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# Directional relative position between objects in image processing: a comparison between fuzzy approaches

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**Abstract**

The importance of describing relationships between objects has been highlighted in works in very different areas, including image understanding. Among these relationships, directional relative position relations are important since they provide an important information about the spatial arrangement of objects in the scene. Such concepts are rather ambiguous, they defy precise definitions, but human beings have a rather intuitive and common way of understanding and interpreting them. Therefore in this context, fuzzy methods are appropriate to provide consistent definitions that integrate both quantitative and qualitative knowledge, thus providing a computational representation and interpretation of imprecise spatial relations, expressed in a linguistic way, and including quantitative knowledge. Several fuzzy approaches have been developed in the literature, and the aim of this paper is to review and compare them according to their properties and according to the types of questions they seek to answer.


*Keywords:* Spatial relationships; Relative directional position; Fuzzy sets; Image understanding

## 1. Introduction

The interest of relationships between objects has been highlighted in very different types of works: in vision, for identifying shapes and objects, in database management systems, for supporting spatial data and queries, in artificial intelligence, for planning and reasoning about spatial properties of objects, in cognitive and perceptual psychology, in geography, in particular for geographic information systems. For instance, in model-based structural pattern recognition, the aim is to recognize objects based on comparison between the characteristics of objects in the scene and objects in the model, and on comparison between relationships of groups of two or more objects in the scene and in the model. Other applications concern spatial reasoning, answering queries, finding and recognizing objects, spatial indexing, information retrieval, in various domains, such as aerial, satellite and medical imaging, robotics, GIS, video, vision.

According to the semantical hierarchy proposed in Ref. [1], spatial relationships can be divided into topological and metric ones (corresponding to levels three and four of this hierarchy). Many authors have stressed the importance of topological relationships, e.g. Refs. [2–9]. But distances and directional relative position (constituting the metric relationships) are also important, e.g. Refs. [10,11,1,12–15]. In this paper, we consider only directional relative position, which provides an important information about the spatial arrangement of objects in the scene. In comparison to topological relations (set relationships, part-whole relationships, adjacency), directional position has received much less attention. Most non-fuzzy approaches are based on a set of basic relations, based on Allen's interval relations [2], for instance Ref. [16], or based on simplifications of objects (e.g. Ref.

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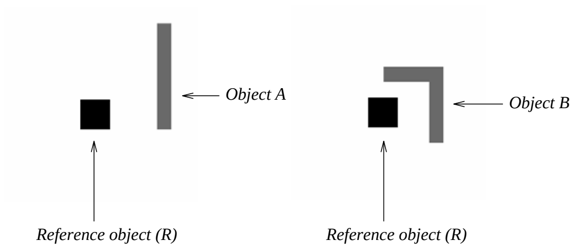


Fig. 1. Two examples where the relative position of objects with respect to the reference object is difficult to define in a "all-or-nothing" manner.

[12]). Qualitative expressions about angular positions are expressed as intervals in Ref. [15]. Stochastic approaches are proposed e.g. in Refs. [17,18], for representing spatial uncertainty in robotics.

Concepts related to directional relative position are rather ambiguous, they defy precise definitions. However, human beings have a rather intuitive and common way of understanding and interpreting them. From our every day experience, it is clear that any "all-or-nothing" definition leads to unsatisfactory results in several situations, even of moderate complexity such as those illustrated in Fig. 1: on the left, the object $A$ is to the right of $R$ but it can also be considered to be to some extent above it; on the right, the object $B$ is strongly to the right of $R$ and above it. Fuzzy set theory appears then as an appropriate tool for such modeling.

Usually vision and image processing make use of quantitative representations of spatial relationships. In artificial intelligence, mainly symbolic representations are developed (see Ref. [19] for a survey). The limitations of purely qualitative reasoning have already been stressed in Ref. [11], as well as the interest of adding semiquantitative extension to qualitative value (as done in the fuzzy set theory for linguistic variables [20,21]) for deriving useful and practical conclusions (as for recognition). On the other hand, purely quantitative representations are limited in the case of imprecise statements, and of knowledge expressed in linguistic terms. The use of fuzzy approaches for representing directional relative position allows to integrate both quantitative and qualitative knowledge, using the semiquantitative interpretation of fuzzy sets. As already mentioned in Ref. [22], this allows to provide a computational representation and interpretation of imprecise spatial relations, expressed in a linguistic way, possibly including quantitative knowledge. We concentrate in this paper on the comparison of fuzzy approaches for defining and assessing such relationships. They allow to cope with both the basic information in images, which is rather numerical, and with the symbolic information usually expressed in such relationships, leading to image understanding systems.

In Section 2, we describe what is intuitively expected from a definition of directional relative position. This will serve as a guide for comparing different approaches. In Section 3, we review existing approaches, and mention their possible extensions to 3D objects (if definitions are given in 2D), and to fuzzy objects. The main contribution of this paper is found in Section 4, where a comparison between these definitions is proposed, according to their formal properties, their behaviors in different situations, and the types of questions they can answer.

## 2. Requirements

In this section, we describe what we intuitively expect from a definition of relative position, and on what type of criteria we can base the comparison between different approaches.

First, from a formal point of view, the definitions should be invariant with respect to geometrical transformations (translation, rotation, scaling). For instance, this is mandatory for applications in pattern recognition. Some symmetry can be expected, for example saying that $A$ is to the left of $B$ should be equivalent to saying that $B$ is to the right of $A$. The domain of application of the definitions is also an important point. Ideally they should apply in any dimension, for crisp objects as well as for fuzzy objects.

From a more intuitive point of view, the influence of the shape of the object (what happens for instance in case of concavities), and the influence of the distance between objects is a point to be considered for various applications.

Comparison will also be provided in Section 4 in other terms, such as the type of elements on which definitions rely, the type of evaluation they provide, the computational complexity, and the type of questions they can answer. These are not really requirements, but they can guide the choice of one specific definition for one specific problem. Indeed, in the context of image understanding, or scene recognition, the specific goal can suggest one particular definition. For instance, assessing the relative position of objects having concavities may call for different answers depending on the application. Or assessing the position of two objects is a different problem from that of finding the area where an object has to be searched in order to satisfy some relationships with a previously recognized object. Another aspect concerns whether the dominant relationship or the relationship in any direction are to be assessed. This may also lead to different approaches.

## 3. Fuzzy definitions of relative position

We start with a review of existing definitions, and their extensions to 3D and to fuzzy objects (either already given in the literature, or suggested here).

To our knowledge, most of existing methods for defining fuzzy relative spatial position rely on angle measurements between points of the two objects of interest [23,24], and concern 2D objects. A fuzzy relationship is defined as a fuzzy set (see Section 3.1), and the correspondence between the relation and the angle measurements is evaluated, according to three methods which are described in Sections 3.2–3.4. A method based on linear cross-sections of the objects instead of only points has been developed in Ref. [25] (Section 3.6). Finally, methods based on whole objects have been proposed, based either on learning from human evaluation [26] (Section 3.5), on projections of the objects [27] (Section 3.7), or on a morphological approach [28,29] (Section 3.8).

In the following, we denote by $\mathscr{S}$ the Euclidean space where the objects are defined. $\mathscr{S}$ is typically a 2D or 3D discrete space (as in image processing).

### 3.1. Fuzzy relations describing relative position

In Refs. [23,24], the angle between the segment joining two points $a$ and $b$ and the $x$-axis of the coordinate frame (in 2D) is computed, according to Fig. 2. This angle, denoted by $\theta(a,b)$, takes values in $[-\pi,\pi]$, which constitutes the domain on which primitive directional relations are defined.

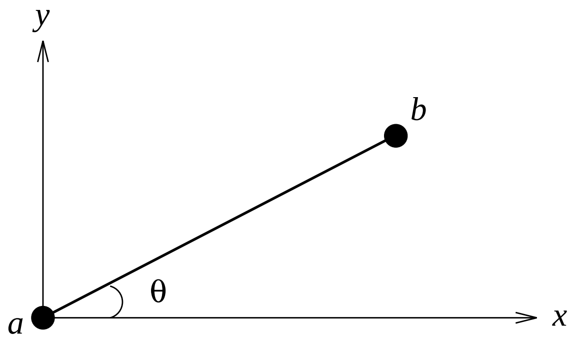


Fig. 2. Definition of angle $\theta$ for two object points $a$ and $b$.

Fig. 3 illustrates four such relations "left", "right", "above" and "below", defined in Ref. [23], as $\cos^2\theta$ and $\sin^2\theta$ functions. Other functions are possible: in Ref. [24] trapezoidal shaped membership functions are used, for the same relations. Whatever the equations, the membership functions for these relations are denoted by $\mu_{left}$, $\mu_{right}$, $\mu_{above}$, and $\mu_{below}$, and are functions from $[-\pi,\pi]$ into $[0,1]$. The equations are chosen according to simplicity (e.g. cos or sin functions), to the fact that they define a fuzzy partition of $[-\pi,\pi]$, and to their invariance properties with respect to rotation (i.e. a rotation should correspond to a translation of the membership functions).

In the work relying on these definitions, only these four basic directions are used, other relations being expressed in terms of these. However, we can propose a straightforward extension to any direction. In 2D, a direction is defined by an angle $\alpha$ with the $x$-axis. Using this convention, the relationship "right" corresponds to $\alpha=0$. From $\mu_{right}=\mu_0$, we derive $\mu_\alpha$, representing the relationship "in direction $\alpha$", for any $\alpha$ as follows:

$$\forall\theta,\quad \mu_\alpha(\theta)=\mu_0(\theta-\alpha) \tag{1}$$

with for instance

$$\mu_0(\theta)=\begin{cases}\cos^2(\theta) & \text{if } \theta\in[-\frac{\pi}{2},+\frac{\pi}{2}],\\ 0 & \text{elsewhere.}\end{cases} \tag{2}$$

This makes the definitions based on angle computation more general. Moreover, as it will be seen in Section 4.4, it guarantees geometric invariance. Note that the relations obtained for $\alpha=\pi$, $\alpha=-\pi/2$ and $\alpha=\pi/2$ are consistent with the definitions of $\mu_{left}$, $\mu_{above}$, and $\mu_{below}$ represented in Fig. 3.

Another solution for defining relations intermediate between the four basic ones has been proposed in Ref. [30]. This solution is based on logical combinations of these four basic relations. For instance, "oblique right" is defined by "(above and right of) or (below and right of)". The

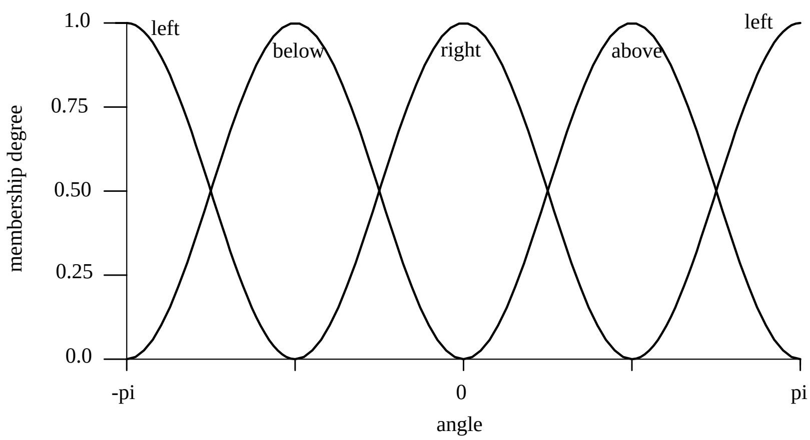


Fig. 3. Definition of fuzzy relations representing relative position.

membership function representing this relationship is computed from $\mu_{right}$, $\mu_{above}$, and $\mu_{below}$ using min (for "and") and max (for "or"). In a more general way, any $t$-norm and $t$-conorm could be used for that purpose. The advantage of this approach is that only four membership functions have to be defined, which is consistent with the usual way of speaking about relative position. The drawback is that, contrary to the definition proposed in Eq. (1), we cannot achieve a great precision in direction using this approach. Also, the shape of the membership function will vary depending on the considered direction, leading to a high anisotropy and therefore a loss of rotation invariance, while it remains the same using Eq. (1).

Another approach to avoid this is to express Eq. (1) as a combination of the four basic operations. The involved aggregation operator is not necessarily a $t$-norm or a $t$-conorm. Let us take the example of $\alpha = \pi/4$. According to Eq. (1), we have

$$\mu_{\pi/4}(\theta) = \begin{cases} \cos^2(\theta - (\pi/4)) = \frac{1}{2} + \cos(\theta)\sin(\theta) & \text{if } \theta \in [-(\pi/4), +3(\pi/4)], \\ 0 & \text{elsewhere.} \end{cases} \quad (3)$$

This can be expressed using the four basic relations as

$$\mu_{\pi/4}(\theta) = \begin{cases} \frac{1}{2} + \sqrt{\mu_0(\theta)\mu_{\pi/2}(\theta)} & \text{if } \theta \in [0, \pi/2], \\ \frac{1}{2} - \sqrt{\mu_0(\theta)\mu_{-\pi/2}(\theta)} & \text{if } \theta \in [-\pi/4, 0], \\ \frac{1}{2} - \sqrt{\mu_\pi(\theta)\mu_{\pi/2}(\theta)} & \text{if } \theta \in [\pi/2, 3(\pi/4)]. \end{cases} \quad (4)$$

It appears from these equations that the involved aggregation operator is rather a mean (a geometrical mean here) than a conjunction or disjunction. Similar computations can be carried out for other angles, but lead to more complex expressions.

Another way to interpret the formulas is to express $\mu_{\pi/4}(\theta)$ as a combination of $\mu_0(\theta)$, the degree to which $\theta$ is to the right, and of $\mu_{\pi/2}(\theta)$, the degree to which $\theta$ is above. Indeed we have, for $\theta \in [0, \pi/2]$ (the other cases can be treated similarly)

$$\begin{aligned} \mu_{\pi/4}(\theta) &= \cos^2\theta\cos^2\frac{\pi}{4}\left(1 + \tan\theta\tan\frac{\pi}{4}\right) \\ &\quad + \sin^2\theta\sin^2\frac{\pi}{4}\left(1 + \cot\theta\cot\frac{\pi}{4}\right), \\ &= \mu_0(\theta)\mu_0\left(\frac{\pi}{4}\right)\omega_0 + \mu_{\pi/2}(\theta)\mu_{\pi/2}\left(\frac{\pi}{4}\right)\omega_1, \end{aligned} \quad (5)$$

with

$$\begin{aligned} \omega_0 &= 1 + \tan\theta\tan\frac{\pi}{4}, \\ \omega_1 &= 1 + \cot\theta\cot\frac{\pi}{4} \end{aligned} \quad (6)$$

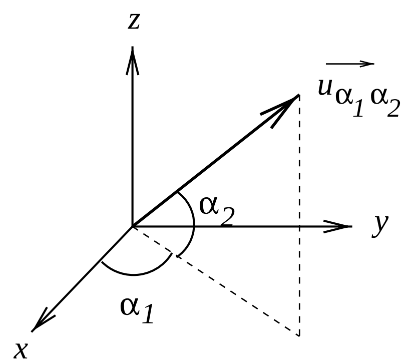


Fig. 4. Definition of a direction in 3D by two angles.

and

$$\frac{\mu_0(\pi/4)\omega_0}{\mu_{\pi/2}(\pi/4)\omega_1} = \frac{\tan\theta}{\tan\pi/4}. \quad (7)$$

This means that the degree to which $\theta$ is in the direction $\pi/4$ is an aggregation of the degrees to which $\theta$ is to the right and above weighted by a function to which $\pi/4$ is right and above, respectively. The ratio between both weights is equal to $\tan\theta/\tan\pi/4$.

More generally for any $\alpha \in [0, \pi/2]$, we have, for $\theta \in [0, \pi/2]$

$$\begin{aligned} \mu_\alpha(\theta) &= \cos^2\theta\cos^2\alpha(1 + \tan\theta\tan\alpha) \\ &\quad + \sin^2\theta\sin^2\alpha(1 + \cot\theta\cot\alpha) \\ &= \mu_0(\theta)\mu_0(\alpha)\omega_0 + \mu_{\pi/2}(\theta)\mu_{\pi/2}(\alpha)\omega_1 \end{aligned} \quad (8)$$

with

$$\begin{aligned} \omega_0 &= 1 + \tan\theta\tan\alpha, \\ \omega_1 &= 1 + \cot\theta\cot\alpha \end{aligned} \quad (9)$$

and

$$\frac{\mu_0(\alpha)\omega_0}{\mu_{\pi/2}(\alpha)\omega_1} = \frac{\tan\theta}{\tan\alpha}. \quad (10)$$

The extension to 3D images calls for a representation of a direction by two angles, as illustrated in Fig. 4. They will be denoted by $\alpha_1$ and $\alpha_2$ (with $\alpha_1 \in [0, 2\pi]$ and $\alpha_2 \in [-\pi/2, \pi/2]$, the 2D case corresponding to $\alpha_2 = 0$).

In Ref. [30], the membership functions defining six basic directions in the 3D space are defined, again using squared cosinus and sinus functions. For instance, the relation "right" is defined as

$$\mu_{right}(\alpha_1, \alpha_2) = \begin{cases} \cos^2(\alpha_1)\cos^2(\alpha_2) & \text{if } \alpha_1 \in [-\frac{\pi}{2}, +\frac{\pi}{2}] \text{ and } \alpha_2 \in [-\frac{\pi}{2}, +\frac{\pi}{2}], \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

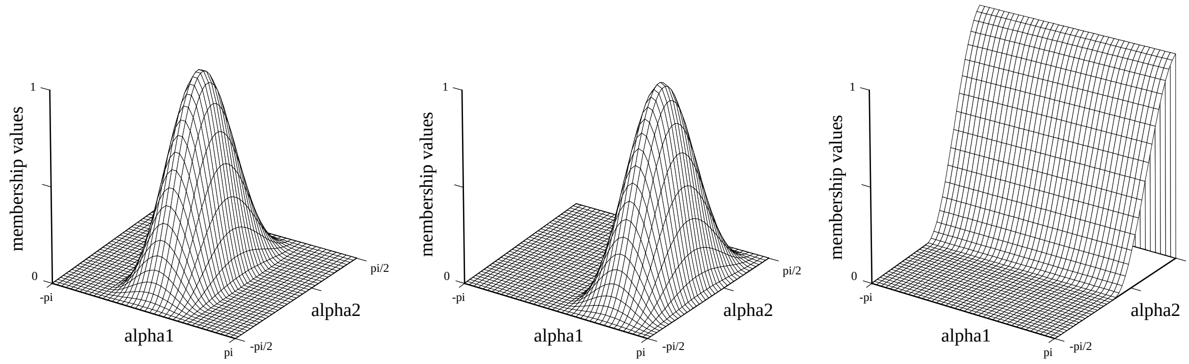


Fig. 5. Definition of fuzzy relations representing relative position in 3D. From left to right: “right”, “behind” and “above”.

The other relations are defined in a similar way. Three of the six basic relations are shown in Fig. 5.

### 3.2. Centroid method

A first simple solution to evaluate a fuzzy relationship between two objects consists in representing each object by a characteristic point. This point is chosen as the object centroid in Refs. [13,24]. Let $c_R$ and $c_A$ denote the centroids of objects $R$ and $A$. The degree of satisfaction of the proposition “$A$ is to the right of $R$” is then defined as

$$\mu^R_{right}(A) = \mu_{right}(\theta(c_R, c_A)), \tag{12}$$

where the membership function $\mu_{right}$ is defined as in Section 3.1.

Note that, although not mentioned in the original paper, this method can be straightforward extended to 3D.

Extension to fuzzy objects can be done in two ways. One way consists in computing a weighted centroid, where the contribution of each object point is equal to its membership value. The second way consists in applying the definition for binary objects on each $\alpha$-cut and then aggregating the results using a summation [31], or the extension principle [32]. However, this second method may be computationally expensive, depending on the quantization of the object membership values.

### 3.3. Histogram of angles: compatibility method

The method proposed in Refs. [33,23,30] consists in computing the normalized histogram of angles and in defining a fuzzy set in $[0,1]$ representing the compatibility between this histogram and the fuzzy relation. More precisely, the angle histogram is computed from the angle between any two points in both objects as defined before, and normalized by the maximum frequency. Let us denote $H^R(A)$ this normalized histogram, where $R$ is the reference object and $A$ the object the position of which with respect to $R$ is evaluated. $H^R(A)$ represents the spatial directional relations of the object $A$ with respect to the reference object $R$. Issues that arise here include expressing this in terms of the basic relations, and extracting a global representative evaluation of this spatial relation.

With respect to the first issue, operations of compatibility and matching of two fuzzy sets are considered. The compatibility set $\mu_{C(H,\mu_\alpha)}$ between $H^R(A)$ and $\mu_\alpha$ is defined, for any $u \in [0,1]$, following the extension principle as

$$\mu_{C(H,\mu_\alpha)}(u) = \begin{cases} 0 & \text{if } \mu_\alpha^{-1}(u) = \emptyset, \\ \sup_{v \mid u=\mu_\alpha(v)} H^R(A)(v) & \text{otherwise.} \end{cases} \tag{13}$$

With respect to the second issue, a global evaluation of the relation can for instance be provided by the center of gravity of the compatibility fuzzy set:

$$\mu^R_\alpha(A) = \frac{\int_0^1 u\mu_{C(H,\mu_\alpha)}(u)\,\mathrm{d}u}{\int_0^1 \mu_{C(H,\mu_\alpha)}(u)\,\mathrm{d}u}. \tag{14}$$

Another solution for the first issue is to use a fuzzy pattern matching approach [34,35] (between $\mu_\alpha$ and $H^R(A)$), as suggested in Ref. [28]. Then the global evaluation is given in the form of a pair necessity/possibility.

Ultimately, this global evaluation, which can be done in many ways, has to be selected according to the type and goals of the application at hand.

Fig. 6 presents the angle histograms for the two examples of Fig. 1.

The compatibility fuzzy sets for two relative positions are presented in Fig. 7 for object $B$ with respect to reference object $R$ (Fig. 1). These fuzzy sets show that object $B$ is right and above of the reference object, and not left nor below. The final values obtained after center of gravity computation are 0.051 for left, 0.550 for right, 0.166 for below and 0.500 for above. These results fit the intuition.

The extension of this method to 3D objects amounts to computing a bi-dimensional histogram, i.e. as a function of two angles, and then applying the same principle using the relations defined in 3D. The computation of the histogram

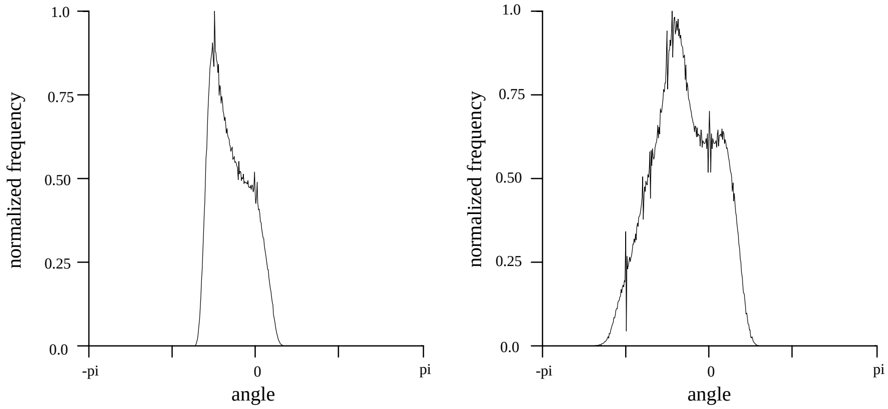


Fig. 6. Angle histograms for the two examples of Fig. 1. Left: object $A$ w.r.t. reference object $R$; right: object $B$ w.r.t reference object $R$.

is heavy in 2D, and becomes even more so in 3D. Another problem when computing bi-histograms is that the domain of possible angle values may be under-represented, depending on the size and the sampling of the considered objects. This may result in a noisy and hole containing histogram. This effect already appears in 2D.

A possible direction for overcoming the computational burden would be to consider 2D restrictions of the 3D space, depending on the direction we are interested in. For instance, when assessing a left or right direction, it might be sufficient to look only at projections of the objects in the horizontal plane. This approach can be considered in particular for almost convex objects without holes. However, for more complex objects, too much information may be lost by this approach.

The fuzzy extension of this method is based on a weighted histogram [23]. Let us denote by $\mu_R$ and $\mu_A$ the membership functions of the fuzzy objects $R$ and $A$. The weighted histogram is computed as

$$H^R(A)(\theta) = \sum_{a,b,\theta(a,b)=\theta} \min[\mu_R(a), \mu_A(b)]. \tag{15}$$

This expression if equivalent to compute a histogram on each $\alpha$-cut and to combine the obtained results by summation as in Ref. [31]. Indeed, let us consider, for instance, a discretization of the values of $\alpha$, as $\alpha_1, \dots \alpha_n$, with $\alpha_1 = 0$, $\alpha_n = 1$, and $\forall i$, $1 \leqslant i \leqslant n-1, \alpha_i < \alpha_{i+1}$ (a similar reasoning holds in the continuous case). Let us denote by $\mu_{R_i}$ and $\mu_{A_i}$ the $\alpha$-cuts of $\mu_R$ and $\mu_A$ at level $\alpha_i$. Then we have

$$\min[\mu_R(a), \mu_A(b)] = \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) \min[\mu_{R_i}(a), \mu_{A_i}(b)]. \tag{16}$$

From this equality, we derive

$$H^R(A)(\theta) = \sum_{a,b,\theta(a,b)=\theta} \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) \min[\mu_{R_i}(a), \mu_{A_i}(b)]$$

$$= \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) \sum_{a,b,\theta(a,b)=\theta} \min[\mu_{R_i}(a), \mu_{A_i}(b)]$$

$$= \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i) H^{\mu_{R_i}}(\mu_{A_i})(\theta). \tag{17}$$

This shows that the weighted histogram is equivalent to the summation of the histograms of the $\alpha$-cuts. However the first form is computationally much less expensive, and does not involve any assumption on the quantification of the values of $\alpha$.

### 3.4. Aggregation method

An aggregation method has been proposed in Refs. [13,24], which uses all points of both objects instead of only one characteristic point. For any pair of points $i$ in $R$ and $j$ in $A$, the angle $\theta(i,j)$ is computed, and the corresponding membership value for a direction $\alpha$ (being one of the 4 considered relations) is computed as previously

$$\mu_{ij} = \mu_\alpha(\theta(i,j)). \tag{18}$$

All these values are then aggregated. The aggregation operator suggested in Ref. [24] is a weighted mean

$$\mu_\alpha^R(A) = \left[ \sum_{i \in R} \sum_{j \in A} w_{ij} \mu_{ij}^p \right]^{1/p}, \tag{19}$$

where $w_{ij}$ are weights the sum of which is equal to 1.

### 3.5. Learning approach

In Ref. [26], a learning approach using neural networks is proposed in order to cope with the complexity and variety of spatial relationships. The idea is to learn membership functions of spatial relationships for a few types of shapes, for which it is possible to easily assign membership values.

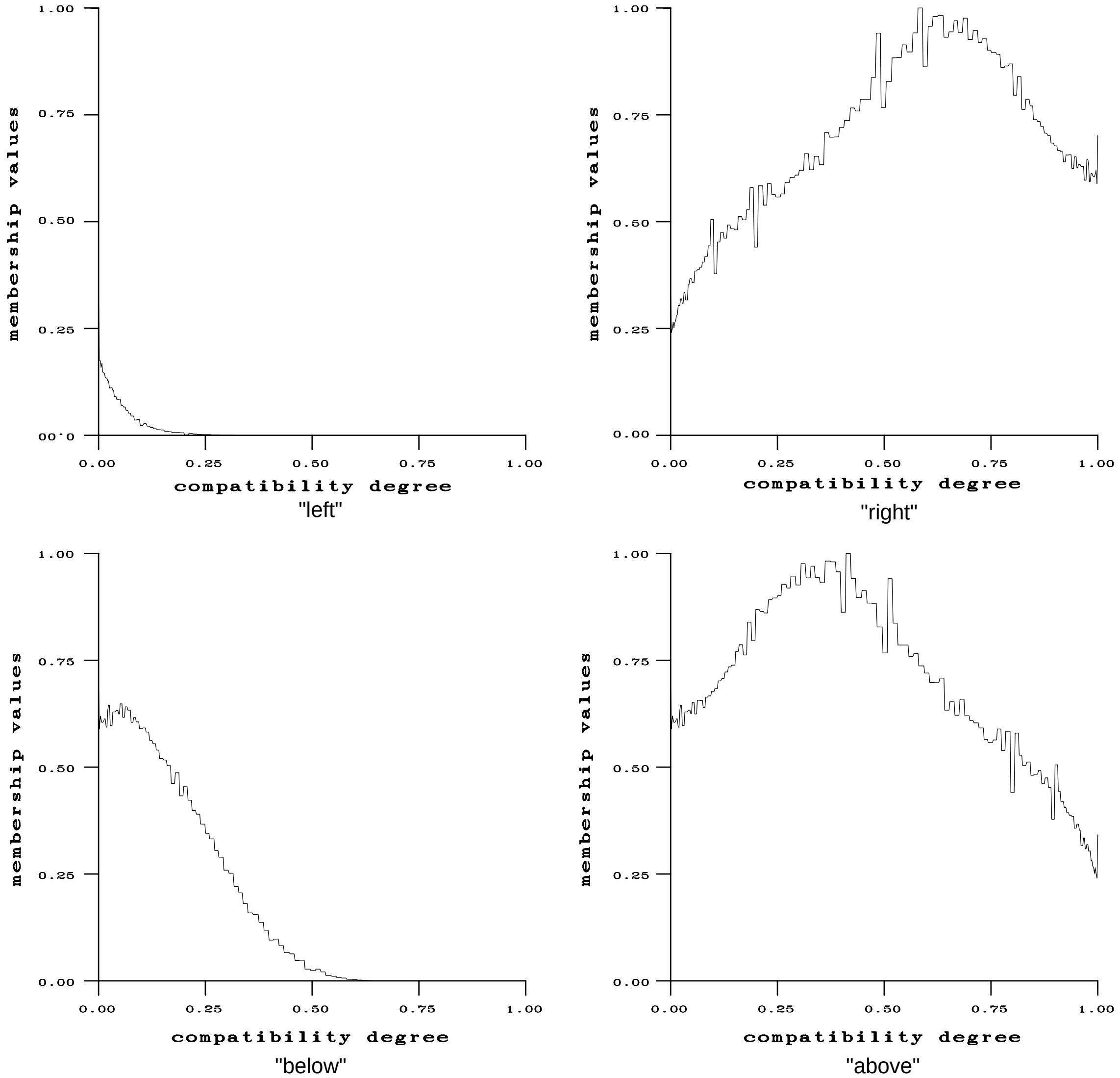


Fig. 7. Compatibility between each of the four relations and the angle histogram of object $B$ with respect to reference object $R$ (both are defined on Fig. 1).

Four basic relations are considered. The angle histograms of the training data are the inputs of neural networks (one such network for each type of shape), the outputs of which are then combined using Choquet fuzzy integrals.

The main problem with this approach is the first assignment for the training data. It does not seem easy to define criteria that allow to distinguish between values such as 0.002 and 0.005, 0.97 and 0.93, etc.

### 3.6. Histogram of forces

Instead of considering pairs of points as in angle histogram approaches, pairs of longitudinal sections are considered in Ref. [25], where the concept of $F$-histogram is introduced. The degree to which an object $A$ is in the direction $\alpha$ with respect to a reference object $R$ is computed using successively points, segments, and longitudinal sections. Information on points is translated by a function $\phi$ acting on the difference of coordinates on the $\alpha$-axis between points of $A$ and points of $R$. Therefore, distance information is explicitly taken into account. A second function $f$ integrates $\phi$ on segments of $A$ and $R$ in the direction $\alpha$. Finally, the contributions of segments constituting the longitudinal sections of $A$ and $R$ in the direction $\alpha$ are summed. A so-called "histogram of forces" allows to compute the weight supporting a proposition like "object $A$ is in direction $\alpha$ from object $R$".

The definitions of the functions involved in this construction are done in an axiomatic way, that guarantees that the obtained relationships have good properties. Note that a similar axiomatic treatment can be adopted for the other approaches without any change in the results. Basically, this approach amounts to considering a weighted angle histogram

$$H^R(A)(\theta) = \sum_{a,b,\theta(a,b)=\theta} \varphi(\|\vec{ab}\|), \tag{20}$$

where $\varphi$ is a decreasing function. Typically, $\varphi(x) = 1/x^r$. For $r = 0$, the weighted histogram is equal to the angle histogram, and for $r \geqslant 1$, it gives more importance to points of $A$ that are close to some points of $R$. This allows to deal with situations where $A$ and $R$ have very different partial extents, and to account only for the closest parts of them. Let us consider the examples in Fig. 1. The angle histograms (or equivalently the force histograms for $r = 0$) are shown in Fig. 6. They are compared in Fig. 8 to force histograms obtained for different values of $r$. For object $A$ with respect to object $R$, it appears that when $r$ increases, the shape of the histogram concentrates more and more around the value 0, i.e. the main relation becomes "to the right", and the part above has less importance. For object $B$ with respect to object $R$, on the contrary the part of $B$ which is above $R$ gains importance, since it is closest to the square than the part that is to the right of $R$.

This approach has been extended to fuzzy objects using their $\alpha$-cuts. Extension to 3D objects could be probably done, but with a high complexity.

### 3.7. Projection-based approach

The approach proposed in Ref. [27] is very different from the previous ones since it does not use any histogram. It is based on a projection of the considered object on the axis related to the direction to be assessed (e.g. the $x$-axis for evaluating the relations "left to" and "right to"). Let us detail the computation for the relation "$A$ is left from $R$". The same construction applies for any direction. Let us denote by $R^f(x)$ the normalized projection of the set $R$ on the $x$-axis. The degree for a point $x$ to be left to $R$ is defined as

$$R^{\leftarrow}(x) = \frac{\int_x^{+\infty} R^f(y)\,\mathrm{d}y}{\int_{-\infty}^{+\infty} R^f(y)\,\mathrm{d}y}. \tag{21}$$

This definition provides a degree of 1 for points that are completely on the left of the support of $R^f$ and a degree of 0 for points that are completely on the right of the support of $R^f$, and the degree decreases in-between.

Let us now introduce a second set $A$. The degree $(A \leftarrow R)^f(x)$ to which $x$ is in the projection of $A$ and to the left of $R$ is expressed as a conjunction of $A^f(x)$ and $R^{\leftarrow}(x)$. The conjunction is taken as a product in Ref. [27]. The degree to which $A$ is left from $R$ is then deduced as the ratio of the areas below $(A \leftarrow R)^f$ and $A^f$

$$\mu_\alpha^R(A) = \frac{\int_{-\infty}^{+\infty} A^f(x) \int_x^{+\infty} R^f(y)\,\mathrm{d}y\,\mathrm{d}x}{\int_{-\infty}^{+\infty} A^f(y)\,\mathrm{d}y \int_{-\infty}^{+\infty} R^f(y)\,\mathrm{d}y}. \tag{22}$$

This approach can be generalized to fuzzy sets [27] by taking each point into account in the projection to the amount of its membership function, leading to similar properties than in the crisp case.

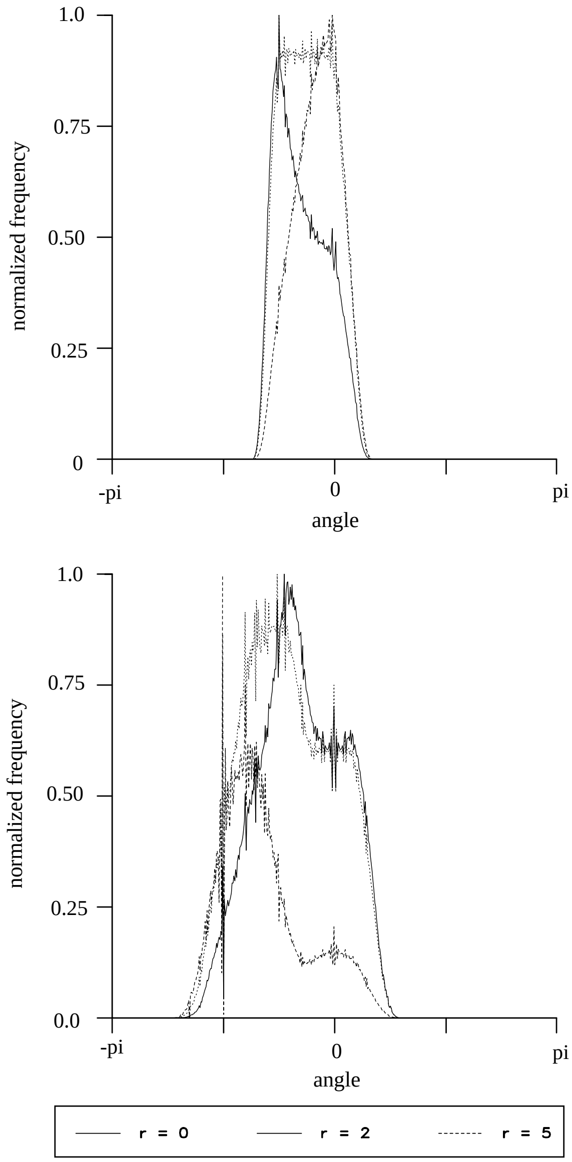


Fig. 8. Force histograms for the two examples of Fig. 1. Top: object $A$ w.r.t. reference object $R$; bottom: object $B$ w.r.t reference object $R$. Solid line: $r = 0$, dotted line: $r = 2$, dashed line: $r = 5$.

### 3.8. Morphological approach

In [28,29,36] a morphological approach has been proposed in order to evaluate the degree to which an object $A$ is in some direction with respect to a reference object $R$, consisting of two steps:

(1) A *fuzzy landscape* is first defined around the reference object $R$ as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation under examination. This makes use here of a spatial representation of fuzzy sets, which already proved to be useful in image processing [37,38]. Therefore, the fuzzy landscape is directly defined in the same space as the considered objects, contrary to the projection method [27], where the fuzzy area is defined on a 1D axis.
(2) Then the object $A$ is compared to the fuzzy landscape attached to $R$, in order to evaluate how well the object matches with the areas having high membership values (i.e. areas that are in the desired direction). This is done using a fuzzy pattern matching approach, which provides an evaluation as an interval or a pair of numbers instead of one number only.

#### 3.8.1. Relative position from fuzzy pattern matching

A 3D direction is defined by two angles $\alpha_1$ and $\alpha_2$ as illustrated in Fig. 4, where $\alpha_1 \in [0, 2\pi]$ and $\alpha_2 \in [-\pi/2, \pi/2]$. The direction in which the relative position of an object with respect to another one is evaluated is denoted by: $\vec{u}_{\alpha_1,\alpha_2} = (\cos\alpha_2 \cos\alpha_1, \cos\alpha_2 \sin\alpha_1, \sin\alpha_2)^t$, and we note $\alpha = (\alpha_1, \alpha_2)$. Let us denote by $\mu_\alpha(R)$ the fuzzy set defined in the image such that points of areas which satisfy to a high degree the relation "to be in the direction $\vec{u}_{\alpha_1,\alpha_2}$ with respect to reference object $R$" have high membership values. In other words, the membership function $\mu_\alpha(R)$ has to be an increasing function of the degree of satisfaction of the relation. It is a spatial fuzzy set (i.e. a function of the image $\mathscr{S}$ into $[0, 1]$) and directly related to the shape of $R$. The precise definition of $\mu_\alpha(R)$ is given below.

Let us denote by $\mu_A$ the membership function of the object $A$, which is a function of $\mathscr{S}$ into [0,1]. The evaluation of relative position of $A$ with respect to $R$ is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all $x \in \mathscr{S}$. An appropriate tool for defining this function is the fuzzy pattern matching approach [35]. Following this approach, the evaluation of the matching between two possibility distributions consists of two numbers, a necessity degree $N$ (a pessimistic evaluation) and a possibility degree $\Pi$ (an optimistic evaluation), as often used in conjunction with fuzzy sets. In our application, they take the following forms:

$$\Pi^R_{\alpha_1,\alpha_2}(A) = \sup_{x\in\mathscr{S}} t[\mu_\alpha(R)(x), \mu_A(x)],$$

$$N^R_{\alpha_1,\alpha_2}(A) = \inf_{x\in\mathscr{S}} T[\mu_\alpha(R)(x), 1 - \mu_A(x)], \tag{23}$$

where $t$ is a $t$-norm (fuzzy intersection) and $T$ a $t$-conorm (fuzzy union) [39]. In the crisp case, these equations reduce to: $\Pi^R_{\alpha_1,\alpha_2}(A) = \sup_{x\in A} \mu_\alpha(R)(x)$, and $N^R_{\alpha_1,\alpha_2}(A) = \inf_{x\in A} \mu_\alpha(R)(x)$.

The possibility corresponds to a degree of intersection between the fuzzy sets $A$ and $\mu_\alpha(R)$, while the necessity corresponds to a degree of inclusion of $A$ in $\mu_\alpha(R)$. They can also be interpreted in terms of fuzzy mathematical morphology, since the possibility $\Pi^R_{\alpha_1,\alpha_2}(A)$ is equal to the dilation of $\mu_A$ by $\mu_\alpha(R)$ at origin, while the necessity $N^R_{\alpha_1,\alpha_2}(A)$ is equal to the erosion, as shown in Ref. [40]. These two interpretations, in terms of set theoretic operations and in terms of morphological ones, explain how the shape of the objects is taken into account.

Several other functions combining $\mu_\alpha(R)$ and $\mu_A(x)$ can be constructed. The extreme values provided by the fuzzy pattern matching are interesting because of their morphological interpretation, and because they provide a pair of extreme values and not only a single value and may better capture the ambiguity of the relation if any. One drawback of these measures is that they are sensitive to noise, since they rely on infimum and supremum computation. An average measure can also be useful from a practical point of view (it is much less sensitive to noise), and is defined as

$$M^R_{\alpha_1,\alpha_2}(A) = \frac{1}{|A|} \sum_{x\in\mathscr{S}} t[\mu_A(x), \mu_\alpha(R)(x)], \tag{24}$$

where $|A|$ denotes the fuzzy cardinality of $A$: $|A| = \sum_{x\in\mathscr{S}} \mu_A(x)$.

#### 3.8.2. Definition of $\mu_\alpha$

The key point in the previous definition relies in the definition of $\mu_\alpha(R)$. The requirements stated above for this fuzzy set are not strong and leave room for a large spectrum of possibilities. This flexibility allows the user to define any membership function according to the application at hand and the context requirements. The following definition looks precisely at the domains of space that are visible from a reference object point in the direction $\vec{u}_{\alpha_1,\alpha_2}$. This applies to any kind of objects, including those having strong concavities.

Let us denote by $P$ any point of $\mathscr{S}$, and by $Q$ any point of $R$. Let $\beta(P, Q)$ be the angle between the vector $\vec{QP}$ and the direction $\vec{u}_{\alpha_1,\alpha_2}$, computed in $[0, \pi]$:

$$\beta(P, Q) = \arccos\left[\frac{\vec{QP} \cdot \vec{u}_{\alpha_1,\alpha_2}}{\|\vec{QP}\|}\right] \quad \text{and} \quad \beta(P, P) = 0. \tag{25}$$

We then determine for each point $P$ the point $Q$ of $R$ leading to the smallest angle $\beta$, denoted by $\beta_{\min}$. In the crisp case, $Q$ is the reference object point from which $P$ is visible in the direction closest to $\vec{u}_{\alpha_1,\alpha_2}$ (see Fig. 9): $\beta_{\min}(P) = \min_{Q\in R} \beta(P, Q)$. The fuzzy landscape $\mu_\alpha(R)$ at point $P$ is then defined as $\mu_\alpha(R)(P) = f(\beta_{\min}(P))$, where $f$ is a decreasing function of $[0, \pi]$ into $[0, 1]$. We can chose for instance a simple linear function (Fig. 9 right): $\mu_\alpha(R)(P) = \max(0, 1 - (2\beta_{\min}(P)/\pi))$.

Illustrations of the definition of $\mu_\alpha(R)$ are given in Fig. 10 for several reference objects. They show the consistency of the proposed approach in case of concavities: since the aim of the proposed definition is not to find only the dominant relationship, an object may satisfy several different relationships with high degrees. Therefore, "to be to the right of $R$" does not mean that the object should be completely to the right of the reference object, but only that at least part of the object is to the right of part of the region. This is the case for instance in Fig. 10, where we obtain high values of being right inside the concavities.

In the fuzzy case, this definition is extended as $Q \in R$ and $f(\beta_{\min}) = \max_{Q\in R} f(\beta(P,Q))$ (since $f$ is decreasing), which translates in fuzzy terms as

$$\mu_\alpha(R)(P) = \max_{Q\in Supp(S)} t[\mu_R(Q), f(\beta(P,Q))], \tag{26}$$

where $t$ is a $t$-norm. Fig. 11 illustrates the obtained result on a fuzzy object.

An advantage of this approach is its interpretation in terms of morphological operations. It can be shown that $\mu_\alpha(R)$ is exactly the fuzzy dilation of $\mu_R$ by $v$, where $v$ is a fuzzy structuring element defined on $\mathcal{S}$ as

$$\forall P \in \mathcal{S}, \quad v(P) = \max\left[0, 1 - \frac{2}{\pi}\arccos\frac{\vec{OP}\cdot\vec{u}_\alpha}{\|\vec{OP}\|}\right], \tag{27}$$

where $O$ is the center of the structuring element. This structuring element is illustrated in 2D in Fig. 12. The following definition is used for the fuzzy dilation (see Ref. [40] for more details about fuzzy morphological operations)

$$\forall P \in \mathcal{S}, \quad D_v(\mu)(P) = \max_{Q\in\mathcal{S}} t[\mu(Q), v(P-Q)], \tag{28}$$

where $t$ is a $t$-norm. This equivalence provides an additional morphological interpretation of this approach.

### 3.8.3. Two simple examples

We illustrate this definition on the two simple 2D examples shown in Fig. 1, and compute the relative position of objects $A$ (rectangle) and $B$ (corner) with respect to reference object $R$ (square), for four directions: left ($\alpha_1 = \pi$), right ($\alpha_1 = 0$), above ($\alpha_1 = \pi/2$) and below ($\alpha_1 = 3\pi/2$). Fig. 13 provides the obtained results, using Eqs. (21) and (22) for the possibility degree, necessity degree and average, respectively. The interval $[N, \Pi]$ represents the range between the minimal and maximal values obtained in

Fig. 11. Left: a fuzzy reference object. Right: fuzzy landscape representing the relationship "to the left of" with the definition expressed by Eq. (26).

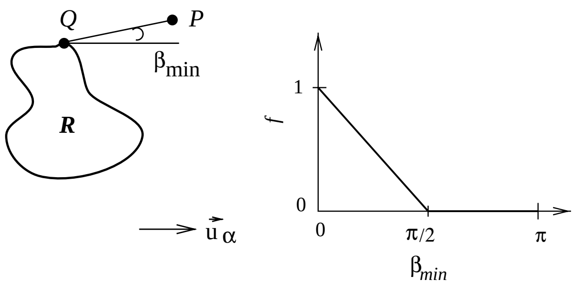


Fig. 9. Definition of $\beta_{\min}$ (left) and of $f(\beta_{\min})$ (right).

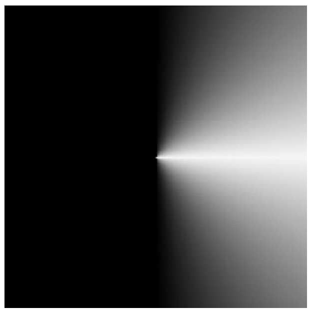

Fig. 12. Structuring element $v$ for $\alpha_1 = 0$.

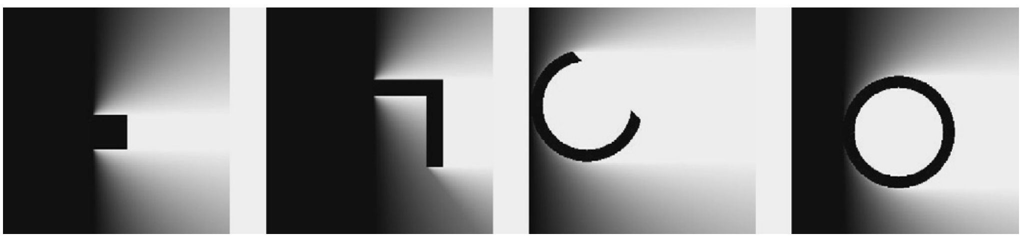

Fig. 10. A few examples of $\mu_\alpha(R)$ for $\alpha_1 = \alpha_2 = 0$ corresponding to the relative position "right" (high grey values correspond to high membership values) using the morphological (angle of visibility) method, for different types of reference objects (reference objects are black).

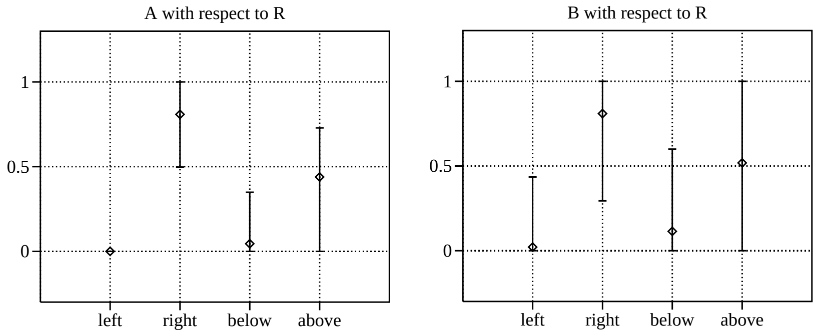


Fig. 13. Left: Results obtained for the object $A$ of Fig. 1 with respect to reference object $R$. The three given values correspond to necessity (lowest value of the bar) and possibility (highest value of the bar) degrees, and to the average value (diamond). Right: Results obtained for the object $B$ of Fig. 1 with respect to reference object $R$.

the object for the degrees of satisfaction of the relation to the reference object. This can be interpreted as the ambiguity of the relationship, or as the ignorance we have about a precise value for this relation.

These results fit well the intuition. Object $A$ is found mainly on the right of reference object $R$ and to some extent also above it. The lower part of $A$ is not above $R$ and therefore the necessity for this relation is equal to 0. Similarly, $B$ is found to be mainly on the right and above object $R$. This last relation is even more ambiguous than in the case of $A$, since a part of $B$ is completely above $R$ while another is completely not above it. We obtain in this case the maximum ambiguity, represented by an interval $[0,1]$ for the necessity and possibility.

The average values provide a summary of the satisfaction of the relationship. Of course, it is but one possible way to provide a global measure. Other measures could be derived from the set of all values taken by $\mu_\alpha(R)(P)$ for $P$ belonging to the considered object, or from the fuzzy histogram of these values.

A propagation algorithm has been proposed in Ref. [36], that considerably reduces the complexity of the algorithm, and that makes it tractable even in the 3D fuzzy case.

This kind of morphological approach is similar to the one proposed in Ref. [41], where the fuzzy landscape is defined as a dilation with a radial structuring element. The assessment of a relative position is then performed by aggregating (as in the aggregation method) the values obtained by performing a conditional dilation, i.e. by taking the intersection of the dilation of the first object with the other object. This method can be extended to fuzzy objects and to 3D objects, in a way similar to the one proposed in Refs. [28,29].

## 4. Comparative study of fuzzy definitions

### 4.1. Types of basic elements

A first perspective along which the various definitions can be compared is the basic information on which they rely, i.e. the geometrical elements that constitute the core of the computation:

- in angle-based approaches, the main elements are points, from which angles and angle histogram are computed; the dimension of these elements is 0, whatever the dimension of the considered space;
- in the force histogram method, the main elements are longitudinal sections, i.e. 1D elements;
- in the projection method, they are projections of the objects in the desired direction; therefore the dimension is 1 for 2D objects, and it would be $n-1$ for an extension of the method to $n$-dimensional objects;
- in the morphological approach, the basic elements are the objects themselves, and all computation is performed in the image space, i.e. in $n$ dimensions for $n$-dimensional objects.

### 4.2. Types of evaluation

Another line of comparison is the type of result that each method provides for the evaluation of a spatial relationship. It may be a number, an interval or a fuzzy number:

- one number is provided by the centroid and the aggregation methods;
- for the angle method, the final result is also one number; however, we can consider the whole compatibility fuzzy set as a result, which is then a fuzzy number, or apply a fuzzy pattern matching approach instead of extension principle to get an interval, as suggested in Ref. [28];
- the method of histogram of forces also provides a number;
- the projection method provides a number;
- the morphological method provides an interval as a result, along with an average value; considering the whole histogram of $\mu_\alpha(R)(x)$, for all $x \in A$ provides an evaluation as a fuzzy set.

Moreover, all methods can be applied for a set of directions, and the result can be considered as a possibility distribution on the space of directions, expressing the degrees to which an object $A$ is in various directions with respect to a reference object $R$. This possibility distribution is a complete description of the relative position between both objects. For some applications, this may be more useful than considering only four basic directions, since it constitutes a richer information.

### 4.3. Algebraic properties

#### 4.3.1. Reflexivity

Reflexivity concerns the direction of an object with respect to itself. A relation is reflexive if we have $\mu_\alpha^A(A) = 1$.

For the centroid method, since $\theta(c_A, c_A)$ is not defined, it could be set to $\alpha$ in order to achieve reflexivity. However, this is not meaningful since the value would depend on $\alpha$.

For the aggregation method and the compatibility method, reflexivity is not satisfied in general. The same holds for the method based on histogram of forces.

For the projection method, reflexivity is not satisfied in general.

For the morphological method, when defining $\mu_{\alpha_1,\alpha_2}(R)$, we have two choices for the points of $R$. The first one is to take $\mu_{\alpha_1,\alpha_2}(R)(P) = 1$ for those points. In this case, the degree of satisfaction of a relationship is reflexive.

However it is not clear if reflexivity property is really necessary in practice, since for most applications, such relationships are assessed between different objects only, and reflexivity is then not meaningful.

#### 4.3.2. Symmetry

Symmetry expresses that the degree to which $A$ is in direction $\alpha$ with respect to $R$ should be equal to the degree to which $R$ is in direction $\alpha + \pi$ with respect to $A$:

$$\mu_\alpha^R(A) = \mu_{\alpha+\pi}^A(R). \tag{29}$$

This property is certainly highly desirable for pattern recognition purposes, since the conclusion should not change if one or the other object is taken as reference.

For the centroid method, we have

$$\mu_{\pi+\alpha}[\theta(c_A, c_R)] = \mu_\alpha[\theta(c_R, c_A)], \tag{30}$$

which implies that symmetry is satisfied. Note that this assumes a consistent definition of the $\mu_\alpha$'s, as the one suggested in Eq. (1).

For the aggregation method, symmetry holds if and only if the weights $w_{xy}$ are chosen in a symmetrical way (i.e. $\forall x, y,\ w_{xy} = w_{yx}$).

For the angle and compatibility method, we have

$$H^R(A)(\theta) = H^A(R)(\theta + \pi), \tag{31}$$

which expresses a symmetry property for the angle histogram, from which the symmetry of the relative position follows.

For the histogram of forces method, the longitudinal sections of both objects are the same for $\alpha$ and for $\alpha + \pi$, but their roles are reversed in the computation. Therefore, the symmetry property holds for this definition.

For the projection method, symmetry holds.

For the morphological approach, the following symmetry property holds for the possibility:

$$\forall \alpha_1 \in [0, 2\pi],\ \forall \alpha_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\Pi_{\alpha_1,\alpha_2}^R(A) = \Pi_{\pi+\alpha_1,-\alpha_2}^A(R), \tag{32}$$

which reduces in 2D to

$$\forall \alpha \in [0, 2\pi], \qquad \Pi_\alpha^R(A) = \Pi_{\pi+\alpha}^A(R). \tag{33}$$

Other properties related to symmetry are stated in Ref. [27] and relate the degree to which $A$ is to the left of $B$ and the degree to which $A$ is to the right of $B$. Here we consider extensions of these properties, by considering, for any direction $\alpha$, $\mu_\alpha^R(A)$ and $\mu_{\alpha+\pi}^R(A)$. Of course, because of the possible shape complexity, no strong links can be expected between these two values. The properties stated in Ref. [27] are:

- S1: $\mu_\alpha^R(A) = 1 \Rightarrow \mu_{\alpha+\pi}^R(A) = 0$;
- S2: $\mu_\alpha^R(A) = 0 \Rightarrow \mu_{\alpha+\pi}^R(A) = 1$;
- S3: $\mu_\alpha^R(A) > 0 \Rightarrow \mu_{\alpha+\pi}^R(A) < 1$;
- S4: $\mu_\alpha^R(A) < 1 \Rightarrow \mu_{\alpha+\pi}^R(A) > 0$.

The idea behind these properties is that an object $A$ cannot be said to be completely on the right of $R$ if it has parts that are to the left of $R$.

For the centroid method, property S1 holds. For S2, a weaker property holds, since we just have $\mu_\alpha^R(A) = 0 \Rightarrow \mu_{\alpha+\pi}^R(A) > 0$ (but not necessarily $\mu_{\alpha+\pi}^R(A) = 1$). For S3, a stronger property holds, since we have $\mu_\alpha^R(A) > 0 \Rightarrow \mu_{\alpha+\pi}^R(A) = 0$. The implication of property S4 does not hold in general.

For the aggregation method, S1 and S3 hold, S2 and S4 do not hold in general.

For the compatibility method, none of these properties hold in general. Similar results are obtained for the histogram of forces method.

For the projection method, all properties S1–S4 holds, as shown in Ref. [27].

For the morphological approach, we have slightly different properties, that may involve both necessity and possibility:

- $\Pi_\alpha^R(A) = 1 \Rightarrow N_{\alpha+\pi}^R(A) = 0$ if $R$ is convex;
- $\Pi_\alpha^R(A) = 0 \Rightarrow N_{\alpha+\pi}^R(A) > 0$;
- $N_\alpha^R(A) = 1 \Rightarrow N_{\alpha+\pi}^R(A) = 0$ if $R$ is convex;
- $N_\alpha^R(A) = 0 \Rightarrow \Pi_{\alpha+\pi}^R(A) > 0$.

### 4.4. Geometrical properties

#### 4.4.1. Invariance with respect to geometric transformations

Invariance with respect to geometric transformations expresses that the relative position between two objects should not change if both objects are translated by the same translation vector, if they are scaled by the same factor or if they are rotated by the same rotation (the direction in which the relative position is assessed being rotated as well).

Since angles are invariant by translation, scaling and rotation (up to a rotation of $\alpha$), these invariance properties hold for all definitions based on angle computation (centroid, aggregation, compatibility methods).

These invariance properties are satisfied also by the method based on histogram of forces.

For the projection method, invariance by translation and rotation is satisfied.

For the morphological method, the definition is invariant with respect to translation, rotation and scaling, for 2D and 3D objects (crisp and fuzzy). This property holds for necessity, possibility, and average value as well.

#### 4.4.2. Characterization of extreme situations

This section aims at characterizing extreme situations, which include two types of problems. The first one consists in determining the situations for $A$ and $R$ for which we have $\mu_\alpha^R(A) = 1$ and $\mu_\alpha^R(A) = 0$, respectively. The second one concerns the evaluation provided by the different methods for objects that have separated supports in the direction $\alpha$, i.e. for which there exists an hyperplane orthogonal to the considered direction, that separates the space in two half-spaces, each of the objects being completely included in different half-space. For instance, $A$ is “completely” to the right of “$R$” (i.e. the supports of $A$ and $R$ are separated in the $x$-direction) if the $x$-coordinates of the points of $A$ are all greater than the $x$-coordinates of the points of $R$. In the example shown in Fig. 1, the supports of $A$ and $R$ are separated in the $x$-direction, while the supports of $B$ and $R$ are not.

Let us consider the first problem. For the centroid method, we have the following results:

$$\mu_\alpha^R(A) = 0 \quad \Leftrightarrow \quad \theta(c_R, c_A) \in \left[-\pi + \alpha, -\frac{\pi}{2} + \alpha\right] \cup \left[\frac{\pi}{2} + \alpha, \pi + \alpha\right], \tag{34}$$

$$\mu_\alpha^R(A) = 1 \quad \Leftrightarrow \quad \theta(c_R, c_A) = \alpha. \tag{35}$$

For the aggregation method, $\mu_\alpha^R(A) = 0$ if and only if $A$ and $R$ have separated supports in the direction $\alpha$, and $\mu_\alpha^R(A) = 1$ if and only if $A$ and $R$ are alignated (and separated) segments in the direction $\alpha$. Note that this condition is severe since it imposes a strict condition on the shape of the objects.

For the histogram of angles with the compatibility method, $\mu_\alpha^R(A) = 0$ if and only if $A$ and $R$ have separated supports in the direction $\alpha$, and $\mu_\alpha^R(A) = 1$ if and only if $H^R(A)(\alpha) = 1$ and $H^R(A)(\theta) = 0$ for $\theta \neq \alpha$ and $\theta$ belonging to the support of $\mu_\alpha$. This means that $A$ and $R$ are composed of separated segments in the direction $\alpha$ but $A$ may also have parts that are opposite to $\alpha$ with respect to $\theta$ (i.e. parts that contribute to the angle histogram outside of the support of $\mu_\alpha$).

Let us now consider the second problem, where $A$ and $R$ have separated supports in the direction $\alpha$.

Due to the previous characterizations of extreme cases, we do not have in general $\mu_\alpha^R(A) = 1$ if $A$ and $R$ have separated supports in the direction $\alpha$, for any of the angle based methods (centroid, aggregation and compatibility).

Both problems are addressed together in Ref. [27] for the projection approach, and the following properties hold: $\mu_\alpha^R(A) = 0$ if and only if the support of $A$ is completely in the direction $\alpha + \pi$ with respect to $R$. In the same way, $\mu_\alpha^R(A) = 1$ if and only if the support of $A$ is completely in the direction $\alpha$ with respect to $R$. This strong result is due to the projection, that summarizes all information on only one axis.

For the morphological method, we have the following results:

- $\Pi_\alpha^R(A) = 0$ if and only if $A$ is completely in the direction opposite to $\alpha$ with respect to $R$.
- $N_\alpha^R(A) = 0$ if and only if there exists at least one point of $A$ that is completely in the direction opposite to $\alpha$ with respect to $R$.
- $N_\alpha^R(A) = 1$ if and only if $A$ is included in a band limited by two parallel lines in the direction $\alpha$ that limit $R$.
- $\Pi_\alpha^R(A) = 1$ if and only if there exists at least one point of $A$ in this band.
- If $A$ and $R$ have separated supports, we do not necessarily obtain 1.

#### 4.4.3. Influence of the distance between objects

Let us assume here that object $A$ is moving in the direction $\beta$ with respect to $R$. The question addressed in this section is the following: what is the behavior of $\mu_\alpha^R(A)$ when the distance between the objects increases? We show that, in general, the limit value can be predicted from the displacement angle.

In 2D, we assume that each point of $A$ undergoes a translation by the vector $\lambda\vec{u}_\beta$. The generalization to 3D is straightforward. The limit value of $\mu_\alpha^R(A + \lambda\vec{u}_\beta)$ is obtained when $\lambda$ goes towards $+\infty$.

For the centroid and the aggregation methods, we have

$$\lim_{\lambda \to +\infty} \mu_\alpha^R(A + \lambda\vec{u}_\beta) = \mu_\alpha(\beta - \alpha). \tag{36}$$

This means that, at the limit, the objects are seen as points.

For the compatibility method, when the distance increases, $H^R(A)$ becomes concentrated around the value

$\beta-\alpha$ and is a Dirac function at $\beta-\alpha$ at the limit. Then the centroid of the compatibility fuzzy set is again $\mu_\alpha(\beta-\alpha)$.

Exactly the same result is obtained for the method based on histogram of forces.

For the projection method, a completely different behavior can be observed. If $\beta=\alpha+\pi/2$ or $\beta=\alpha-\pi/2$, then the distance has no influence since the projections do not change. On the contrary, if $\beta$ takes any other value, then the supports are going farther from each other. Two cases have to be considered:

(1) If $\beta\in]\alpha-\pi/2,\alpha+\pi/2[$, then $A$ is more and more in the direction $\alpha$ with respect to $R$. At the limit, the projections have disjoint supports and $\mu_\alpha^R(A)=1$.
(2) If $\beta\in]\alpha+\pi/2,\alpha+3\pi/2[$, then $A$ is less and less in the direction $\alpha$ with respect to $R$. At the limit, the projections have disjoint supports and $\mu_\alpha^R(A)=0$.

For the morphological approach, the limit of $\mu_\alpha(R)$ when the point goes to infinity in the direction $\beta$ is equal to

$$f(\beta-\alpha)=1-\frac{2(\beta-\alpha)}{\pi}. \tag{37}$$

This limit does not depend on the considered point. Therefore all three values $\Pi_\alpha^R(A)$, $N_\alpha^R(A)$, and $M_\alpha^R(A)$ have the same limit when $A$ is at infinite distance of $R$ in the direction $\beta$. This result is equivalent to the value $\mu_\alpha(\beta-\alpha)$ obtained with the other approaches. In 3D, we obtain similar results. For instance, when the $x$ coordinate of a point goes to infinity (i.e. the point moves in the horizontal direction away from the reference object), then the limit of $\mu_\alpha(R)$ is equal to

$$1-\frac{2}{\pi}\arccos(\cos\alpha_1\cos\alpha_2). \tag{38}$$

Again the three values defining the relative position have an equal limit.

These results show that when the distance between the objects increases, the objects are seen as points. The value of their relative position can be predicted only from the direction of interest and the direction in which one object goes far away from the reference object. Therefore, the shape of the objects no longer plays a role in the assessment of their relative position.

### 4.5. Implementation issues

Let us consider now the computation complexity of the different approaches presented above. We assume here that both objects have numbers of points of the same order, say $N$, which is also of the same order as the number of points in the image (it is enough to restrict the image to the support of the objects to achieve this condition). [1]

[1] A more precise complexity evaluation for the morphological approach is provided in Refs. [29,36].

For the centroid method, the computation of the center of gravity has a linear complexity, and therefore the whole computation is $O(N)$.

For angle based methods (aggregation and compatibility), the complexity is $O(N^2)$ for the computation of the histogram. For the compatibility method, considerable computation time comes also from the computation of the compatibility fuzzy set (say $C$), and is directly related to the discretization of angles that is used. Therefore the global complexity is $O(N^2+C)$. Note than in general $C$ is small in comparison to $N^2$.

For the method based on histogram of forces, the algorithm proposed in Ref. [25] is $O(N\sqrt{N})$.

For the projection based approach, the computation of the projection is $O(N)$. If we assume that the average number of points in the projection is $\sqrt{N}$, then the computation of $A^{\leftarrow}$ is $O(\sqrt{N})$. However, in the worst case (very elongated structure in the direction $\alpha$), it is $O(N)$. Therefore, the double integration involved in the computation of $\mu_\alpha^R(A)$ is $O(N^2)$, which makes a global computation of the order of $O(N^2)$.

For the morphological approach, the complexity is $O(N(1+2n_V))$ where $n_V$ denotes the size of the neighborhood used for the propagation (typically only a few points). Therefore the complexity is of the order of $N$. One advantage of this approach is that the fuzzy landscape $\mu_\alpha^R$ can be computed only once if the position of several objects has to be assessed with respect to the same reference object $R$. Once $\mu_\alpha^R$ is computed, the complexity is strictly linear for each object. Moreover, with the propagation algorithm described in Ref. [36], the complexity is in general much lower than this worst-case complexity. This computation time gain is not possible with the other approaches, since the histogram of angles or of forces has to be re-computed for any pair of objects.

Let us now look at the extension to 3D objects and to fuzzy objects, and at its complexity.

Although not always stated in the original papers presenting the definitions, all definitions can be potentially extended to 3D. A detailed description of these extensions is provided for the compatibility method in Ref. [30] and for the morphological approach in Refs. [29,36]. The computation time becomes rather important for angle based methods. For the morphological approach, experiments on various types of volumes have shown reasonable computation time with the propagation algorithm.

Extensions to fuzzy objects can be obtained by integrating results of the method performed on each $\alpha$-cut [13]. However this becomes very heavy. For instance for a $O(N^2)$ method, if $n$ $\alpha$-cuts are taken, then the complexity becomes $O(N^2 n)$. One advantage of the morphological approach is that it is directly applicable to fuzzy objects, without increase of the complexity [36]. Histogram-based approaches can also be applied without additional complexity, just by taking each point into account in the histogram to the amount of its membership to the considered fuzzy set [23]. An extension to fuzzy objects is also proposed in Ref. [27] for the projection method, that seems to be rather expensive.

### 4.6. Comparison on some examples

In this section, we give some numerical results, on a few simple examples. Five pairs of objects are studied, illustrated in Figs. 1 and 14. The first two examples ($A$ with respect to $R$ and $B$ with respect to $R$) already served to illustrate the main definitions. The three other examples are constituted by three rectangles for which the relative position with respect to a concave object is assessed. This aims to show the influence of concavities and of distance to the reference object.

Fig. 14. An example of a concave reference object and a rectangle having an increasing distance to it.

Tables 1 and 2 show the results for object $A$ with respect to object $R$, according to various methods. They all agree to say that $A$ is mainly to the right of $R$. The degree of being to the right increases with the value of $r$, since the part of $A$ which is to the right of $R$ is the closest one to $R$. On the contrary, the degree of being above decreases with $r$. The values are somewhat different for all approaches, but since the ranking and the general behavior is the same, no conclusion concerning a more favorable approach can be derived from this example.

Tables 3 and 4 show the results for object $B$ with respect to object $R$. For these objects, two main relations are satisfied: right and above. The centroid method does not account well for the above relation, for which it gives a very low value. This shows one of the limitations of this approach which is too simple in that it reduces the data too much. Since the part of $B$ which is above $R$ is closer than the one to its right, the values of being right decrease with $r$ while the values of being above increase. The morphological approach highlights the ambiguity of the relations for these objects. Parts of $B$ satisfy completely the above relation for instance, while other parts do not satisfy it at all. The non zero degrees obtained for the relation below for instance are due to some points of $B$ that are indeed partially below $R$.

Table 1
Relative position of object $A$ (rectangle) with respect to object $R$ (square) of Fig. 1, using centroid, aggregation and compatibility methods. Angle or force histograms are computed using $r = 0, 2$ and 5 (the angle histogram method corresponds to $r = 0$)

| Object $A$ with respect to object $R$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Relation | Centroid | Aggregation | | | Compatibility | | |
| | | $r=0$ | $r=2$ | $r=5$ | $r=0$ | $r=2$ | $r=5$ |
| Left | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Right | 0.76 | 0.73 | 0.79 | 0.86 | 0.62 | 0.67 | 0.75 |
| Below | 0.00 | 0.00 | 0.01 | 0.01 | 0.05 | 0.06 | 0.06 |
| Above | 0.24 | 0.27 | 0.20 | 0.13 | 0.38 | 0.33 | 0.25 |

Table 3
Relative position of object $B$ with respect to object $R$ of Fig. 1, using centroid, aggregation and compatibility methods. Angle or force histograms are computed using $r = 0, 2$ and 5

| Object $B$ with respect to object $R$ | | | | | | | |
|---|---|---|---|---|---|---|---|
| Relation | Centroid | Aggregation | | | Compatibility | | |
| | | $r=0$ | $r=2$ | $r=5$ | $r=0$ | $r=2$ | $r=5$ |
| Left | 0.00 | 0.00 | 0.00 | 0.01 | 0.05 | 0.06 | 0.08 |
| Right | 0.83 | 0.63 | 0.54 | 0.33 | 0.55 | 0.49 | 0.35 |
| Below | 0.00 | 0.03 | 0.02 | 0.01 | 0.17 | 0.16 | 0.15 |
| Above | 0.17 | 0.34 | 0.43 | 0.66 | 0.45 | 0.51 | 0.65 |

Table 2
Relative position of object $A$ (rectangle) with respect to object $R$ (square) of Fig. 1, using fuzzy pattern matching approach (FPM) between relationships and angle or force histograms computed using $r = 0, 2$ and 5, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Object $A$ with respect to object $R$ | | | | |
|---|---|---|---|---|
| Relation | FPM | | | Morphological approach |
| | $r=0$ | $r=2$ | $r=5$ | |
| Left | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] $M = 0.00$ |
| Right | [0.37, 0.68] | [0.39, 0.98] | [0.58, 1.00] | [0.50, 1.00] $M = 0.81$ |
| Below | [0.00, 0.10] | [0.00, 0.12] | [0.00, 0.13] | [0.00, 0.35] $M = 0.05$ |
| Above | [0.32, 0.63] | [0.02, 0.61] | [0.00, 0.42] | [0.00, 0.73] $M = 0.44$ |

Table 4
Relative position of object $B$ with respect to object $R$ of Fig. 1, using fuzzy pattern matching approach (FPM) between relationships and angle or force histograms computed using $r = 0, 2$ and 5, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Object $B$ with respect to object $R$ | | | | |
|---|---|---|---|---|
| Relation | FPM | | | Morphological approach |
| | $r = 0$ | $r = 2$ | $r = 5$ | |
| Left | [0.00, 0.06] | [0.00, 0.10] | [0.00, 0.11] | [0.00, 0.44] $M = 0.02$ |
| Right | [0.34, 0.81] | [0.10, 0.75] | [0.00, 0.39] | [0.29, 1.00] $M = 0.81$ |
| Below | [0.00, 0.28] | [0.00, 0.26] | [0.00, 0.11] | [0.00, 0.60] $M = 0.11$ |
| Above | [0.19, 0.66] | [0.25, 0.91] | [0.61, 1.00] | [0.00, 1.00] $M = 0.52$ |

Table 5
Relative position of the first rectangle with respect to the concave object of Fig. 14, using centroid, aggregation and compatibility methods. Angle or force histograms are computed using $r = 0, 2$ and 5

| Rectangle 1 with respect to concave object | | | | | | | |
|---|---|---|---|---|---|---|---|
| Relation | Centroid | Aggregation | | | Compatibility | | |
| | | $r = 0$ | $r = 2$ | $r = 5$ | $r = 0$ | $r = 2$ | $r = 5$ |
| Left | 0.00 | 0.14 | 0.16 | 0.12 | 0.44 | 0.46 | 0.37 |
| Right | 0.50 | 0.36 | 0.36 | 0.23 | 0.50 | 0.50 | 0.38 |
| Below | 0.00 | 0.14 | 0.13 | 0.15 | 0.47 | 0.45 | 0.55 |
| Above | 0.50 | 0.36 | 0.36 | 0.50 | 0.50 | 0.51 | 0.63 |

Tables 5 and 6 show the results for the first rectangle with respect to the concave object of Fig. 14. This rectangle is well inside the concavity, and therefore is expected to satisfy several relations. Again the centroid approach is too restrictive and does not give any positive value for left and below, while the other measures do. The part that is the closest to the concave object is the bottom part of the rectangle, that is just above the bottom part of the reference object. This explains why the evaluation for the relation above increases with $r$. In general, the compatibility method gives higher values than the aggregation method, and thus better represents the ambiguity of the relative position between these objects and the fact that the concave object almost surrounds the rectangle. Concerning pattern matching approaches, the morphological one (working in the spatial domain) provides better results than the one working on histograms. For instance, the relation right is completely satisfied since every point of the rectangle is visible in this direction from the reference object.

Tables 7 and 8 show the results for the second rectangle with respect to the concave object of Fig. 14. This rectangle is now completely outside the concavity. So the left

Table 7
Relative position of the second rectangle with respect to the concave object of Fig. 14, using centroid, aggregation and compatibility methods. Angle or force histograms are computed using $r = 0, 2$ and 5

| Rectangle 2 with respect to concave object | | | | | | | |
|---|---|---|---|---|---|---|---|
| Relation | Centroid | Aggregation | | | Compatibility | | |
| | | $r = 0$ | $r = 2$ | $r = 5$ | $r = 0$ | $r = 2$ | $r = 5$ |
| Left | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Right | 0.98 | 0.81 | 0.78 | 0.84 | 0.69 | 0.67 | 0.73 |
| Below | 0.00 | 0.03 | 0.03 | 0.04 | 0.16 | 0.16 | 0.16 |
| Above | 0.02 | 0.15 | 0.20 | 0.12 | 0.31 | 0.33 | 0.27 |

Table 6
Relative position of the first rectangle with respect to the concave object of Fig. 14, using fuzzy pattern matching approach (FPM) between relationships and angle or force histograms computed using $r = 0, 2$ and 5, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Rectangle 1 with respect to concave object | | | | |
|---|---|---|---|---|
| Relation | FPM | | | Morphological approach |
| | $r = 0$ | $r = 2$ | $r = 5$ | |
| Left | [0.00, 0.58] | [0.00, 0.51] | [0.00, 0.33] | [0.44, 1.00] $M = 0.85$ |
| Right | [0.08, 1.00] | [0.00, 0.71] | [0.00, 0.33] | [1.00, 1.00] $M = 1.00$ |
| Below | [0.00, 0.63] | [0.00, 0.50] | [0.00, 0.30] | [0.39, 1.00] $M = 0.89$ |
| Above | [0.00, 0.92] | [0.29, 1.00] | [0.67, 1.00] | [1.00, 1.00] $M = 1.00$ |

Table 8
Relative position of the second rectangle with respect to the concave object of Fig. 14, using fuzzy pattern matching approach (FPM) between relationships and angle or force histograms computed using $r = 0, 2$ and 5, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Rectangle 2 with respect to concave object | | | | |
|---|---|---|---|---|
| Relation | FPM | | | Morphological approach |
| | $r = 0$ | $r = 2$ | $r = 5$ | |
| Left | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] $M = 0.00$ |
| Right | [0.46, 0.97] | [0.42, 1.00] | [0.74, 1.00] | [1.00, 1.00] $M = 1.00$ |
| Below | [0.00, 0.29] | [0.00, 0.24] | [0.00, 0.20] | [0.11, 0.61] $M = 0.32$ |
| Above | [0.03, 0.54] | [0.00, 0.58] | [0.00, 0.26] | [0.15, 0.83] $M = 0.48$ |

relation should not be satisfied anymore. This is indeed what is obtained by all methods. Also the relation below has now a very low degree, and the main relation is right. Again this relation is found to be completely satisfied, without any ambiguity, by the morphological approach.

Tables 9 and 10 show the results for the third rectangle with respect to the concave object of Fig. 14. The rectangle is still completely outside the concavity, but now it is farther from the reference object. This leads to results that become more and more binary in favor of the relation right, the other ones having very low values. This fits well the intuition. From this example, it can be observed than when the relation becomes less ambiguous, all methods work quite similarly. Again the morphological approach gives a completely non-ambiguous result for the relation right.

Table 9
Relative position of the third rectangle with respect to the concave object of Fig. 14, using centroid, aggregation and compatibility methods. Angle or force histograms are computed using $r = 0, 2$ and 5

| Rectangle 3 with respect to concave object | | | | | | | |
|---|---|---|---|---|---|---|---|
| Relation | Centroid | Aggregation | | | Compatibility | | |
| | | $r = 0$ | $r = 2$ | $r = 5$ | $r = 0$ | $r = 2$ | $r = 5$ |
| Left | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| Right | 0.99 | 0.94 | 0.94 | 0.94 | 0.90 | 0.89 | 0.89 |
| Below | 0.00 | 0.01 | 0.01 | 0.00 | 0.06 | 0.06 | 0.05 |
| Above | 0.01 | 0.05 | 0.05 | 0.06 | 0.10 | 0.11 | 0.11 |

## 5. Summary and conclusion

A summary of the main properties discussed in the previous section is provided in Table 11.

Another aspect that is very important is the type of questions each definition is able to answer, or dedicated to. These questions can take different forms, e.g.:

- What are the spatial relationships between two given objects?
- To which degree a given spatial relation holds between two objects?
- What are the regions of the space where a spatial relationship is satisfied (to some degree) with respect to a reference object?

Table 10
Relative position of the third rectangle with respect to the concave object of Fig. 14, using fuzzy pattern matching approach (FPM) between relationships and angle or force histograms computed using $r = 0, 2$ and 5, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Rectangle 3 with respect to concave object | | | | |
|---|---|---|---|---|
| Relation | FPM | | | Morphological approach |
| | $r = 0$ | $r = 2$ | $r = 5$ | |
| Left | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] | [0.00, 0.00] $M = 0.00$ |
| Right | [0.76, 0.98] | [0.76, 0.98] | [0.76, 0.98] | [1.00, 1.00] $M = 1.00$ |
| Below | [0.00, 0.14] | [0.00, 0.13] | [0.00, 0.08] | [0.06, 0.31] $M = 0.19$ |
| Above | [0.02, 0.24] | [0.02, 0.24] | [0.02, 0.24] | [0.08, 0.38] $M = 0.23$ |

Table 11
Comparison table. The sign $\surd$ means that the property is satisfied. The sign—means that it is not satisfied in general

| Method | Basic elements | Evaluation as a |
|---|---|---|
| Centroid | Points | Number |
| Aggregation | Points | number |
| Compatibility | Points | Number/fuzzy number |
| Hist. of Forces | Segments | Number |
| Projection | Projections | Number |
| Morphology | Whole objects | interval/fuzzy set |

| Method | Reflexivity | Symmetry | Geom. invariance | Distance |
|---|---|---|---|---|
| Centroid | — | $\surd$ | $\surd$ | $\mu_\alpha(\beta-\alpha)$ |
| Aggregation | — | $\surd$ | $\surd$ | $\mu_\alpha(\beta-\alpha)$ |
| Compatibility | — | $\surd$ | $\surd$ | $\mu_\alpha(\beta-\alpha)$ |
| Hist. of Forces | — | $\surd$ | $\surd$ | $\mu_\alpha(\beta-\alpha)$ |
| Projection | — | $\surd$ | $\surd$ | 0, 1 or no influence |
| Morphology | $\surd$ | $\surd$ (for $\Pi$) | $\surd$ | $1-\frac{2(\beta-\alpha)}{\pi}(=\mu_\alpha(\beta-\alpha))$ |

| Method | Complexity | 3D objects | Fuzzy objects |
|---|---|---|---|
| Centroid | $O(N)$ | $\surd$ | $\surd$ |
| Aggregation | $O(N^2)$ | $\surd$ | $\surd$ |
| Compatibility | $O(N^2+C)$ | $\surd$ | $\surd$ |
| Hist. of Forces | $O(N\sqrt{N})$ | Could be extended | $\surd$ |
| Projection | $O(N^2)$ | Could be extended | $\surd$ |
| Morphology | $O(N^2)$ | $\surd$ | $\surd$ (applies directly) |

An important feature of angle histogram and force histogram is that they provide a general description of the directional relationships. From this general information, several ones can be deduced, as the degree of satisfaction of one specific relationship (for a particular direction), or the dominant relationship. This is not easy to obtain with the morphological approach, that needs one computation for each direction of interest. This approach is more dedicated to cases where we are interested in specified relations.

For problems where we have to assess the relative position of several objects with one reference object, the morphological approach may be more appropriate if the computation time is a strong requirement.

Most works in spatial reasoning aim at defining and assessing spatial relationships between objects, given these objects. But we may take another point of view, and address the problem of the representation of knowledge about expected relationships, in order to guide the reasoning process in the space, for exploring the space and search for the object that satisfies some relationships with respect to already known objects [42,43]. For the example of model-based pattern recognition, this leads to progressive recognition, where each object is detected and recognized by gathering constraints given by the model expressing relationships that this object has to satisfy with respect to previously recognized objects [44]. For this problem where the relationships are considered as constraints with respect to one object (rather than a relation between two objects), we can make use of a spatial representation, as fuzzy sets in the space. Each relationship is expressed as one fuzzy set, corresponding to a spatial constraint, restricting the space to the only regions where the relationship is satisfied [42]. This can be directly obtained using the morphological approach (it is the result of the first step), but is more difficult to obtain with other approaches, that do not work directly in the image space.

Applications can be anticipated for structural pattern recognition, image content description and spatial reasoning, inference of more complex relationships, reasoning with relationships.

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