied, we have:

- i and u satisfy the rules of excluded middle and of non-contradiction;
- i and u satisfy the condition for the combination relation, i.e. $i[a, u(b,c)] \leq u[b, u(a,c)]$.

Conversely, for any T-norm i and T-conorm u such that they satisfy the condition for the combination relation $i[a, u(b,c)] \leq u[b, u(a,c)]$ and the rules of non-contradiction and excluded middle, then i and u satisfy the condition for extensivity and idempotence, i.e. $i[b, u[c(b), a]] \leq a$. Note that the proof of this result involves properties T'3 (unit element) and T'2 (associativity), discarding, by the way, weak T-norms and T-conorms.

6.2.5. *Refinement and weak inclusion.* A fuzzy set v is said to be a refinement of another fuzzy set μ or an enhanced version of μ if and only if:[40]

$$\forall x \in S, \begin{cases} \mu(x) \leq \frac{1}{2} \Rightarrow v(x) \leq \mu(x) \\ \mu(x) \geq \frac{1}{2} \Rightarrow v(x) \geq \mu(x). \end{cases}$$

It can be shown[29] that if v is an enhanced version of μ, then we have:

$$\forall x \in S, \quad E_{v \cup c(v)}(v \cap c(v))(x) \leq E_{\mu \cup c(\mu)}(\mu \cap c(\mu))(x),$$

for the fuzzy erosion defined by definition 6, $\cap$ and $\cup$ being fuzzy intersection and union defined by any

T-norm and T-conorm (not necessarily the same as those used for the erosion definition), and c being a fuzzy complementation. The interest of this equation from a fuzzy set point of view relies in the interpretation of $E_{\nu \cup c(\nu)}(\nu \cap c(\nu))(0) = I[\nu \cup c(\nu), \nu \cap c(\nu)]$ in terms of fuzzy entropy of the fuzzy set ν.[28,31] Note that this definition of entropy not always satisfies the requirements for a fuzzy entropy as stated by de Luca *et al.*,[40] depending on the choice of the T-norm.

A fuzzy set ν is said to be weakly included in a fuzzy set μ iff:[28]

$$\forall x \in S, \quad \nu(x) \leq \tfrac{1}{2} \quad \text{or} \quad \mu(x) > \tfrac{1}{2}.$$

If ν is weakly included in μ, then $E_\nu(\mu)(0) = I(\nu, \mu) \geq \frac{1}{2}$. In the same way, if ν_x is weakly included in μ, then $E_\nu(\mu)(x) \geq \frac{1}{2}$. If it holds for any x, then $E_\nu(\mu) \geq \frac{1}{2}$. These equations may also be interpreted in a context of fuzzy measures.[28]

6.3. *Comparison with respect to decision theory and data fusion*

Our aim in constructing fuzzy morphologies was originally to provide tools for data fusion and decision making in a fuzzy set framework. From this point of view, definition 6 has many advantages, which will be explained now. They rely on two main properties of T-norms and T-conorms: the first is their interpretation in terms of fuzzy intersection and union (directly used in Section 4); the second is their interpretation in terms of combination operators, commonly exploited for fusing information represented by fuzzy sets and for decision making.

6.3.1. *Interpretation of T-norms and T-conorms.* T-norms are originated in stochastic geometry: a statistical metric space[41] is defined as a set S such that a function $\Pi(x; p, q)$ is associated with any points p and q of S. This function may be interpreted as the probability that the distance $d(p, q)$ be less than or equal to x. One of the conditions imposed to $\Pi(x; p, q)$ is expressed as:

$$T[\Pi(x; p, q), \quad \Pi(y; q, r)] \leq \Pi(x + y; p, r),$$

and extends the classical triangular inequality on distances.* The study of statistical metric spaces, carried out e.g. in Schweizer and Sklar,[42,43] leads to the characterization of associative functions on the unit square, by using results on functional equations[44] and transformations from semi-groups into new semi-groups.

Let us now try to interpret the various elements involved in T-norm theory with respect to the processing of uncertain, imprecise or ambiguous information, and particular for data fusion and decision theory.

At first, the *elements of* $[0, 1]$ can be interpreted as measures derived from data given by one or several sensors, for instance, a measure of membership to a class, a measure of evidence of presence, a measure of satisfaction of a criterion. This measure can be quantity of information, a probability, a membership degree, a mass function, a plausibility function, etc., depending on the theoretical framework considered. Additional constraints can be imposed on these measures according to this framework, but here we will consider membership degrees to fuzzy sets without any additional particular constraint. The value 0 and 1 play particular roles. The value 0 means that, for an event, the sensor provides a null measure, either because it considers the event as impossible, or because it has no information or a complete ignorance about the event. On the contrary, the value 1 means that the sensor considers the information, or the event as sure and thus represents a total certainty. Values which are strictly between 0 and 1 represent degrees of partial knowledge on the information. They can also be interpreted as imprecision, or as a quantity of information available about the event.

In this context, T-norms and T-conorms appear as *operators for combining information, or for aggregating criteria*, represented by measures in $[0, 1]$. A T-norm i is necessarily less than the "min" and thus, as a conjunction operator, represents a consensus between information, or its common or redundant part. It reduces the less certain information and has at most confidence in the sensor which gives the smallest measure. It searches for a simultaneous satisfaction of criteria or objectives. On the contrary, a T-conorm u, necessarily greater than the "max", increases the certainty we have about an information and has at least confidence in the sensor which gives the greatest or the most certain measure, or the most information. It is disjunction operation, which expresses redundancy between criteria. Note that these two operators rule out any compromise operator (mean operator), i.e comprised between "min" and "max", for which the global measure is intermediate between the partial measures provided by each sensor.[39]

The *commutativity* (T1, T'1) and *associativity* (T2, T'2) properties express that the result of the combination is independent of the order in which the information is combined. These properties are commonly satisfied by information combination operators (for instance, the Dempster's orthogonal rule of combination verifies these properties). These properties are even often imposed as axioms governing the construction of operators, as they are commonly recognized as minimal properties the operators should satisfy, although human reasoning not always combines information in a commutative and associative way.

The *existence of a unit element* (T3, T'3) expresses that if a sensor provides this unit value, it will not change the result of the combination and so will have no influence on the final decision. For T-norms, the unit element is 1. The combination of such an information with any other one well matches the idea of a consensus between a certainty about an event and another measure of this event. For T-conorms, the

* The name triangular norm assigned to T is due to the role played by T in this inequality.

unit element is 0, which corresponds to a complete ignorance of a sensor, or the absence of information and thus has no influence on a disjunction operator.

The *increasingness property* (T4, T'4) corresponds to a constraint generally imposed on the operators: if two sensors given information or measures x' and y' greater than x and y, respectively, we expect from the combination of x' and y' a result that is also greater than the result obtained from x and y (representing more information, or more certainty).

The *limit conditions* (T0, T'0) govern the behaviour of the combination of measures in $\{0, 1\}$, and impose it to be compatible with the binary case. Thus, their interpretation is the same as for classical logic, where the reasoning deals only with values "true" and "false".

The *continuity property* (T5, T'5) assures the robustness of the information combination. If a sensor provides an information or measure x' slightly different from x, the combination of x' with any other value should not be very different from that obtained with x. This property is not always imposed (for instance, i_0 and u_0 are not continuous). It is, for example, possible to impose that some values completely determine the result and that small changes in these values drastically change the result and the derived decision.

Complementation generalizes the negation of propositions ($\neg$) of classical logic. It modelizes the notion of "contrary" of an information or a measure. The property C0 expresses a compatibility condition with classical negation ($\neg$ true = false, $\neg$ false = true) and thus has the same interpretation. The decreasingness property (C1) expresses that two measures are ordered in the reverse sense with respect to their contrary. If we have more cetainty or more information about an event, we have less about its contrary. The involutive characteristic of the complementation (C2) fits the common sense. The most used complementation is $c(x) = 1 - x$. Others have been proposed (see e.g. Dubois[38] and Yager[45]).

Duality property (D) between a T-norm and a T-conorm expresses the equivalence between conjunction of information and disjunction of its contrary.

The *existence of a null element* for an operator means that this value completely determines the result of the combination. It is enough that a sensor provides this value for the result of any combination being this value. For T-norms, the null element is 0, which is consistent with the idea that a consensus cannot provide any information from a set of measures with 0 being among them. For T-conorms, the null element is 1: if a sensor provides a total certainty about an event, its combination by disjunction with any other information will also be a total certainty.

If the *idempotence property* is satisfied, then measuring again an already known information will not change the already derived deduction. This property is not necessarily imposed for data fusion. For instance, the Dempster rule of combination is not idempotent. We may want on the contrary that the combination of two (uncertain) identical data reinforces or weakens the global confidence in the considered event. This is formalized as the archimedian property. For T-norms, it expresses that the confidence decreases if we have twice the same uncertain information. This behaviour is close to the probabilistic logic where when multiplying probabilities, probability decreases. On the contrary for T-conorms, the archimedian property expresses that the confidence in an information is reinforced if this information occurs twice. As opposed to T-norms and T-conorms, mean operators (in between the "min" and the "max") always satisfy the idempotence property. The kind of stability expressed by these compromise operators is incompatible with the archimedian property. The only idempotent T-norm and T-conorm are "min" and "max", respectively. Examples of archimedian T-norm and T-conorm are the product and the algebraic sum, respectively. All archimedian strictly monotonous T-norms have the following general form:[39]

$$\forall (x, y) \in [0, 1]^2, \quad i(x, y) = f^{-1}[f(x) + f(y)],$$

where f is a continuous decreasing bijection from $[0, 1]$ into $[0, +\infty]$ such that $f(0) = +\infty$ and $f(1) = 0$. The corresponding T-conorms have the following general form:

$$\forall (x, y) \in [0, 1]^2, \quad u(x, y) = \phi^{-1}[\phi(x) + \phi(y)],$$

with ϕ a continuous increasing bijection from $[0, 1]$ into $[0, +\infty]$ such that $\phi(0) = 0$ and $\phi(1) = +\infty$.

Every additive generating function f of an archimedian strictly monotonous T-norm i has an equivalent multiplicative generating function h[42] and thus i can also be expressed as:

$$\forall (x, y) \in [0, 1]^2, \quad i(x, y) = h^{-1}[h(x)h(y)],$$

where h is a strictly increasing function $[0, 1]$ into $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$ (h may be deduced from f by $h = e^{-f}$).

The *nilpotence property* means that the accumulation of n information leads to the null element (0 for T-norms and 1 for T-conorms). For instance, for T-conorms, a total certainty about an event is gained if we obtain a sufficient number of non-null measures supporting that event, even if uncertain. The operators $i(x, y) = \max(0, x + y - 1)$ and $u(x, y) = \min(1, x + y)$ are examples of nilpotent T-norm and T-conorm. Any nilpotent T-norm has the following general form:[39]

$$\forall (x, y) \in [0, 1]^2, \quad i(x, y) = f^*[f(x) + f(y)],$$

where f is a decreasing bijection from $[0, 1]$ into $[0, 1]$ such that $f(0) = 1$ and $f(1) = 0$, and $f^*(x) = f^{-1}(x)$ if $x \in [0, 1]$, $f^*(x) = 0$ if $x \geq 1$. The general form of nilpotent T-conorms can be deduced by duality. Again we have an equivalence between additive generating functions and multiplicative ones.

The rules of *excluded middle* and *non-contradiction* have an interpretation in terms of reasoning in particular in the domain of artificial intelligence and approximate reasoning. They are not necessarily imposed and may or may not be in conflict with other properties.

For instance, the satisfaction of these principles and the idempotence property are mutually exclusive. On the contrary, nilpotent operators always satisfy these two principles.

T-norms can also be constructed from another T-norm. Let i be a T-norm and h a continuous strictly increasing function from $[0,1]$ into $[a,1]$ such that $h(0)=a$ and $h(1)=1$. Then the operator i' defined by:

$$\forall (x,y)\in[0,1]^2,\quad i'(x,y)=h^*[i(h(x),h(y))],$$

with $h^*(x)=0$ if $x<a$ and $h^*(x)=h^{-1}(x)$ else is a T-norm.[42] Note that the "min" and i_0 are invariant by this construction, for any function h. Archimedian T-norms, for instance, can be obtained by this construction for $i(x,y)=xy$.

The *generating functions* of T-norms and T-conorms (functions f, ϕ, h introduced above) can be interpreted as functions which modify the initial data, for instance, by reinforcing or decreasing uncertainty or imprecision. The inverse function used after combination (f^{-1}, ϕ^{-1}, h^{-1}) or "pseudo-inverse" (f^*, h^*) can be interpreted as reverse modification of the result in order to recover the initial interpretation of the membership scale. The generating functions have thus an interpretation very similar to that already mentioned for the λ function involved in definition 5. The combination of the T-norm $\max(0, x+y-1)$ with the function $h(x)=1-\lambda(x)$ provides the weak T-norm $\max(0, 1-\lambda(x)-\lambda(y))$. This construction is similar to the one given above ($h^*[i(h(x),h(y))]$, except that function h^* is not used (the obtained value is not reversely transformed). This is why the result is only a weak T-norm. Moreover, this leads to a different interpretation as λ acts only on the individual values and not on their combination, as opposed to h.

ΦMM inherits all these interpretations of T-norms and T-conorms. Since properties are directly used (duality, for instance) and lead to morphological properties. Others are addtional properties for morphological operators, of another kind, no more strictly morphological. For example, the idempotent T-norm and T-conorm provide a particular definition for ΦMM transformation (definition 2). Then going from these extreme combination operators to archimedian or nilpotent ones, the information is progressively weakened (respectively, reinforced) by these operators and we obtain definitions for ΦMM dilations (respectively, erosions) which reflect this behaviour, and which have progressively less effect on the initial fuzzy sets. This leads to the important property of spatial ordering of ΦMM operators (see the last results of Section 5). Thus, this is an example where properties of fuzzy combination operators are inherited by ΦMM and can be used in a morphological way.

6.3.2. *Relationship with approximate reasoning.* One important feature of approximate reasoning using fuzzy sets is the generalization of implication. Fuzzy implication is often defined as:[46]

$$Imp(a,b)=u[c(a),b].$$

Fuzzy inclusion is related to implication by means of the following equation:

$$\mathrm{Imp}(\nu,\mu)=\inf_{x\in S} Imp[\nu(x),\mu(x)],$$

which leads to the definition of the inclusion operator used in this paper:

$$I(\nu,\mu)=\inf_{x\in S} u[c(\nu(x)),\mu(x)].$$

A second definition for fuzzy implication is the following:

$$Imp(a,b)=sup\{\varepsilon\in[0,1]/i(a,\varepsilon)\le b\},$$

which would provide the following expression for the inclusion indicator:

$$I(\nu,\mu)=\inf_{x\in S} sup\{\varepsilon\in[0,1]/i[\nu(x),\varepsilon]\le\mu(x)\}.$$

These second definitions coincide with the ones if i is an archimedian T-norm with nilpotent elements.

By using for the T-conorm u is the first implication definition the operators $\max(x,y)$, $\min(1,x+y)$, $x+y-xy$, we obtain, respectively, Kleene–Diene's implication, Lukasiewicz's implication and Reichenbach's implication; by using for the T-norm i in the second implication definition the operators $\min(x,y)$, $\max(0,x+y-1)$, xy, we obtain, respectively, Brower–Gödel's implication, Lukasiewicz's implication and Goguen's implication.[46,47] Thus, direct relationships are established with the implications commonly used in approximate reasoning. Then from the implication, a complete set of entailment operators can be deduced (fuzzy equivalence, etc.) and thus related to fuzzy inclusion. An example of expressing an entailment scheme by means of fuzzy inclusion indicator can be found in Sinha.[28]

6.3.3. *Decision making with uncertain and imprecise spatial information.* This section is dedicated to a few examples showing how features or properties of ΦMM can be interpreted in terms of decision making in the context of image processing.

The basic information in image processing has an important spatial nature, which is affected by imprecision and/or uncertainty. We distinguish two kinds of fuzziness in image processing. The first deals with crisp objects whose observation is corrupted by noise. Thus, fuzziness represents the imprecision and uncertainty due to that noise. This is the approach considered by Dougherty.[37] Based on it, the authors require commutativity of operators with decision thresholding "as most as possible" (since exact commutativity is impossible in general, but for very particular definitions as proved in Section 5). In this context, fuzzy morphology is of important use since fuzzy operators are less sensitive to small changes in shapes. From a crisp point of view, two slightly different objects X and X' may provide drastically different results when set relationships like intersection or inclusion are considered. On the contrary, fuzzy set operators would provide degrees

of inclusion or intersection which are slightly different for slightly different shapes and thus fit better the intuitive idea of small changes in objects corrupted by noise.

On the other hand, imprecision may be inherent to the observed objects and to the images, as mentioned in the Introduction. This leads to the second kind of fuzziness, which cannot be modelized by a noise combined with a crisp object. For dealing with such intrinsically fuzzy objects, the requirement of commutativity with decision thresholding is no longer justified and would even be inadequate. Moreover, the fuzzy results obtained on such objects by fuzzy transformations provide degrees of recognition, of detection, or confidence in the decision, either directly or through measures like fuzzy entropy.[40]

An example of this approach is described in reference (21). The application concerns a data fusion problem in medical imaging. The aim was to combine several magnetic resonance images to improve the detection of spatial information (edges, structure location, etc.). Taking the example of edges in such images, imprecision is due to both fuzziness of contours in each image, and to imperfect registration between images. The first type is taken into account by representing edges in each image by a fuzzy set μ_E, depending on their strength. The second type is modelled as a fuzzy structuring element ν_R representing the registration imprecision. The fuzzy dilated edge set $D_{\nu_R}(\mu_E)$ then provides the location of edges with gradations which represent both sources of imprecision. The fusion of such dilated fuzzy sets obtained from several images then allows us to actually take a decision with all the information about the problem and thus avoids the contractions or conflicts obtained from the fusion of crisp edges. This simple example shows how imprecision in spatial data can be introduced and managed in a fusion and decision process by means of ΦMM.

Another example is described in references (48, 49) and concerns three-dimensional reconstruction of blood vessels by a data fusion approach. The three-dimensional reconstruction of vessels has a great medical interest for understanding and interpreting vessel morphology and atheromateous vascular lesions. To avoid the limitations of reconstruction methods based on angiographic images only, an original approach has been proposed for three-dimensional reconstruction based on fusion of digital angiography and endovascular echography data, without any geometrical *a priori* knowledge of the vessel model. A geometrical fusion step leads to the determination of the unknown rotation and translation parameters, which allows one to align all data in a common reference frame, leading to a binary reconstruction from the echographic slices. Another binary reconstruction is obtained from the angiographies using a probabilistic approach. A method is then proposed for a reconstruction integrating both angiographic and echographic data, where imprecision on the geometrical parameters is taken into account in an original process which associates fuzzy number modelling and ΦMM for the reconstruction. The obtained fuzzy reconstructions are combined by a fuzzy operator before a binary decision is taken. Taking into account all information about the problem, along with its imprecision, the method avoids ambiguities of a reconstruction based only on one modality and solves the possible contradictions between both acquisitions. For this application, ΦMM proved to be very useful for introducing the imprecision on the geometrical parameters in an efficient way: the different positions of a point in three-dimensional space, along with their possibility degrees (depending on the imprecision on the geometrical parameters) are represented through a fuzzy structuring element, and included in the reconstruction through a fuzzy dilation. This application shows also how a fuzzy structuring element can be built directly from the data, without any arbitrary choice.

Using a fuzzy structuring element ν such that $\nu(0) < 1$ also has an interpretation for decision making. As dilation is no more extensive and erosion no more anti-extensive for such a structuring element, almost certain values of μ become less certain in $D_\nu(\mu)$, zero value (certainty about the absence of the considered event at such points) become fuzzy values (see Fig. 16). Thus, in general the certainty about the presence of an event expressed by μ decreases as the result of a morphological operation which contains more fuzziness than the initial fuzzy set μ. It is obvious that this will have some influence on the decision we can make from the result. Such an effect can be used to model an additional imprecision or uncertainty with respect to the initial data. Let us consider again a fuzzy set expressing the strength of edges in an image (high membership values represent high confidence in the presence of an edge and low membership values represent high confidence in the absence of an edge). For representing uncertainty or imprecision in the location of the edges in the image, we can use a fuzzy structuring element, possibly with $\nu(0) < 1$ expressing that even points with certain values (very low or very high) should be reconsidered for decision.

ΦMM provides operators which are useful for pattern recognition from a morphological point of view (see Section 6.1.4). In an imprecise or uncertain spatial context ΦMM can be moreover interpreted in terms of fuzzy pattern recognition. Let us take example of fuzzy pattern matching, which is used for reasoning by analogy and taking decision in that way. Fuzzy erosion provides a degree to which a fuzzy structuring element (a fuzzy pattern) matches a fuzzy shape of interest, at all possible space locations. In terms of possibility theory,[39,50] fuzzy erosion corresponds exactly to the necessity that some given data μ satisfy a filter ν.[51] For instance, the definition of this necessity described by Salotti[52] corresponds to the fuzzy erosion obtained with the T-conorm "max" (i.e. definition 2). In the same way, fuzzy dilation represents the degree of intersection between the fuzzy pattern ν and the fuzzy shape μ. It

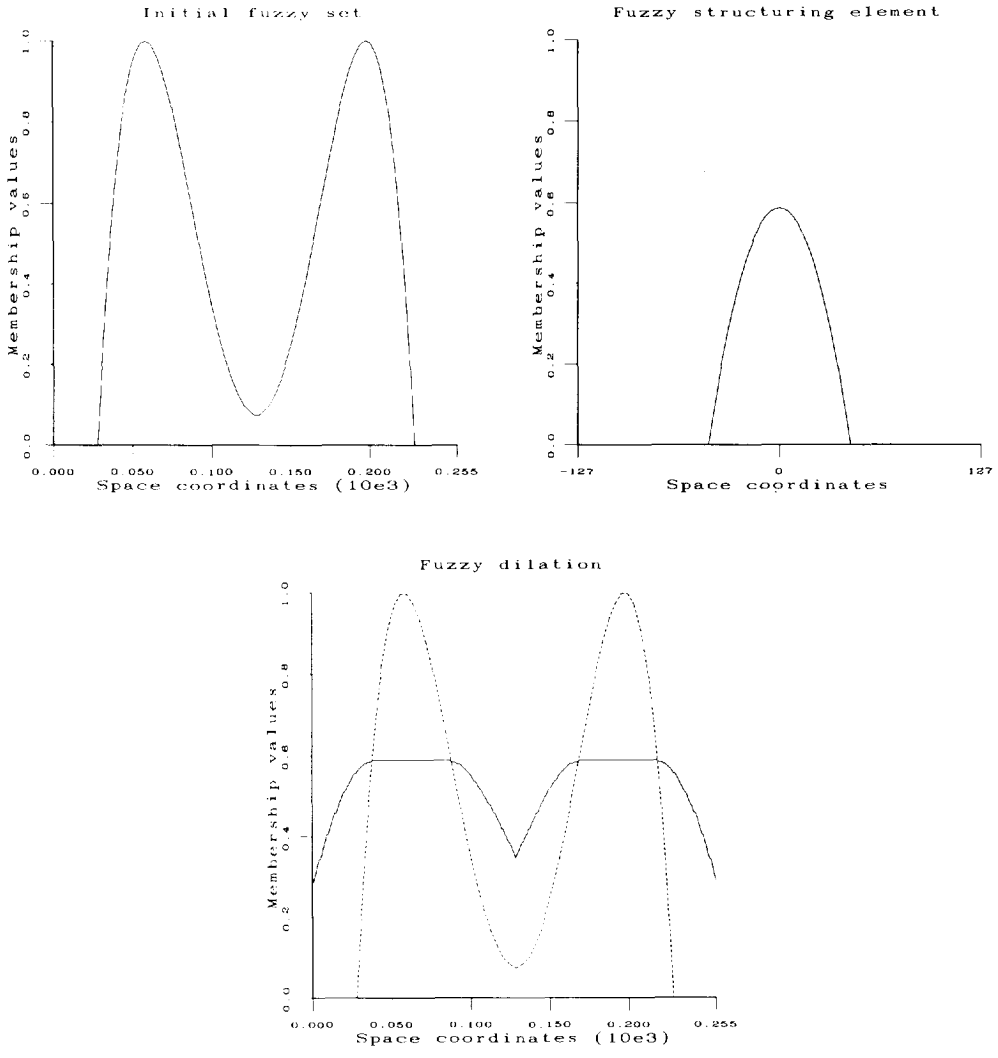


Fig. 16. Fuzzy dilation with $\nu(0) < 1$.

corresponds to the possibility that the data μ satisfy a filter ν.

In a similar way, ΦMM can be interpreted in terms of evidence theory:[53] fuzzy erosion corresponds to a belief function and fuzzy dilation to a plausibility function.

These results about the interpretation of ΦMM for decision problems in image processing under imprecision and uncertainty are clearly not exhausitive. Future works aim at developing these ideas and applying them in satellite and medical image processing, for data fusion problems. In particular, the advantages of ΦMM concerning morphological properties (Section 6.1) and fuzzy properties (Section 6.2) may be combined in the framework of decision making (Section 6.3).

7. CONCLUSION

In this communiation, we made a review of the requirements which have to be fulfilled when creating a morphology able to manipulate fuzzy set membership functions. We have shown that some constraints arise from morphological considerations, when others are dictated by fuzzy set or decision theory. Under this light, we examined the six existing definitions, and compared their properties. We first demonstrated that no definition was able to fulfill all the demands. We have then shown that the methodological framework of T-norms and T-conorms is the most general and leads to the construction of an infinity of ΦMM (this general definition contains four other definitions as special cases). These fuzzy mathematical morphologies are structured in families with specific properties. From an adequate choice of their free parameters (the T-norm, the complementation and the associated T-conorm), they may be adapted to a broad variety of problems with specific constraints. Furthermore, the use of T-norms as a key feature makes ΦMM inherit all the properties of these operators from a data fusion and decision theory point of view. In order to guide the user

in the selection of the most adapted fuzzy mathemtical morphology with respect to its own will, we discussed in detail the interpretation of the requirements in terms of fuzzy set and in terms of decision theory. We advocated for a coherent information processing framework for picture processing which takes into account information representation by fuzzy sets, spatial processing via ΦMM and decision making.

REFERENCES

1. L. A. Zadeh, Fuzzy sets, *Inform. Control* **8**, 338–353 (1965).
2. A. Rosenfeld, Fuzzy digital topology, *Inform. Control* **40**, 76–87 (1979).
3. A. Rosenfeld, The fuzzy geometry of image subsets, *Pattern Recognition Lett.* **2**, 311–317 (1984).
4. V. Goetcherian, From binary to grey tone image processing using fuzzy logic concepts, *Pattern Recognition* **12**, 7–15 (1980).
5. A. Kaufmann and M. M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*. North-Holand, Amsterdam (1988).
6. M. Wu, Fuzzy morphology and image analysis, in *Proceedings of the 9th ICPR*, 453–455, Rome (1988).
7. C. R. Giardina and D. Sinha, Image processing using pointed fuzzy sets, SPIE Vol. 1192, Intelligent Robots and Computer Vision VIII: *Algorithms Techn* 659–668 (1989).
8. P. A. Laplante and C. R. Giardina, Fast dilation and erosion of time varying grey valued images with uncertainty, SPIE Vol. 1568 *Image Algebra and Morphological Image Processing II*, 295–302 (1991).
9. V. di Gesu, Mathematical morphology and image analysis; a fuzzy approach, in *Proceedings of the Workshop on Knowledge-Based Systems and Models of Logical Reasoning*. Cairo, Egypt (1988).
10. V. di Gesu, M. C. Maccarone and M. Tripiciano, MMFUZZY Mathematical morphology based on fuzzy operators, in R. Lowen, M. Roubens, eds *Proceedings of the 4th IFSA Congress*, 29–32, Brussels (1991).
11. V. di Gesu, M. C. Maccarone and M. Tripiciano, *Mathematical Morphology based on Fuzzy Operators, Fuzzy Logic*, R. L owen and M. Roubens, eds 477–486, Kluwer Academic Publishers (1993).
12. M. Nakatsuyama, Fuzzy mathematical morphology for image processing, Proc. ANZIIS-93. pp. 75–78 Perth, Western Australia (1993).
13. A. Yuille, L. Vincent and D. Geiger, Statistical morphology, SPIE Vol. 1568 *Image Algebra and Morphological Image Processing* II, 271–282 (1991).
14. L. Koskinen, J. Astola and Y. Neuvo, Soft morphological filters, SPIE Vol. 1568 *Image Algebra and Morphological Image Processing* II, 262–270 (1991).
15. F. Prêteux and N. Merlet, New concepts in mathematical morphology: the topographical and differential distance function, SPIE Vol. 1568 *Image Algebra Morphological Image Processing* II, 66–77 (1991).
16. I. Bloch and H. Maître, *Fuzzy Sets and Mathematical Morphology*. Internal Report (in French), Télécom Paris 92D007, March (1992).
17. I. Bloch and H. Maître, Constructing a fuzzy mathematical morphology: alternative ways, *Second IEEE International Conference on Fuzzy Systems*, FUZZ IEEE 93, pp. 1303–1308. San Francisco, California, 28 March–1 April (1993).
18. D. Sinha and E. R. Dougherty, An Intrinsically Fuzzy Approach to Mathematical Morphology, SPIE Vol. 1607. Boston, Massachusetts, November (1991).
19. D. Sinha and E. R. Dougherty: Fuzzy Mathematical Morphology, *J. Vis. Commun. Image Rep.* **3**(3), 286–302 (1992).
20. D. Sinha and E. R. Dougherty, Fuzzification of set inclusion SPIE Vol. 1708. Orlando, Florida, April (1992).
21. I. Bloch, *Triangular Norms and Conorms: A General Tool for Generating Fuzzy Mathematical Morphologies*, Internal Report (in French), Télécom Paris 93D001, January (1993).
22. I. Bloch and H. Maître, Fuzzy mathematical morphology, *Ann. Math. Art.* Intell. **10**, 55–84 (1994).
23. I. Bloch and H. Maître, Mathematical morphology on fuzzy sets *Int. Workshop on Mathematical Morphology and its Applications to Signal Processing*, pp. 151–156. Barcelona, (12–14 May 1993).
24. H. Bandemer and W. Näther, *Fuzzy data analysis*, Theory and Decision Library, Serie B: *Mathematical and Statistical Methods*. Kluwer Academic Publishers, Dordrecht (1992).
25. B. De Baets and E. Kerre, An introduction to fuzzy mathematical morphology, *Proc. of NAFIPS'93* pp. 129–133. North American Fuzzy Information Processing Society, Allentown, Pennsylvania, USA (1993).
26. I. Bloch and H. Maître, Triangular norms as a tool for constructing fuzzy mathematical morphologies; *Int. Workshop on Mathematical Morphology and its Applications to Signal Processing*, pp. 157–161. Barcelona (12–14 May 1993).
27. J. Serra, *Image Analysis and Mathematical Morphology*. Academic Press, London (1982).
28. D. Sinha and E. R. Dougherty, Fuzzification of set inclusion: theory and applications, *Fuzzy Sets Syst.* **55**, 15–42 (1993).
29. I. Bloch, *About Properties of Fuzzy Mathematical Morphologies: Proofs of Main Results*, Technical Report, Télécom Paris 93D023 (December 1993).
30. E. R. Dougherty and D. Sinha, Need for Fuzzy Morphology: Erosion as a Fuzzy Marker, SPIE Vol. 1708, pp. 431–439 Orlando Florida (April 1992).
31. B. Kosko, Fuzziness vs. probability, *Int. J. Gen. Syst.* **17**, 211–240 (1990).
32. M. Ishizuka, Inference methods based on extended Dempster & Shafer's theory for problems with uncertainty/fuzziness. *New Generation Comput.* **1**, 159–168 (1983).
33. I. Bloch, Fuzzy connectivity and mathematical morphology, *Pattern Recognition Lett.* **14**, 483–488 (June 1993).
34. J. Serra (Ed), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*. Academic Press, London (1988).
35. H. Heijmans, Theoretical Aspects of Gray-Level Morphology, *IEEE Transactions on Pattern Analysis and Machine Intelligence* **13**(6), 568–582 (1991).
36. D. Sinha and E. R. Dougherty, Characterization of Fuzzy Minkowski Algebra. SPIE Vol. 1769, 59–69. San Diego (July 1992).
37. E. R. Dougherty, D. Sinha and P. Sinha, Fuzzy morphological filters, *SPIE Intelligent Robots and Computer Vision: Algorithms and Techniques* Boston (1992).
38. D. Dubois and H. Prade, A review of fuzzy set aggregation connectives, *Inf. Sci.* **36**, 85–121 (1985).
39. D. Dubois and H. Prade: *Théorie des possibilités, applications à la représentation des connaissances en informatique*. Masson, Paris (1988).
40. A. De Luca and S. Termini, A definition of non-probabilistic entropy in the setting of fuzzy set theory, *Inf. Control* **20**, 301–312 (1972).
41. K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. USA* **28**, 535–537 (1942).
42. B. Schweizer and A. Sklar, Associative functions and abstract semi-groups, *Publ. Math. Debrecen* **10**, 69–81 (1963).
43. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*. North Holland, Amsterdam (1983).
44. J. Aczél, *Lectures on Functional Equations and Applications*. Academic Press, New York (1973).
45. R. R. Yager, Connectives and quantifiers in fuzzy sets, *Fuzzy Sets Syst.* **40**, 39–75 (1991).
46. D. Dubois and H. Prade, Fuzzy sets in approximate reasoning, part 1: inference with possibility distributions, *Fuzzy Sets Syst.* **40**, 143–202 (1991).

47. B. Bouchon-Meunier, Fuzzy logic and knowledge representation using linguistic modifiers, in *Fuzzy Logic for the Management of Uncertainty*, L. Zadeh, J. Kacprzyk eds. pp. 399–414. J. Wiley & Sons, New York (1992).
48. F. Sureda, I. Bloch, C. Pellot and A Herment: Reconstruction 3D de vaisseaux sanguins par fusion de données à partir d' images angiographiques et échographiques, *Traitement du Signal* To be published (1995).
49. I. Bloch, C. Pellot, F. Sureda and A. Herment, 3D Reconstruction of blood vessels by multi-modality data fusion using fuzzy and Markovian modelling, *First International Conference on Computer Vision, Virtual Reality and Robotics in Medicine*, pp. 392–398 Nice (April 1995).
50. L. A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* **1**, 3–28 (1978).
51. M. Cayrol, H. Farreny and H. Prade, *Fuzzy Pattern Matching, Kybernetes* **11**, 103–116 (1982).
52. S. Salotti, Filtrage flou sur des objets pour raisonner, par analogie: le système FLORAN, *Deuxièmes journées nationales sur les applications des ensembles flous*, Nîmes, 335–342 (2–3 November 1992).
53. G. Shafer, *A mathematical Theory of Evidence*. Princeton University Press, New Jersey (1976).
54. A. Kaufmann, *Introduction à la théorie des sous-ensembles flous*. Masson (1977).

APPENDIX I: PROPERTIES SUGGESTED BY CLASSICAL MORPHOLOGY

This appendix refers to Section 3.1. It provides precise definitions of the properties summarized in this Section, as well as some interpretations. For each property, the operations (dilation, erosion, morphological opening and morphological closing) for which they are required, in order that ΦMM inherits this property, are also specified.

Four fundamental principles

In the framework of mathematical morphology, four fundamental principles are assumed.[27] Here, we translate them in terms of fuzzy sets.

Property 1. Translation invariance. A transformation Ψ on fuzzy sets is translation invariant iff:

$$\forall \mu \in M, \quad \forall t \in S, \quad \Psi(\mu + t) = [\Psi(\mu)] + t,$$

where $\mu + t$ is the fuzzy set with membership function μ translated by $t (\forall x \in S, (\mu + t)(x) = \mu(x + t))$.

This property means that transformations like erosion or dilation do not depend on the origin of the space S. It is satisfied by the four basic operations (erosion, dilation, opening, closing) in BMM, GMM and FMM.

Property 2. Compatibility with homotheties. A transformation Ψ on fuzzy sets satisfies this principle iff:

$$\forall \mu \in M, \quad \forall \lambda \in]0, 1], \quad \Psi(\lambda\mu) = \lambda\Psi(\mu).$$

Compatibility with homotheties guarantees that the transformations do not depend on a scale parameter. In the case of fuzzy sets, this scale parameter is limited to $]0, 1]$ in order for the result to remain a fuzzy set (homotheties act here on the membership values: $(\lambda\mu)(x) = \lambda\mu(x)$).

Property 3. Local knowledge. Ψ satisfies to this principle iff the knowledge of μ in a mask Z is sufficient to know $\Psi(\mu)$ in a mask Z'. It is satisfied by the four operations in BMM, GMM and FMM.

Property 4. Semi-continuity. An increasing operation Ψ on fuzzy sets is upper semi-continuous iff, for any decreasing series $(\mu_i)_{i \in N}$ such that:

$$\lim_{i \to +\infty} \mu_i = \mu$$

the series $(\Psi(\mu_i))_{i \in N}$ is decreasing with limit $\Psi(\mu)$, or, equivalently:

$$\overline{\lim_{i \to +\infty}} \Psi(\mu_i) \subset \Psi(\mu),$$

where $\overline{\lim}_{i \to +\infty} \Psi(\mu_i)$ denotes the upper limit of $\Psi(\mu_i)$, i.e. union of adherent points.

Lower semi-continuity is defined in a similar way and involves the lower limit of $\Psi(\mu_i)$. An operator is continuous iff it is upper semi-continuous and lower semi-continuous.

This principle governs the robustness of the transformations. In the set theory, it is related to the hit or miss topology, which is adapted to mathematical morphology. This property is fundamental from a theoretical point of view to assure a good analytical context. However, from a practical point of view, this property is no longer mentioned when working in a discrete space.

Algebraic properties

In addition to these basic properties, classical mathematical morphology operations have important algebraic properties which are effectively used for the applications. They are given below.

Property 5. Duality with respect to complementation. This means that transforming a set or function with a given operation Ψ or transforming the complementary set with the dual operation Φ and taking the complementary of the result are equivalent, i.e. $\Psi(\mu) = [\Phi(\mu^C)]^C$. Duality must hold between erosion and dilation and between opening and closing.

Property 6. Increasingness. An operation Ψ is increasing iff:

$$\forall (\mu, \mu') \in M^2, \quad \mu \leq \mu' \Rightarrow \Psi(\mu) \leq \Psi(\mu'),$$

where "$\leq$" denotes the inclusion on fuzzy sets. Dilation and closing have to be increasing with respect to both set and structuring element, while erosion and opening have to be increasing with respect to the set and decreasing with respect to the structuring element.

Property 7. Extensivity or anti-extensivity. An operation Ψ is extensive iff:

$$\forall \mu \in M, \quad \Psi(\mu) \geq \mu,$$

and anti-extensive iff the converse inclusion holds.

Opening has to be anti-extensive, closing has to be extensive. As well as in BMM, on fuzzy sets we can concede that erosion and dilation may have this property under some limiting conditions (for example, for a binary structuring element B, it holds for the origin 0 of S belonging to B).

Property 8. Idempotence. An idempotent transformation Ψ verifies:

$$\forall \mu \in M, \quad \Psi[\Psi(\mu)] = \Psi(\mu).$$

Morphological opening $D_v[E_v(\mu)]$ (respectively, morphological closing $E_v[D_v(\mu)]$) has to be idempotent to be an algebraic opening, i.e. an anti-extensive, increasing and idempotent mapping (respectively, algebraic closing, i.e. an extensive, increasing and idempotent mapping). The notation $\bar{v}$ stands for the symmetrical of v with respect to the origin of the space S[i.e. $\bar{v}_x(x) = v_x(-x)$. where $-x$ denotes the symmetrical of x with respect to the origin of S].

Property 9. Pseudo-commutativity of dilation. Dilation verfies, for X being a set or more generally a function:

$$D_B(X) = D_{\check{X}}(\check{B}) = \overline{D_{\check{X}}(B)}.$$

Pseudo-commutativity reduces to commutativity if B and X are symmetrical. For erosion, we have the following relation:

$$E_B(X) = \overline{E_{\check{X}^c}(B^C)}.$$

Property 10. Fitting characterization. This property means

that it is possible to find for the fuzzy erosion a relation similar to the following fitting characterization for binary erosion:

$$x \in E_B(X) \Leftrightarrow B_x \subset X$$

where B_x denotes the structuring element B centred at x.

Property 11. Compatibility with union and intersection, or with "max" and "min" on functions. This means that following equalities or inequalities hold:

- P.11.1: $D_\nu(\mu \cup \mu') = D_\nu(\mu) \cup D_\nu(\mu')$,
- P.11.2: $D_{\nu \cup \nu'}(\mu) = D_\nu(\mu) \cup D_{\nu'}(\mu')$,
- P.11.3: $D_\nu(\mu \cap \mu') \leq D_\nu(\mu) \cap D_\nu(\mu')$,
- P.11.4: $D_{\nu \cap \nu'}(\mu) \leq D_\nu(\mu) \cap D_{\nu'}(\mu')$,
- P.11.5: $E_\nu(\mu \cup \mu') \geq E_\nu(\mu) \cup E_\nu(\mu')$,
- P.11.6: $E_{\nu \cup \nu'}(\mu) = E_\nu(\mu) \cap E_{\nu'}(\mu')$,
- P.11.7: $E_\nu(\mu \cap \mu') = E_\nu(\mu) \cap E_\nu(\mu')$,
- P.11.8: $E_{\nu \cap \nu'}(\mu) \geq E_\nu(\mu) \cup E_{\nu'}(\mu')$.

Property 12. Iteration and combination. From dilation and erosion, other operations can be constructed by iteration and combination. For example, dilation with a larger structuring element can result by iterating dilation with a given structuring element. Mathematical morphology with binary structuring element (BMM and GMM) verifies:

$$D_{B'}[D_B(f)] = D_{B \oplus B'}(f) = D_{B'(B)}(f). \quad \text{(P.12.1)}$$

$$E_{B'}[E_B(f)] = E_{B \oplus B'}(f). \quad \text{(P.12.2)}$$

where $\oplus$ denotes Minkowski addition, and

$$D_{B'}[E_B(f)] \leq (\subset) E_B[D_{B'}(f)]. \quad \text{(P.12.3)}$$

APPENDIX 2: TRIANGULAR NORMS AND CONORMS

This appendix refers to Section 4.2.2., where T-norms and T-conorms constitute the basis for constructing ΦMM operators. It provides the definitions and main properties of T-norms and T-conorms.

In the context of stochastic geometry,[41,42] a triangular norm (or T-norm) i is defined as a function of two variables from $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying several properties:

T1: commutativity,
T2: associativity,
T3: 1 is unit element,
T4: increasingness with respect to the two variables.

From these properties, limit conditions can be derived (T0): $i(0, 1) = i(0, 0) = i(1, 0) = 0$ and $i(1, 1) = 1$, and it is easily shown that 0 is null element ($\forall x \in [0, 1]$, $i(x, 0) = 0$). A continuity property (T5) is often added to these properties.

The similarity of these properties with the properties of binary intersection allows to define a fuzzy intersection as a T-norm.[38,39,45]

In this framework, a complementation is defined as an operation c from $[0, 1]$ to $[0, 1]$ such that:

C0: $c(0) = 1$, $c(1) = 0$,
C1: c is decreasing,
C2: c is involutive.

The most used complementation is $c(x) = 1 - x$. In particular, it is commonly used as fuzzy complementation.[1,54] Other ones are described by Dubois[38] and Yager.[45]

From a T-norm i and a complementation c, another function u can be constructed which satisfies the de Morgan law:

$$u(x, y) = c[i(c(x), c(y))]. \quad \text{(D)}$$

u is called the T-conorm associated with i with respect to c. Any T-conorm is commutative (T'1), associative (T'2), monotonic (T'4), admits 0 as unit element (T'3), verifies limit conditions [$u(0, 1) = u(1, 1) = u(1, 0) = 1$ and $u(0, 0) = 0$] (T'0), and admits 1 as null element. Thus, a fuzzy union can be defined as a T-conorm.

Adding some properties like continuity, idempotence, distributivity, Archimedian property, or existence of nilpotent elements, gives rise to different families of T-norms and T-conorms. For example, it can be shown that the "min" and "max" operators are the only T-norm and T-conorm, which are idempotent and distributive over each other. These operators play a particular role because the "min" is the greatest T-norm and the "max" is the smallest T-conorm. On the other end, the smallest T-norm and the greatest T-conorm are defined by:

$$i_0(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{else.} \end{cases} \qquad u_0(x, y) = \begin{cases} x & \text{if } y = 0, \\ y & \text{if } x = 0 \\ 1 & \text{else.} \end{cases}$$

These additional properties can guide the choice of a particular T-norm, according to the application at hand and the properties it requires.

Figure 3 illustrates the most used T-norms and T-conorms.

Another property, which will be important for ΦMM, is the distributivity over "min" and "max": any T-norm i and T-conorm u distribute over "min" and "max" (but, in general, "min" and "max" are not distributive over i and u) and we have:

$$\forall (x, y, z) \in [0, 1]^3, \quad i[x, \min(y, z)] = \min[i(x, y), i(x, z)]$$
$$\forall (x, y, z) \in [0, 1]^3, \quad i[x, \max(y, z)] = \max[i(x, y), i(x, z)]$$
$$\forall (x, y, z) \in [0, 1]^3, \quad u[x, \min(y, z)] = \min[u(x, y), u(x, z)]$$
$$\forall (x, y, z) \in [0, 1]^3, \quad u[x, \max(y, z)] = \max[u(x, y), u(x, z)]$$

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