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Logical dual concepts based on mathematical morphology in stratified institutions: applications to spatial reasoning

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ABSTRACT

Several logical operators are defined as dual pairs, in different types of logics. Such dual pairs of operators also occur in other algebraic theories, such as mathematical morphology. Based on this observation, this paper proposes to define, at the abstract level of institutions, a pair of abstract dual and logical operators as morphological erosion and dilation. Standard quantifiers and modalities are then derived from these two abstract logical operators. These operators are studied both on sets of states and sets of models. To cope with the lack of explicit set of states in institutions, the proposed abstract logical dual operators are defined in an extension of institutions, the stratified institutions, which take into account the notion of open sentences, whose satisfaction is parametrised by sets of states. A hint on the potential interest of the proposed framework for spatial reasoning is also provided.

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1. Introduction

There exists a profusion of logics but all of them satisfy the same structure defined by a syntax, a semantics and a calculus. Syntax gives both the language (signatures) and the formal rules that define well-formed formulas and theories. Semantics, so-called model theory, gives the mathematical meaning of all these syntactic notions, among others the rules that associate truth values to formulas. Finally, calculus, so-called proof theory, gives the inference rules that govern the reasoning and thus translate semantics into syntax as correctly as possible. To cope with the explosion of logics, a categorical abstract model-theory, the theory of institutions (Diaconescu, 2008; Goguen & Burstall, 1992), has been proposed, that generalises Barwise's 'Translation Axiom' (Barwise, 1974). Institutions then define both syntax and semantics of logics at an abstract level, independently of commitment to any particular logic. Later, institutions have been extended to propose a syntactic approach to truth (Diaconescu, 2006, 2008; Fiadeiro & Sernadas, 1988; Meseguer, 1989). For the sake of generalisation, in institutions signatures are simply defined as objects of a category

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and formulas built over signatures are simply required to form a set. All other contingencies such as inductive definition of formulas are not considered. However, in concrete logics (anyway all the particular logics considered in this paper as examples), the reasoning (both syntactic and semantic) is defined by induction on the structure of formulas. Indeed, usually, formulas are built from 'atomic' formulas by applying iteratively operators such as connectives, quantifiers or modalities. What we can then observe is that most of these logical operators come through dual pairs (conjunction and disjunction $\wedge$ and $\vee$, quantifiers $\forall$ and $\exists$, modalities $\Box$ and $\diamond$).

When looking at the algebraic properties of mathematical morphology (Bloch, Heijmans, & Ronse, 2007; Serra, 1982) on the one hand, and of all these dual operators on the other hand, several similarities can be shown, and suggest that links between institutions and mathematical morphology are worth to be investigated. This has already been done in the restricted framework of modal propositional logic (Bloch, 2002). In Bloch (2002), it was shown that modalities $\Box$ and $\diamond$ can be defined as morphological erosion and dilation. An interesting feature, based on properties of morphological operators, is that this leads to a set of axioms and inference rules which are de facto sound. In this paper, we propose to extend this work by defining, at the abstract level of institutions, a pair of abstract operators as morphological erosion and dilation. We will then show how to obtain standard quantifiers and modalities from these two abstract operators.

In mathematical morphology, erosion and dilation are operations that are defined, in a general deterministic and algebraic setting, on lattices, for instance on sets. Thus, they can be applied to formulas by identifying formulas with sets. We have two ways of doing this, either given a model M identifying a formula φ by the set of states η that satisfy φ, and classically denoted by $M \models_\eta \varphi$, or identifying φ by the set of models that satisfy it. As usual in logic, our abstract dual operators based on morphological erosion and dilation will be studied both on sets of states and sets of models. The problem is that institutions do not explicitly provide, given a model M, its set of states. This is why we will define our abstract logical dual operators based on erosion and dilation in an extension of institutions, the stratified institutions (Aiguier & Diaconescu, 2007). Stratified institutions have been defined in Aiguier and Diaconescu (2007) as an extension of institutions to take into account the notion of open sentences, whose satisfaction is parametrised by sets of states. For instance, in first-order logic, the satisfaction is parametrised by the valuation of unbound variables, while in modal logics it is further parametrised by possible worlds. Hence, stratified institutions allow for a uniform treatment of such parametrizations of the satisfaction relation within the abstract setting of logics as institutions.

Another interesting feature of the approach proposed in this paper is that mathematical morphology provides tools for spatial reasoning. Until now, mathematical morphology has been used mainly for quantitative representations of spatial relations, or semi-qualitative ones, in a fuzzy set framework (see e.g. Bloch, 2005). For qualitative spatial reasoning, several symbolic and logical approaches have been developed (see e.g. Aiello, Pratt-Hartman, & van Benthem, 2007; Aiello & van Benthem, 2002; Ligozat, 2012), but mathematical morphology has not been much used in this context to our knowledge. In this paper, inspired by the work that was done in Bloch (2002), Bloch (2006), Bloch et al. (2007), Bloch and Lang (2002) in the propositional and modal logic framework, we show how logical connectives based on morphological operators can be used for symbolic representations of spatial relations. Indeed, spatial relations are a main component of spatial reasoning Aiello et al. (2007), and several frameworks have been proposed to model spatial relations and reason about them in logical frameworks (see e.g. Bennett & Duntsch, 2007; Clementini & Felice, 1997; Cohn, Bennett, Gooday, & Gotts, 1997; Randell, Cui, & Cohn, 1992; van Benthem & Bezhanishvili, 2007 for topological relations, Ligozat, 2012; Mossakowski & Moratz, 2015 for directional relations, Renz & Nebel, 2007 for constraint based techniques for topology, distances and directions, and Bloch, 2005, 2006 for semi-qualitative representations in the framework of fuzzy sets). Since it is usual to introduce uncertainty in qualitative spatial reasoning, we propose to extend our abstract logical connectives based on erosion and dilation to the fuzzy case. This first requires to develop fuzzy reasoning in stratified institutions. Fuzzy (or many-valued) reasoning has an institutional semantics (Diaconescu, 2013, 2014). The approach proposed here is substantially similar to that proposed in Diaconescu (2013), although developed in stratified institutions.

The paper is organised as follows. Section 2 reviews some concepts, notations and terminology about institutions and stratified institutions which are used in this work. In Section 3 we propose to define abstractly the important concepts of Boolean connectives, quantifiers, and fuzzy reasoning in stratified institutions. Section 4 introduces a new way to build dual operators from the notion of morphological erosion and dilation operators. We study two ways to build such dual operators. We first define them from morphological dilation and erosion of formulas based on a structuring element, and then as algebraic erosion and dilation over the lattice of formulas. This last point allows us to define modalities when they are interpreted topologically as algebraic erosion and dilation. Finally, in Section 5, we show how these modalities can be interpreted for abstract spatial reasoning using qualitative representations of spatial relationships derived from mathematical morphology.

2. Stratified institutions

The notions introduced here make use of basic notions of category theory (category, functors, natural transformations, etc.). We do not present these notions in these preliminaries, but interested readers may refer to textbooks such as Barr and Wells (1990), MacLane (1971).

2.1. Institutions

Let us start by recalling the definition of institutions, over which stratified institutions are defined as an extension, by introducing the notion of states for models.

Definition 2.1 (Institution Goguen & Burstall, 1992): An **institution** $\mathcal{I} = (Sig, Sen, Mod, \models)$ consists of

- a category *Sig* whose objects are called *signatures* and are denoted Σ,
- a functor $Sen : Sig \to Set$ giving for each signature Σ a set $Sen(\Sigma)$ whose elements are called *sentences*,

- a contravariant functor $Mod : Sig^{op} \to Cat$ giving[1] for each signature a category, whose objects and arrows are called Σ-*models* and Σ-*morphisms* respectively, and
- a *Sig*-indexed family of relations $\models_\Sigma \subseteq Mod(\Sigma) \times Sen(\Sigma)$ called *satisfaction relation*, such that the following property, called the *satisfaction condition*, holds: $\forall \sigma : \Sigma \to \Sigma', \forall M' \in Mod(\Sigma'), \forall \varphi \in Sen(\Sigma)$,

$$M' \models_{\Sigma'} Sen(\sigma)(\varphi) \Leftrightarrow Mod(\sigma)(M') \models_\Sigma \varphi$$

Notation 2.2: The functor *Mod* can be extended to formulas. Hence, given a signature Σ and two formulas $\varphi, \psi \in Sen(\Sigma)$, we denote:

- $Mod(\varphi) = \{M \in Mod(\Sigma) \mid M \models_\Sigma \varphi\}$,
- $\varphi \models \psi \iff Mod(\varphi) \subseteq Mod(\psi)$, and
- $\varphi \equiv \psi \iff Mod(\varphi) = Mod(\psi)$.

Example 2.3: The following examples of institutions are of particular importance both in computer science and in this paper. Many other examples can be found in the literature (e.g. Diaconescu, 2008; Goguen & Burstall, 1992; Tarlecki, 1999).

Propositional Logic (PL) The category of signatures is *Set*, the category of sets and functions.
Given a signature P, the set of P-sentences is the least set of sentences finitely built over propositional variables in P, which are the *atomic formulas* for **PL**, and Boolean connectives in $\{\neg, \vee, \wedge, \Rightarrow\}$. Given a signature morphism $\sigma : P \to P'$, $Sen(\sigma)$ translates P-formulas to P'-formulas by renaming propositional variables according to σ.
Given a signature P, the category of P-models is $(\{0, 1\}^P, \leq)$ such that 0 and 1 are the usual truth values, and $\leq$ is a partial ordering such that $\nu \leq \nu'$ iff $\forall p \in P, \nu(p) \leq \nu'(p)$. Given a signature morphism $\sigma : P \to P'$, the forgetful functor $Mod(\sigma)$ maps a P'-model ν' to the P-model $\nu = \nu' \circ \sigma$.
Finally, satisfaction is the usual propositional satisfaction.

Many-sorted First Order Logic (FOL) Signatures are triplets (S, F, P) where S is a set of sorts, and F and P are sets of function and predicate names respectively, both with arities in $S^* \times S$ and S^+ respectively.[2] Signature morphisms $\sigma : (S, F, P) \to (S', F', P')$ consist of three functions between sets of sorts, sets of functions and sets of predicates respectively, the last two preserving arities.
Given a signature $\Sigma = (S, F, P)$, the atomic formulas (so-called Σ-atoms) are $p(t_1, \ldots, t_n)$ where $p : s_1 \times \ldots \times s_n \in P$ and $t_i \in T_F(X)_{s_i}$ $(1 \leq i \leq n, s_i \in S)$[3]. The set of Σ-sentences is the least set of formulas built over the set of Σ-atoms by finitely applying Boolean connectives in $\{\neg, \vee, \wedge, \Rightarrow\}$ and the quantifiers $\forall$ and $\exists$. Given a signature morphism $\sigma : \Sigma \to \Sigma'$, $Sen(\sigma)$ is the mapping defined by renaming functions and predicates according to σ.
Given a signature $\Sigma = (S, F, P)$, a Σ-model $\mathcal{M}$ is a family $\mathcal{M} = (M_s)_{s \in S}$ of sets (one for every $s \in S$), each one equipped with a function $f^{\mathcal{M}} : M_{s_1} \times \ldots \times M_{s_n} \to M_s$ for every $f : s_1 \times \ldots \times s_n \to s \in F$ and with a n-ary relation $p^{\mathcal{M}} \subseteq$

$M_{s_1} \times \ldots \times M_{s_n}$ for every $p : s_1 \times \ldots \times s_n \in P$. A model morphism $\mu : \mathcal{M} \to \mathcal{M}'$ is a mapping $\mu : M \to M'$ that preserves sorts (i.e. $\mu(M_s) \subseteq M'_s$ for each $s \in S$) such that for every $f : s_1 \times \ldots \times s_n \to s \in F$ and every $(a_1, \ldots, a_n) \in M_{s_1} \times \ldots \times M_{s_n}$, $\mu(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{M}'}(\mu(a_1), \ldots, \mu(a_n))$, and for every $p : s_1 \times \ldots \times s_n \in P$ and every $(a_1, \ldots, a_n) \in M_{s_1} \times \ldots \times M_{s_n}$, $(a_1, \ldots, a_n) \in p^{\mathcal{M}} \Longrightarrow (\mu(a_1), \ldots, \mu(a_n)) \in p^{\mathcal{M}'}$.

Given a signature morphism $\sigma : \Sigma = (S, F, P) \to \Sigma' = (S', F', P')$ and a Σ'-model $\mathcal{M}'$, $Mod(\sigma)(\mathcal{M}')$ is the Σ-model $\mathcal{M}$ defined for every $s \in S$ by $M_s = M'_s$, and for every function name $f \in F$ and every predicate name $p \in P$, by $f^{\mathcal{M}} = \sigma(f)^{\mathcal{M}'}$ and $p^{\mathcal{M}} = \sigma(p)^{\mathcal{M}'}$.

Finally, satisfaction is the usual first-order satisfaction.

Modal Propositional Logic (MPL) The category of signatures is the same as **PL**. For each set P, the P-sentences are formed from the elements of P by closing under Boolean connectives and unary modal connectives $\Box$ (necessity) and $\Diamond$ (possibility). A model (I, W, R) for a signature P, called *Kripke model*, consists of

- an index set I,
- a family $W = \{W^i\}_{i \in I}$ of 'possible worlds', which are functions from P to $\{0, 1\}$ (or equivalently subsets of P),
- an 'accessibility' relation $R \subseteq I \times I$.

A model homomorphism $h : (I, W, R) \to (I', W', R')$ consists of a function $h : I \to I'$ which preserves the accessibility relation, i.e. $(i, j) \in R$ implies $(h(i), h(j)) \in R'$, and such that $W^i \subseteq W'^{h(i)}$ for each $i \in I$. Given a signature morphism $\sigma : P \to P'$ and a P'-model (I', W', R'), $Mod(\sigma)((I', W', R'))$ is the P-model (I, W, R) such that $I = I'$, $R = R'$ and $W^i = \{\nu' \circ \sigma \mid \nu' \in W'^i\}$ for each $i \in I$.

The satisfaction of P-sentences by the Kripke P-models, $(I, W, R) \models_P \varphi$, is defined by $(I, W, R) \models^i_P \varphi$ for each $i \in I$, where $\models^i_P$ is defined by induction on the structure of the sentences as follows:

- $(I, W, R) \models^i_P p$ iff $p \in W^i$ for each $p \in P$,
- $(I, W, R) \models^i_P \varphi_1 \wedge \varphi_2$ iff $(I, W, R) \models^i_P \varphi_1$ and $(I, W, R) \models^i_P \varphi_2$; and similarly for the other Boolean connectives in $\{\vee, \Rightarrow, \neg\}$,
- $(I, W, R) \models^i_P \Box\varphi$ iff $(I, W, R) \models^j_P \varphi$ for each j such that $(i, j) \in R$, and
- $\Diamond\varphi$ is the same as $\neg\Box\neg\varphi$.

Topological MPL (TMPL) In **MPL**, the modalities $\Box$ and $\Diamond$ are interpreted relationally (i.e. in Kripke models). Here, they will be interpreted topologically. Hence, the category of signatures and the functor *Sen* are the same as **MPL**. Conversely, given a signature P, a P-model M is a topological space (X, τ) equipped with a valuation function $\nu : P \to \mathcal{P}(X)$.[4] Such models are called *topos-models*. A model morphism $h : (X, \tau, \nu) \to (X', \tau', \nu')$ is a continuous mapping such that for every $p \in P$, $h(\nu(p)) \subseteq \nu'(p)$. Given a signature morphism $\sigma : P \to P'$ and a P'-model (X', τ', ν'), $Mod(\sigma)((X', \tau', \nu'))$ is the P-model (X, τ, ν) such that $X = X'$, $\tau = \tau'$ and $\nu = \nu' \circ \sigma$.

The satisfaction of sentences by the topological models, $(X, \tau, \nu) \models_P \varphi$, is defined by $(X, \tau, \nu) \models^x_P \varphi$ for each $x \in X$, where $\models^x_P$ is defined by induction on the structure of the sentences as follows:

- $(X, \tau, \nu) \models^x_P p$ iff $x \in \nu(p)$ for each $p \in P$,

- $(X, \tau, \nu) \models^x_P \varphi_1 \wedge \varphi_2$ iff $(X, \tau, \nu) \models^x_P \varphi_1$ and $(X, \tau, \nu) \models^x_P \varphi_2$, and similarly for the other Boolean connectives in $\{\vee, \Rightarrow, \neg\}$,
- $(X, \tau, \nu) \models^x_P \Box\varphi$ iff there exists $O \in \tau$ s.t. $x \in O$ and $(X, \tau, \nu) \models^y_P \varphi$ for each $y \in O$, and
- $\Diamond\varphi$ is the same as $\neg\Box\neg\varphi$.

Hence, $\Box$and $\diamond$ are interpreted as both topological notions of interior and closure, respectively.

Metric MPL (MMPL) Here, modalities will be interpreted in a metric space. The institution **MMPL** has the same signatures and sentences as **MPL** and **TMPL**. Conversely, given a signature P, a P-model is a metric space (X, d) equipped with a valuation function $\nu : P \to \mathcal{P}(X)$. Such models are called *metric models*. A model morphism $h : (X, d, \nu) \to (X', d', \nu')$ is a continuous mapping such that for every $p \in P$, $h(\nu(p)) \subseteq \nu'(p)$. Given a signature morphism $\sigma : P \to P'$ and a P'-model (X', d', ν'), $Mod(\sigma)((X', d', \nu'))$ is the P-model (X, d, ν) such that $X = X'$, $d = d'$ and $\nu = \nu' \circ \sigma$.

The satisfaction of sentences by metric models $(X, d, \nu) \models_P \varphi$ is defined by $(X, d, \nu) \models^x_P \varphi$ for each $x \in X$, where $\models^x_P$ is defined by induction on the structure of the sentences as follows:

- atomic sentences and Boolean connectives are satisfied standardly;
- $(X, d, \nu) \models^x_P \Box\varphi$ iff $\exists\varepsilon > 0, \forall y \in X, d(x, y) < \varepsilon \Rightarrow (X, d, \nu) \models^y_P \varphi$;
- $\Diamond\varphi$ is the same as $\neg\Box\neg\varphi$.

2.2. Stratified institutions

Stratified institutions refine institutions by introducing the notion of states for models. Hence, each model M is equipped with a set $[\![M]\!]$, whose elements are called states, such as possible worlds for Kripke models.

The definition of stratified institutions given in Definition 2.4 slightly improves the original one in Aiguier and Diaconescu (2007) by considering a concrete category to equip models with states rather than the category of sets. This is motivated by the different applications developed in this paper such as the extensions of stratified institutions to modalities or to qualitative spatial reasoning, which require to consider in the first case sets equipped with binary relations, and in the second one topological or metric spaces.

Definition 2.4 (Stratified institution): A **stratified institution** consists of:

- a category Sig of signatures;
- a sentence functor $Sen : Sig \to Set$;
- a model functor $Mod : Sig^{op} \to Cat$;
- a 'stratification' $[\![\_]\!]$ which consists of a functor $[\![\_]\!]_\Sigma : Mod(\Sigma) \to \mathcal{C}$ for each signature $\Sigma \in Sig$ (**states of models**) where $\mathcal{C}$ is a concrete category (i.e. $\mathcal{C}$ is equipped with a faithful functor $\mathcal{U} : \mathcal{C} \to Set$), and a natural transformation $[\![\_]\!]_\sigma : [\![\_]\!]_{\Sigma'} \to [\![\_]\!]_\Sigma \circ Mod(\sigma)$ for each signature morphism $\sigma : \Sigma \to \Sigma'$ such that $\mathcal{U}([\![M']\!]_\sigma)$ is surjective for each $M' \in Mod(\Sigma')$ (and then by standard results in the category theory, $[\![M']\!]_\sigma$ is an epimorphism in $\mathcal{C}$)[5]. To simplify the notations

and when this does not raise ambiguities, we use in the rest of this paper the notation $[\![M]\!]_\Sigma$, given a signature Σ and a model $M \in Mod(\Sigma)$, to denote both the object in the concrete category $\mathcal{C}$ and the underlying set $\mathcal{U}([\![M]\!]_\Sigma)$. Similarly, given a signature morphism $\sigma : \Sigma \to \Sigma'$ and a Σ'-model M', we will use the notation $[\![M']\!]_\sigma$ to denote both the morphism $[\![M']\!]_\sigma$ in $\mathcal{C}$ and the mapping $\mathcal{U}([\![M']\!]_\sigma)$ in *Set*;

- a satisfaction relation between models and sentences which is parametrised by model states, $M \models^\eta_\Sigma \varphi$ where $\eta \in [\![M]\!]_\Sigma$ such that, $\forall \sigma : \Sigma \to \Sigma', \forall M \in Mod(\Sigma'), \forall \eta \in [\![M]\!]_{\Sigma'}, \forall \varphi \in Sen(\Sigma)$, the two following properties are equivalent:
  1. $Mod(\sigma)(M) \models^{[\![M]\!]_\sigma(\eta)}_\Sigma \varphi$,
  2. $M \models^\eta_{\Sigma'} Sen(\sigma)(\varphi)$.

Then, we can define for every $\Sigma \in Sig$, the satisfaction relation $\models_\Sigma \subseteq Mod(\Sigma) \times Sen(\Sigma)$ as follows:

$$M \models_\Sigma \varphi \text{ if and only if } M \models^\eta_\Sigma \varphi \text{ for all } \eta \in [\![M]\!]_\Sigma.$$

In Diaconescu (2017), a more concise definition of the stratification $[\![\_]\!]$ has been given based on the advanced notion of lax natural transformation. In our framework, following Diaconescu (2017), by considering *Cat* and *Sig* as 2-categories[6], $[\![\_]\!]$ can be defined as the lax natural transformation from the 2-functor *Mod* to the 2-functor $Conc : Sig \to \mathcal{C}$ which maps any signature to the concrete category $\mathcal{C}$. Hence, for every signature morphism $\varphi : \Sigma \to \Sigma'$, $[\![\_]\!]_\varphi : Id_\mathcal{C} \circ [\![\_]\!]_{\Sigma'} \to [\![\_]\!]_\Sigma \circ Mod(\varphi)$ is a natural transformation.

Now, this new definition of the stratification $[\![\_]\!]$ is slighty different from the one in Diaconescu (2017). The reason is that we project models on concrete categories and not on simple sets. Indeed, we provide a structure to sets of states.

Notation 2.5: Given a signature $\Sigma \in Sig$, a model $M \in Mod(\Sigma)$ and a formula $\varphi \in Sen(\Sigma)$, we denote by $[\![M]\!]_\Sigma(\varphi) = \{\eta \in [\![M]\!]_\Sigma \mid M \models^\eta_\Sigma \varphi\}$.

Example 2.6: **PL** is the stratified institution with *Set* as concrete category and $[\![\nu]\!]_P = \mathbb{1}$ ($\mathbb{1}$ is any singleton up to isomorphism) for each set P of propositional variables and each P-model ν.

Example 2.7 (Internal stratification Aiguier & Diaconescu, 2007): From any institution $\mathcal{I}$, we can define a stratified institution $St(\mathcal{I})$. Before giving its definition, let us recall the definitions of weak amalgamation square and quasi-representable signature which will be useful in this regard.

Definition 2.8 ((Weak) amalgamation square): A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma_1} & \Sigma_1 \\ {\scriptstyle \sigma_2}\downarrow & & \downarrow{\scriptstyle \theta_1} \\ \Sigma_2 & \xrightarrow[\theta_2]{} & \Sigma' \end{array}$$

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $Mod(\sigma_1)(M_1) = Mod(\sigma_2)(M_2)$, there exists a unique Σ'-model M' such that $Mod(\theta_1)(M') = M_1$ and $Mod(\theta_2)(M') = M_2$.

When dropping the uniqueness condition, we say this is a *weak* amalgamation square.

It is common in actual institutions that all pushout squares of signature morphisms are weak amalgamation squares, in fact most often they are amalgamation squares.

Definition 2.9 (Quasi-representable signature Diaconescu, 2008): A signature morphism $\chi : \Sigma \to \Sigma'$ is **quasi-representable** if and only if each model homomorphism $h : Mod(\chi)(M') \to N$ has a unique χ-expansion $h' : M' \to N'$.

Let $\mathcal{I} = (Sig, Sen, Mod, \models)$ be an institution. Let us denote by $St(\mathcal{I}) = (Sig', Sen', Mod', [\![\_]\!], \models)$ the stratified institution defined as follows:

- Sig' is the category whose objects and morphisms are, respectively, quasi-representable signatures $\chi : \Sigma \to \Sigma'$ and pairs of base institution signature morphisms $(\varphi : \Sigma \to \Sigma_1, \varphi' : \Sigma' \to \Sigma'_1) : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1)$ such that:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\chi} & \Sigma' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \Sigma_1 & \xrightarrow[\chi_1]{} & \Sigma'_1 \end{array}$$

  is a weak amalgamation square,
- $Sen' : Sig' \to Set$ is the functor that maps every $\chi : \Sigma \to \Sigma'$ to $Sen(\Sigma')$,
- $Mod' : Sig'^{op} \to Cat$ is the functor that maps $\chi : \Sigma \to \Sigma'$ to $Mod(\Sigma)$, and
- $[\![\_]\!]$ is the Sig'-indexed family of functors $[\![\_]\!]_\chi : Mod'(\chi) \to Set$ that maps every χ-model M to its set of states $[\![M]\!]_\chi = \{M' \in Mod(\Sigma') \mid Mod(\chi)(M') = M\}$.

Given $\chi : \Sigma \to \Sigma'$ and a χ-model M, for each state $M' \in [\![M]\!]_\chi$, we define the satisfaction of $\varphi \in Sen'(\chi)$ by M at M', denoted $M \models^{M'}_\chi \varphi$, by:

$$M \models^{M'}_\chi \varphi \text{ iff } M' \models_{\Sigma'} \varphi$$

Finally, a χ-model M satisfies φ, denoted $M \models_\chi \varphi$ if and only if $M \models^{M'}_\chi \varphi$ for every $M' \in [\![M]\!]_\chi$.

$St(\mathcal{I})$ is a stratified institution where the concrete category is Set. Indeed, for each signature morphism $(\varphi, \varphi' : (\chi : \Sigma \to \Sigma') \to (\chi_1 : \Sigma_1 \to \Sigma'_1))$, the natural transformation $[\![\_]\!]_{(\varphi,\varphi')}$ is defined by $[\![M]\!]_{(\varphi,\varphi')}(M') = Mod(\varphi')(M')$ for each state $M' \in [\![M]\!]_{\chi'}$. The definition of $[\![\_]\!]_\chi$ on model homomorphisms uses the quasi-representable property of χ. The surjectivity of $[\![\_]\!]_{(\varphi,\varphi')}$ is assured by the weak amalgamation property of the square defining (φ, φ').

Example 2.10: MPL is the stratified institution where the concrete category is *Graph*, $[\![(I, W, R)]\!]_P = (I, R)$ for each set P of propositional variables and each P-model (I, W, R), and for each signature morphism $\sigma : P \to P'$ and each P'-model (I', W', R'), $[\![(I', W', R')]\!]_\sigma$ is simply the identity morphism on (I', R').

Example 2.11: TMPL is the stratified institution which follows the same definition as **MPL** by replacing $[\![(I, W, R)]\!]_P = (I, R)$ by $[\![(X, \tau, \nu)]\!]_P = (X, \tau)$. Hence, the concrete category is the category of topological spaces *Top*.

Proposition 2.12 (Aiguier & Diaconescu, 2007): *Any stratified institution is an institution.*

(The proof of Proposition 2.12 is substantially similar to that given in Aiguier & Diaconescu, 2007.)

By this proposition, we will also denote by $\mathcal{I}$ the generic stratified institution $(Sig, Sen, Mod, [\![\_]\!], \models)$.

3. Internal logic and extension to fuzzy case

Here, we propose to define abstractly the important logic concepts of Boolean connectives, quantifiers, and fuzzy reasoning. By 'abstractly' we mean independently of any stratified institution. Boolean connectives and quantifiers have already been defined internally to any institution (Diaconescu, 2008), and since recently have been defined internally to any stratified institutions (Diaconescu, 2017). Here, we recall their definitions, given in Diaconescu (2017).

Fuzzy (or many-valued) reasoning has also received an institutional semantics (Diaconescu, 2013, 2014). The approach proposed here is substantially similar to that proposed in Diaconescu (2013) although defined in the framework of stratified institutions. Hence, the definitions of fuzzy semantic connectives will be a combination between the many-valued connectives of Diaconescu (2013) with the stratified institution-theoretic connectives defined below.

3.1. Internal logic and quantifiers

Let $\mathcal{I}$ be a stratified institution. Let Σ be a signature of $\mathcal{I}$. Let M be a Σ-model. A Σ-sentence φ' is in M a

- **semantic negation** of φ when $[\![M]\!]_\Sigma(\varphi') = [\![M]\!]_\Sigma \setminus [\![M]\!]_\Sigma(\varphi)$;
- **semantic conjunction** of φ_1 and φ_2 when $[\![M]\!]_\Sigma(\varphi') = [\![M]\!]_\Sigma(\varphi_1) \cap [\![M]\!]_\Sigma(\varphi_2)$;
- **semantic disjunction** of φ_1 and φ_2 when $[\![M]\!]_\Sigma(\varphi') = [\![M]\!]_\Sigma(\varphi_1) \cup [\![M]\!]_\Sigma(\varphi_2)$;
- **semantic implication** of φ_1 and φ_2 when $[\![M]\!]_\Sigma(\varphi') = ([\![M]\!]_\Sigma \setminus [\![M]\!]_\Sigma(\varphi_1)) \cup [\![M]\!]_\Sigma(\varphi_2)$.

A stratified institution $\mathcal{I}$ has (semantic) negation when each Σ-formula has a negation in each Σ-model. It has (semantic) conjunction (respectively disjunction and implication) when any two Σ-formulas have a conjunction (respectively disjunction and implication) in each Σ-model. Obviously, Boolean conjunctions and disjunctions can be extended in the same way to infinite conjunctions and disjunctions.

As usual, we denote negation, conjunction, disjunction and implication by $\neg$, $\wedge$, $\vee$ and $\Rightarrow$, respectively. Unlike institutions that deal with sentences, stratified institutions such as **MPL**, **MMPL** and **TMPL** have now semantic negation, disjunction and implication.

In the same way, it is equally easy to introduce abstract quantifiers in stratified institutions by following the same construction as in the definition of internal stratification given in Example 2.7. Hence, let $\mathcal{I} = (Sig, Sen, Mod, [\![\_]\!], \models)$ be a stratified institution, let $\chi : \Sigma \to \Sigma'$ be a signature morphism in Sig and let $M \in Mod(\Sigma)$ be a model. Then, $M \models^{\eta}_{\Sigma} (\forall\chi)\varphi$ if and only if for every Σ'-model M' such that $Mod(\chi)(M') = M$ and every state $\eta' \in [\![M']\!]_{\Sigma'}$ such that $[\![M']\!]_{\chi}(\eta') = \eta$ we have that $M' \models^{\eta'}_{\Sigma'} \varphi$. Existential quantification is defined dually by replacing 'every model M'' and 'every state η'' by 'some model M'' and 'some state η'' in the definition of universal quantification.

Remark 3.1: It is worth noting that the notations $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\neg\varphi$, $\bigwedge_i \varphi_i$, $\bigvee_i \varphi_i$, etc. are semantic notations which do not necessarily have syntactic equivalent in *Sen* (Diaconescu, 2017). For instance, all the examples of logic presented in Section 2 according to their respective grammar do not have syntactic formulas for infinite conjunctions and disjunctions. Only finite conjunctions and dijunctions of formulas are allowed. However, all of them have the semantic notations $\bigwedge_i \varphi_i$ and $\bigvee_i \varphi_i$, denoting respectively semantic possibly infinite conjunction and disjunction. In the following, given a signature Σ, we will denote by $Sen_{inf}(\Sigma)$ the set of formulas defined like $Sen(\Sigma)$ according to its grammar and closed under infinite conjunctions and disjunctions.

3.2. Multi-valued and fuzzy case

3.2.1. Residuated lattice

The algebraic structures underlying many-valued logic, fuzzy (or more generally L-fuzzy) logic, are usually residuated lattices. Residuated lattices (Ward & Dilworth, 1938) generalise Boolean algebras for classical logic by considering a set of truth values which may contain more than two values or that are not necessarily scalar values.

Definition 3.2 (Residuated lattice): A **residuated lattice** $(L, \bigwedge, \bigvee, \otimes, \to, 0, 1)$ is:

- a bounded lattice $(L, \bigwedge, \bigvee, 0, 1)$ where $\bigwedge$ and $\bigvee$ are the supremum and infimum operators associated with a partial ordering $\leq$, and 0 and 1 are the least and the greatest elements, respectively;
- $\otimes$ and $\to$ are binary operators such that:
  - $(L, \otimes, 1)$ is a monoid, that is, $\otimes$ is a commutative and associative operation with the identity $a \otimes 1 = a$;
  - $\otimes$ is isotone in both arguments;
  - the operation $\to$ is a residuation operation with respect to $\otimes$, i.e.

$$a \otimes b \leq c \text{ iff } a \leq b \to c$$

Usually fuzzy logic assumes $L = [0,1]$ with the usual ordering on real numbers (although this is but a convention), and L-fuzzy sets (or logic) are more general (Goguen, 1967). In the remainder of this paper, we will use the term fuzzy sets or fuzzy logic in a general sense, without assumption on the particular form of L, since we rely only on properties that are true in both cases, in particular those directly derived from the lattice structure and the adjunction (or residuation) property between $\otimes$ and $\rightarrow$. Examples of L-fuzzy sets can be found for instance for dealing with bipolar information (having a positive part and a negative part), or membership values expressed as intervals (Bloch, 2012; Dubois & Prade, 2008; Sussner et al., 2012).

The most famous examples of residuated lattices are Goguen algebra and Lukasiewicz algebra, defined respectively as follows:

- **Goguen algebra.** $([0,1], \bigwedge, \bigvee, \otimes, \rightarrow, 0, 1)$ where $\otimes$ is the ordinary product of reals and

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ \frac{b}{a} & \text{otherwise} \end{cases}$$

- **Lukasiewicz algebra.** $([0,1], \bigwedge, \bigvee, \otimes, \rightarrow, 0, 1)$ where:

$$a \otimes b = 0 \bigvee (a + b - 1)$$

$$a \rightarrow b = 1 \bigwedge (1 - a + b)$$

3.2.2. Institutional semantics

Let $\mathcal{I} = (Sig, Sen, Mod, [\![\_]\!], \models)$ be a stratified institution. Let $\mathcal{L} = (L, \bigwedge, \bigvee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. We can consider that for every signature Σ, the truth of Σ-formulas $\varphi \in Sen(\Sigma)$ is a value in L, i.e. for every Σ-model $M \in Mod(\Sigma)$, $[\![M]\!]_\Sigma(\varphi)$ is a fuzzy subset of $[\![M]\!]_\Sigma$ over L. Hence, whereas in $\mathcal{I}$, the satisfaction relation $M \models_\Sigma \varphi$ can be seen as a mapping from $[\![M]\!]_\Sigma$ to $\{0,1\}$, in a fuzzy extension of $\mathcal{I}$, $M \models_\Sigma \varphi$ is a mapping from $[\![M]\!]_\Sigma$ to L. For every $\eta \in [\![M]\!]_\Sigma$, we will rather use the notation $(M \models^\eta_\Sigma \varphi)$ than $M \models_\Sigma \varphi(\eta)$ to denote the value in L yielded by the mapping $M \models_\Sigma \varphi$. Of course, to preserve the satisfaction condition, we have to impose the following equivalence: for each signature morphism $\sigma : \Sigma \rightarrow \Sigma'$, every Σ'-model M', every Σ-formula φ and every $\eta' \in [\![M']\!]_\Sigma$,

$$(M' \models^{\eta'}_{\Sigma'} Sen(\sigma)(\varphi)) = (Mod(\sigma)(M') \models^{[\![M']\!]_\sigma(\eta')}_\Sigma \varphi)$$

Standardly, Boolean connectives and quantifiers can be internally defined in any fuzzy extension of a stratified institution $\mathcal{I}$. To give a meaning to negation, we suppose that $\mathcal{L}$ is with complements $\overline{(.)}$. Hence, a Σ-sentence ψ is, in a Σ-model M, a

- **fuzzy semantic negation** of φ when for every $\eta \in [\![M]\!]_\Sigma$, $(M \models^\eta_\Sigma \psi) = \overline{(M \models^\eta_\Sigma \varphi)}$;
- **fuzzy semantic conjunction** of φ_1 and φ_2 when for every $\eta \in [\![M]\!]_\Sigma$, $(M \models^\eta_\Sigma \psi) = (M \models^\eta_\Sigma \varphi_1) \bigwedge (M \models^\eta_\Sigma \varphi_2)$;
- **fuzzy semantic disjunction** of φ_1 and φ_2 when for every $\eta \in [\![M]\!]_\Sigma$, $(M \models^\eta_\Sigma \psi) = (M \models^\eta_\Sigma \varphi_1) \bigvee (M \models^\eta_\Sigma \varphi_2)$;

- **fuzzy semantic implication** of φ_1 and φ_2 when for every $\eta \in [\![M]\!]_\Sigma$, $(M \models^\eta_\Sigma \psi) = (M \models^\eta_\Sigma \varphi_1) \to (M \models^\eta_\Sigma \varphi_2)$.

The following connective $\otimes$ is often added, whose fuzzy semantics is:

$$\forall \eta \in [\![M]\!]_\Sigma, (M \models^\eta_\Sigma \varphi_1 \otimes \varphi_2) = ((M \models^\eta_\Sigma \varphi_1) \otimes (M \models^\eta_\Sigma \varphi_2)).$$

First-order quantifiers can also be easily represented in a fuzzy way. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism in Sig and let $M \in Mod(\Sigma)$ be a model. A Σ-sentence φ' is a **(fuzzy semantic) universal χ-quantification** in M when for every $\eta \in [\![M]\!]_\Sigma$, $(M \models^\eta_\Sigma \varphi') = \bigwedge\{(M' \models^{\eta'}_\Sigma \varphi) \mid Mod(\chi)(M') = M \text{ and } [\![M']\!]_\chi(\eta') = \eta\}$. Existential quantification is defined dually by replacing the infimum $\bigwedge$ by the supremum $\bigvee$. In Section 4.3.2, we will give a more general definition which allows us to extend a large family of dual logical operators, such as modalities, to the fuzzy case.

Fuzzy logics allow us to reason about formulas according to uncertainty. This leads to extend the satisfaction relation $\models_\Sigma$ to a binary relation between models in $Mod(\Sigma)$ and couples in $Sen(\Sigma) \times L$ as follows:

$$M \models_\Sigma (\varphi, l) \iff l \leq \bigwedge\{(M \models^\eta_\Sigma \varphi) \mid \eta \in [\![M]\!]_\Sigma\} \tag{1}$$

where $\leq$ is the ordering defined on L.

We have then the following result that proves that fuzzy extensions of stratified institutions are institutions.

Proposition 3.3: *Let $\mathcal{I} = (Sig, Sen, Mod, [\![\_]\!], \models)$ be a stratified institution. Let $\mathcal{L} = (L, \bigwedge, \bigvee, \otimes, \to, 0, 1)$ be a residuated lattice. Then, $\mathcal{I}' = (Sig', Sen', Mod', \models')$ where:*

- *$Sig' = Sig$,*
- *$\forall \Sigma \in Sig', Sen'(\Sigma) = Sen(\Sigma) \times L$,*
- *$Mod' = Mod$, and*
- *for every $\Sigma \in Sig'$, the binary relation $\models'_\Sigma \subseteq Mod'(\Sigma) \times Sen'(\Sigma)$ satisfies Equation (1).*

is an institution, i.e. for every signature morphism $\sigma : \Sigma \to \Sigma'$, every Σ'-model M' and couple $(\varphi, l) \in Sen'(\Sigma)$, we have:

$$M' \models_{\Sigma'} (Sen(\sigma)(\varphi), l) \iff Mod(\sigma)(M') \models_\Sigma (\varphi, l).$$

Proof: By definition, we have that:

$$(M' \models^{\eta'}_{\Sigma'} Sen(\sigma)(\varphi)) = (Mod(\sigma)(M') \models^{[\![M']\!]_\sigma(\eta')}_\Sigma \varphi).$$

As $[\![M']\!]_\sigma$ is surjective, we also have that:

$$\bigwedge\{(M' \models^{\eta'}_{\Sigma'} Sen(\sigma)(\varphi)) \mid \eta' \in [\![M']\!]_{\Sigma'}\} = \bigwedge\{(Mod(\sigma(M') \models^\eta_\Sigma \varphi) \mid \eta \in [\![M]\!]_\Sigma\},$$

and we can conclude that:

$$l \leq \bigwedge\{(M' \models^{\eta'}_{\Sigma'} Sen(\sigma)(\varphi)) \mid \eta' \in [\![M']\!]_{\Sigma'}\} \iff$$
$$l \leq \bigwedge\{(Mod(\sigma(M') \models^\eta_\Sigma \varphi) \mid \eta \in [\![M]\!]_\Sigma\}. \qquad \blacksquare$$

4. Duality from morphological dilations and erosions in stratified institutions

In this section, we show that mathematical morphology (Bloch et al., 2007; Serra, 1982) can be used for defining systematically and uniformly the different logical concepts such as quantifiers and modalities. Indeed, we can observe that most of unary modalities and quantifiers have always a dual, and they commute with conjunction and disjunction. This then enables us to define such logic concepts via algebraic dilations and erosions.

Before giving the definition of our abstract dual operators based on dilation and erosion, let us recall the basic concepts and results of mathematical morphology.

4.1. Basic operators of mathematical morphology on complete lattices and structuring elements

The most abstract way to define dilation and erosion is as follows. Let $(L, \preceq)$ and $(L', \preceq')$ be two (complete) lattices. Let $\vee$ and $\vee'$ denote the supremum in L and in L', associated with $\preccurlyeq$ and $\preceq'$, respectively. Similarly, let $\wedge$ and $\wedge'$ denote the infimum in L and in L', respectively. An algebraic dilation is an operator $\delta : L \to L'$ that commutes with the supremum, i.e.

$$\forall (a_i)_{i \in I} \in L, \quad \delta(\vee_{i \in I} a_i) = \vee'_{i \in I} \delta(a_i)$$

where I denotes any index set (not fixed). An algebraic erosion is an operator $\varepsilon : L' \to L$ that commutes with the infimum, i.e.

$$\forall (a_i)_{i \in I} \in L', \varepsilon(\wedge'_{i \in I} a_i) = \wedge_{i \in I} \varepsilon(a_i)$$

where I denotes any index set (not fixed). It follows that both operators are increasing (i.e. $\forall (a,b) \in L, a \preceq b \Rightarrow \delta(a) \preceq \delta(b)$, and a similar equation for erosion), δ preserves the least element $\bot$ in L ($\delta(\bot) = \bot$), and ε preserves the greatest element $\top'$ in L' ($\varepsilon(\top') = \top'$).

Now, in binary mathematical morphology, morphological operators are often defined on sets (i.e. L and L' are the powersets or finite powersets of given sets S and S', and often $S = S'$ and $L = L'$) through a structuring element designed in advance. Mathematical morphology has been mainly applied in image processing. In this particular case, the set S is an Abelian group equipped with an additive internal law $+$, and its elements represent image points. Let us recall here the basic definitions of dilation and erosion D_B and E_B in this particular case, where B is a set called structuring element, under an additionnal hypothesis of invariance under translation. Let X and B be two subsets of S. The dilation and erosion of X by the structuring element B, denoted respectively by $D_B(X)$ and $E_B(X)$, are defined as follows:

$$D_B(X) = \{x \in S \mid \check{B}_x \cap X \neq \emptyset\}$$
$$E_B(X) = \{x \in S \mid B_x \subseteq X\}$$

where $B_x = \{x + b \in S \mid b \in B\}$ where $+$ is the additive law associated with S (e.g. translation), and $\check{B}$ is the symmetrical of B with respect to the origin of space. An example

Figure 1. From left to right: binary image, structuring element, dilation, erosion.

is given in Figure 1 in $\mathbb{Z}^2$ for a binary image. This simple example illustrates the intuitive meaning of dilation (expanding the white objects according to the size and shape of the structuring element) and erosion (reducing the white objects according to the structuring element).

Now, in a more generally setting, we consider any set S, not necessarily endowed with a particular structure, the structuring element B can then be seen as a binary relation R_B on the set S ($R_B \subseteq S \times S$) as follows: $(x, y) \in R_B \iff y \in B_x$ (Bloch et al., 2007; Madrid, Ojeda-Aciego, Medina, & Perfilieva, 2019).[7] This is the way we will consider structuring elements in this paper, as done in previous work, in particular for mathematical morphology on graphs (see e.g. Bloch, Bretto, & Leborgne, 2015; Cousty, Najman, Dias, & Serra, 2013; Meyer & Stawiaski, 2009, among others) or logics (see e.g. Aiguier, Atif, Bloch, & Hudelot, 2018; Aiguier, Atif, Bloch, & Pino Pérez, 2018; Bloch, 2002; Bloch et al., 2007; Bloch & Lang, 2002; Gorogiannis & Hunter, 2008).

The most important properties of dilation and erosion based on a structuring element are the following ones (Bloch et al., 2007; Najman & Talbot, 2010; Serra, 1982):

- **Monotonicity:** if $X \subseteq Y$, then $D_B(X) \subseteq D_B(Y)$ and $E_B(X) \subseteq E_B(Y)$; if $B \subseteq B'$, then $D_B(X) \subseteq D_{B'}(X)$ and $E_{B'}(X) \subseteq E_B(X)$.
- If for every $x \in E$, $x \in B_x$ (and this condition is actually necessary and sufficient), then
  - ***D_B* is extensive:** $X \subseteq D_B(X)$;
  - ***E_B* is anti-extensive:** $E_B(X) \subseteq X$.
- **Commutativity:** $D_B(X \cup Y) = D_B(X) \cup D_B(Y)$ and $E_B(X \cap Y) = E_B(X) \cap E_B(Y)$ (and similar expressions for infinite or empty families of subsets).
- **Adjunction:** $X \subseteq E_B(Y) \Leftrightarrow D_B(X) \subseteq Y$.
- **Duality:** $E_B(X) = [D_{\check{B}}(X^C)]^C$ where $\_^C$ is the set-theoretical complementation.

Hence, D_B and E_B are particular cases of general algebraic dilation and erosion on the lattice $(\mathcal{P}(S), \subseteq)$.

In this paper, we rely on both the general algebraic framework, and on the definitions using structuring elements considered as binary relations. We exploit the duality (here with respect to the complementation) and its parallel notions in logical settings for designing the proposed framework. Note that the general algebraic definition of dilation and erosion assumes commutativity with supremum and infimum, respectively, for any family of elements of L or L'. This includes empty and infinite families. If $\bot$ is the least element of a lattice and $\top$ the greatest element, then we have $\vee(\emptyset) = \bot$ and $\wedge\emptyset = \top$. Considering infinite families requires the lattice to be complete.

Particular cases of lattices have been considered when defining mathematical morphology on fuzzy sets (see e.g Bloch, 2009 and the references therein), L-fuzzy sets (see e.g. Sussner et al., 2012), etc. Such extensions will be used as well.

4.2. Lattice of formulas

We saw in Section 4.1 that the operators erosion and dilation can be defined in an algebraic framework given by complete lattices. Here, to abstractly define our dual connectives on formulas from dilation and erosion, we need to consider a lattice of fomulas.

Let $M \in Mod(\Sigma)$ be a model. Considering the inclusion on the power set $\mathcal{P}([\![M]\!]_\Sigma)$, the poset $(\mathcal{P}([\![M]\!]_\Sigma), \subseteq)$ is a complete lattice. Similarly, a complete lattice can be defined on the set $Sen_{inf}(\Sigma)_{/\equiv_M}$ where $\equiv_M$ is the equivalence relation defined by:

$$\varphi \equiv_M \psi \Longleftrightarrow [\![M]\!]_\Sigma(\varphi) = [\![M]\!]_\Sigma(\psi)$$

Then $(Sen_{inf}(\Sigma)_{/\equiv_M}, \preceq_M)$ is the lattice where $\preceq_M$ is the partial ordering defined by:

$$[\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M} \Longleftrightarrow [\![M]\!]_\Sigma(\varphi) \subseteq [\![M]\!]_\Sigma(\psi)$$

Any subset $\Phi = \{[\varphi_i]_{\equiv_M} \mid i \in J\}$ (where J is an index set) of $Sen_{inf}(\Sigma)_{/\equiv_M}$ has as supremum $\bigvee \Phi$ and infimum $\bigwedge \Phi$, corresponding to union and intersection in the complete lattice $(\mathcal{P}([\![M]\!]_\Sigma), \subseteq)$, and then, following the definitions given in Section 3.1, $\bigvee \Phi$ and $\bigwedge \Phi$ are the equivalence classes of the semantic (possibly infinite) disjunction $[\bigvee_{i \in J} \varphi_i]_{\equiv_M}$ and of the semantic (possibly infinite) conjunction $[\bigwedge_{i \in J} \varphi_i]_{\equiv_M}$, respectively. Hence, $(Sen_{inf}(\Sigma)_{/\equiv_M}, \preceq_M)$ is a complete lattice. Greatest and least elements are respectively $\top$ and $\bot$, corresponding to equivalence classes of tautologies and antilogies (i.e. contradictions), and we have $\vee\emptyset = \bot$ and $\wedge\emptyset = \top$. Now, for most of logics (anyway all the logics presented in this paper), such supremum and infimum over infinite sets of formulas may not occur. The reason is that in most of logics such supremum and infimum are defined from disjunction and conjunction, which are only syntactically defined for finite set of formulas. Hence, for a set $\Phi = \{[\varphi_i]_{\equiv_M} \mid i \in J\}$ where J is an infinite index set, $\bigvee \Phi$ and $\bigwedge \Phi$ have no syntactic equivalent in $Sen(\Sigma)_{/\equiv_M}$ because $\bigvee_{i \in J} \varphi_i$ and $\bigwedge_{i \in J} \varphi_i$ do not exist in $Sen(\Sigma)$ (formulas are only of finite size). This is why we will also consider in the following the sublattice $(Sen(\Sigma)_{/\equiv_M}, \preceq_M)$ of $(Sen_{inf}(\Sigma)_{/\equiv_M}, \preceq_M)$ restricted to formulas in $Sen(\Sigma)$. This sublattice is not finite (only formulas are of finite size not their number), and then it is not complete anymore. However, it is bounded because $\bot$ and $\top$ are the identity elements for $\vee$ and $\wedge$, respectively.

4.3. Morphological dilations and erosions of formulas based on structuring elements

In the rest of the paper, we consider a stratified institution $\mathcal{I}$ which has conjunction, disjunction and negation.

4.3.1. Definitions

In stratified institutions, given a Σ-model M, $[\![M]\!]_\Sigma$ is an element of a concrete category $\mathcal{C}$ (i.e. $\mathcal{C}$ comes with a faithful functor $\mathcal{U}$ such that $\mathcal{U}([\![M]\!]_\Sigma)$ is a set[8]). Therefore, let us suppose that for each model $M \in Mod(\Sigma)$, $[\![M]\!]_\Sigma$ is equipped with a structuring element B (i.e. $\forall \eta \in [\![M]\!]_\Sigma, B_\eta \subseteq [\![M]\!]_\Sigma$) which represents a relationship between states, i.e. $\eta' \in B_\eta$ iff η' satisfies some relationship to η (see the next section to have examples of structuring elements for given stratified institutions), and $\check{B}_\eta$ is defined by $\eta' \in \check{B}_\eta \Leftrightarrow \eta \in B_{\eta'}$. Drawing inspiration from Bloch & al. in Bloch (2002), Bloch and Lang (2002), dilation and erosion of a formula $\varphi \in Sen(\Sigma)$ then give rise to two formulas $D_B(\varphi)$ and $E_B(\varphi)$ satisfying the following equivalences:

$$\begin{aligned} M \models^\eta_\Sigma D_B(\varphi) &\Longleftrightarrow \check{B}_\eta \cap \{\eta' \in [\![M]\!]_\Sigma \mid M \models^{\eta'}_\Sigma \varphi\} \neq \emptyset \\ &\Longleftrightarrow \exists \eta' \in \check{B}_\eta, M \models^{\eta'}_\Sigma \varphi \\ &\Longleftrightarrow \check{B}_\eta \cap [\![M]\!]_\Sigma(\varphi) \neq \emptyset M \models^\eta_\Sigma E_B(\varphi) \\ &\Longleftrightarrow B_\eta \subseteq \{\eta' \in [\![M]\!]_\Sigma \mid M \models^{\eta'}_\Sigma \varphi\} \\ &\Longleftrightarrow \forall \eta' \in B_\eta, M \models^{\eta'}_\Sigma \varphi \\ &\Longleftrightarrow B_\eta \subseteq [\![M]\!]_\Sigma(\varphi) \end{aligned}$$

We obviously have that:

$$\forall \varphi, \quad \psi \in Sen(\Sigma), \quad \varphi \equiv_M \psi \Longrightarrow \begin{cases} E_B(\varphi) \equiv_M E_B(\psi), & \text{and} \\ D_B(\varphi) \equiv_M D_B(\psi) \end{cases}$$

Hence, the logical operators E_B and D_B can be extended into two operators, also denoted E_B and D_B over $Sen_{inf}(\Sigma)_{/\equiv_M}$ (and then also over $Sen(\Sigma)_{/\equiv_M}$) defined as:

$$E_B([\varphi]_{\equiv_M}) = [E_B(\varphi)]_{\equiv_M} \quad \text{and} \quad D_B([\varphi]_{\equiv_M}) = [D_B(\varphi)]_{\equiv_M}$$

In $(Sen_{inf}(\Sigma)_{/\equiv_M}, \preceq_M)$, the operators E_B and D_B are then algebraic erosion and dilation.

4.3.2. Extension to the fuzzy case

From our extension of stratified institutions to fuzzy reasoning, we can also define fuzzy dilation and erosion of formulas based on structuring elements. Several definitions of mathematical morphology on fuzzy sets with fuzzy structuring elements have been proposed in the literature, since the early work in Baets (1995), Bloch and Maître (1993) (see e.g Bloch, 2009; Bloch & Maître, 1995; Nachtegael & Kerre, 2000 for reviews). Here, we follow the approach developed in Bloch (2009), Bloch (2012), Sussner (2016) using conjunctions and implications in [0, 1] or more generally in residuated lattices. Hence, given a Σ-model M with a structuring element B such that for every $\eta \in [\![M]\!]_\Sigma$, B_η is a fuzzy set, the dilation of a fuzzy formula by B is defined for every $\eta \in [\![M]\!]_\Sigma$ as follows:

$$(M \models^{\eta}_{\Sigma} D_B(\varphi)) = \bigvee \{\check{B}_\eta(\eta') \otimes (M \models^{\eta'}_{\Sigma} \varphi) \mid \eta' \in [\![M]\!]_\Sigma\}.$$

The erosion of a fuzzy formula by B is defined for every $\eta \in [\![M]\!]_\Sigma$ as follows:

$$(M \models^{\eta}_{\Sigma} E_B(\varphi)) = \bigwedge \{B_\eta(\eta') \to (M \models^{\eta'}_{\Sigma} \varphi) \mid \eta' \in [\![M]\!]_\Sigma\}.$$

If we note $\mathcal{F}([\![M]\!]_\Sigma)$ the set of all fuzzy sets on $[\![M]\!]_\Sigma$, the couple $(\mathcal{F}([\![M]\!]_\Sigma), \leq)$ where $\leq$ denotes the fuzzy inclusion, is a complete lattice. Therefore, we can consider the lattices $(Sen_{inf}(\Sigma)/_{\equiv_M}, \preceq_M)$ and $(Sen(\Sigma)/_{\equiv_M}, \preceq_M)$ where $\equiv_M$ and $\preceq_M$ are the fuzzy extensions of the two relations $\equiv_M$ and $\preceq_M$ defined in Section 4.2. Here again, it is easy to show that the fuzzy versions of D_B and E_B commute with union and intersection of fuzzy sets of states, respectively, i.e. for every $\varphi_1, \varphi_2 \in Sen(\Sigma)$, we have:

- $D_B(\varphi_1 \vee \varphi_2) \equiv_M D_B(\varphi_1) \vee D_B(\varphi_2)$,
- $E_B(\varphi_1 \wedge \varphi_2) \equiv_M E_B(\varphi_1) \wedge E_B(\varphi_2)$,

and the extensions of these equivalences to any family of fomulas. Therefore, D_B and E_B, interpreted in a fuzzy sets setting, are algebraic dilation and erosion, respectively. As for the crisp case, it is quite straightforward to show that these fuzzy dilation and erosion are monotonous, extensive and anti-extensive when $\eta \in B_\eta$, and dual (resp. adjoint) if $\otimes$ and $\to$ are dual (resp. adjoint).

Proof: Let us prove that D_B and E_B are algebraic dilation and erosion, i.e. that they commute with the supremum and infimum, respectively. The proof follows the same lines as the corresponding proof in classical mathematical morphology. For the dilation, it is sufficient to prove that $\otimes$ commutes with the disjunction in the residuated lattice L, i.e. that $\vee_i(a_i \otimes b) = (\vee_i a_i) \otimes b$, for any family a_i and any b in L. Since $\otimes$ and $\to$ are adjoint (i.e. $\to$ is the residuated implication of $\otimes$), we have, for all c:

$$\begin{aligned} \vee_i(a_i \otimes b) \leq c &\Leftrightarrow \forall i, a_i \otimes b \leq c \\ &\Leftrightarrow \forall i, a_i \leq b \to c \\ &\Leftrightarrow \vee_i\, a_i \leq b \to c \\ &\Leftrightarrow (\vee_i a_i) \otimes b \leq c \end{aligned}$$

(by applying the adjunction property for the second and fourth equivalences).

Now take $c_1 = \vee_i(a_i \otimes b)$. Then the first inequality holds (it is even an equality), and we can derive that $(\vee_i a_i) \otimes b \leq c_1$, i.e. $(\vee_i a_i) \otimes b \leq \vee_i(a_i \otimes b)$.

Then take $c_2 = (\vee_i a_i) \otimes b$. The second inequality holds, and we can derive that $\vee_i(a_i \otimes b) \leq c_2$, i.e. $\vee_i(a_i \otimes b) \leq (\vee_i a_i) \otimes b$.

Since both inequalities hold for all a_i and b, we can conclude that $\vee_i(a_i \otimes b) \leq (\vee_i a_i) \otimes b$.

It follows immediately that D_B commutes with the supremum and is hence an algebraic dilation.

The proof for erosion is similar. ■

4.3.3. Examples

We show in this section that the two dual logical operators E_B and D_B can be instantiated to define both first-order quantifiers $\forall, \exists$ and modalities $\Box, \Diamond$. Moreover, from Section 4.3.2, all these operators can naturally be extended to fuzzy cases.

4.3.3.1. First-order quantifiers. Let $St(\mathbf{Fol})$ be the stratified institution of the first-order logic Let $\chi : (S, F, P) \hookrightarrow (S, F \coprod X, P)$ be a signature inclusion and let x be a variable in X. For every (S, F, P)-model M, let us define the structuring element B^x as follows:

$$\forall M' \in [\![M]\!]_\Sigma, \quad B^x_{M'} = \{M'' \in [\![M]\!]_\Sigma \mid \forall y \neq x \in X, y^{M''} = y^{M'}\},$$

i.e. the set of models identical to M' on all variables except possibly x. This structuring element is symmetrical (i.e. $M'' \in B^x_{M'} \Leftrightarrow M' \in B^x_{M''}$) and contains the origin (i.e. $M' \in B^x_{M'}$).

We can then define the first-order quantifiers $\forall x$ and $\exists x$ as erosion and dilation from B^x as follows:

$$\forall x.\varphi \equiv E_{B^x}(\varphi),$$
$$\exists x.\varphi \equiv D_{B^x}(\varphi).$$

More generally, in any internal stratification $St(\mathcal{I})$ of an institution $\mathcal{I}$, both quantifiers $\forall\chi$ and $\exists\chi$ for a signature $\chi : \Sigma \to \Sigma'$ can be defined similarly. Indeed, for every χ-model M, let us define the structuring element B^χ as follows:

$$\forall M' \in [\![M]\!]_\chi, \quad B^\chi_{M'} = [\![M]\!]_\chi$$

Again, the structuring element is symmetrical and contains the origin, and we have:

$$\forall \chi.\varphi \equiv E_{B^\chi}(\varphi),$$
$$\exists \chi.\varphi \equiv D_{B^\chi}(\varphi).$$

4.3.3.2. Modalities for Kripke models. Let $\mathcal{I}$ be a stratified institution whose concrete category is *Graph*. Hence for each Σ-model M, $[\![M]\!]_\Sigma$ is a directed graph $([\![M]\!]_\Sigma, R_M)$. Obviously, this accessibility relation R_M naturally leads to the structuring element B defined as follows:

$$R_M(\eta, \eta') \Longleftrightarrow \eta' \in B_\eta.$$

The modalities $\Box$and $\diamond$ are then defined as follows:[9]

$$\Box\varphi \equiv E_B(\varphi),$$
$$\Diamond\varphi \equiv D_{\check{B}}(\varphi).$$

4.3.4. Properties

The following properties are the direct extensions of properties of dilation and erosion on sets to formulas.

- **Monotonicity:** if $[\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$, then $[D_B(\varphi)]_{\equiv_M} \preceq_M [D_B(\psi)]_{\equiv_M}$ and $[E_B(\varphi)]_{\equiv_M} \preceq_M [E_B(\psi)]_{\equiv_M}$.
- **Extensivity of dilation:** $[\varphi]_{\equiv_M} \preceq_M [D_B(\varphi)]_{\equiv_M}$ and **anti-extensivity of erosion:** $[E_B(\varphi)]_{\equiv_M} \preceq_M [\varphi]_{\equiv_M}$ if and only if for every $\eta \in [\![M]\!]_\Sigma$, $\eta \in B_\eta$.
- **Adjunction:** $[\varphi]_{\equiv_M} \preceq_M [E_B(\psi)]_{\equiv_M} \Leftrightarrow [D_B(\varphi)]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$.
- **Commutativity with supremum or infinum:** $D_B(\varphi_1 \vee \varphi_2) \equiv_M D_B(\varphi_1) \vee D_B(\varphi_2)$ and $E_B(\varphi_1 \wedge \varphi_2) \equiv_M E_B(\varphi_1) \wedge E_B(\varphi_2)$, and extensions of these equivalences to any finite or empty family of formulas.
- **Duality:** $E_B(\varphi) \equiv_M \neg D_{\check{B}}(\neg\varphi)$.

It follows that D_B and E_B are respectively algebraic dilation and erosion over $(Sen(\Sigma)_{/\equiv_M}, \preceq_M)$, i.e. in $(Sen(\Sigma)_{/\equiv_M}, \preceq_M) D_B$ and E_B commute with supremum and infimum (for finite or empty families), respectively. Moreover, by a standard result of mathematical morphology (Bloch et al., 2007), E_B (respectively D_B) is the unique erosion (respectively the unique dilation) associated with D_B (respectively E_B) by the adjunction property. From standard results of mathematical morphology and the adjunction property, we also have the following properties:

Corollary 4.1:

- $E_B(\top) \equiv_M \top$
- $D_B(\bot) \equiv_M \bot$
- $[\varphi]_{\equiv_M} \preceq_M [E_B(D_B(\varphi))]_{\equiv_M}$
- $[D_B(E_B(\varphi))]_{\equiv_M} \preceq_M [\varphi]_{\equiv_M}$
- $E_B(D_B(E_B(\varphi))) \equiv_M E_B(\varphi)$
- $D_B(E_B(D_B(\varphi))) \equiv_M D_B(\varphi)$
- $[E_B(\varphi)]_{\equiv_M} = \bigvee\{[\psi]_{\equiv_M} \mid [D_B(\psi)]_{\equiv_M} \preceq_M [\varphi]_{\equiv_M}\}$
- $[D_B(\varphi)]_{\equiv_M} = \bigwedge\{[\psi]_{\equiv_M} \mid [\varphi]_{\equiv_M} \preceq_M [E_B(\psi)]_{\equiv_M}\}$

It follows that $E_B D_B$ (closing) and $D_B E_B$ (opening) are morphological filters (i.e. increasing and idempotent operators). Moreover, closing and opening are dual (i.e. $D_B(E_B(\varphi)) \equiv_M \neg E_{\check{B}}(D_{\check{B}}(\neg\varphi))$.

Theorem 4.2: *The following properties are satisfied by dilation and erosion of formulas. Note that now properties are expressed independently of a model M.*

(1) $E_B(\top) \equiv \top$ *and* $D_B(\bot) \equiv \bot$.
(2) $\varphi \models E_B(\varphi)$.
(3) *If for every model* $M \in Mod(\Sigma)$ *and every* $\eta \in [\![M]\!]_\Sigma$, $\eta \in B_\eta$, *then* $\varphi \models D_B(\varphi)$ *and* $E_B(\varphi) \models \varphi$.
(4) $D_B(\varphi \vee \psi) \equiv D_B(\varphi) \vee D_B(\psi)$ *and* $E_B(\varphi \wedge \psi) \equiv E_B(\varphi) \wedge E_B(\psi)$. *Moreover, we have:* $D_B(\varphi \wedge \psi) \models D_B(\varphi) \wedge D_B(\psi)$ *and* $E_B(\varphi) \vee E_B(\psi) \models E_B(\varphi \vee \psi)$. *Similar expressions hold for any families of formulas.*
(5) $E_B(\varphi) \equiv \neg D_{\check{B}}(\neg\varphi)$, *or dually* $D_B(\varphi) \equiv \neg E_{\check{B}}(\neg\varphi)$.
(6) *If the stratified institution has implication, then*
    (a) $E_B(\varphi \Rightarrow \psi) \models E_B(\varphi) \Rightarrow E_B(\psi)$,
    (b) $(E_B(\varphi) \Rightarrow D_B(\varphi)) \equiv \top$ *if for every* $M \in Mod(\Sigma)$ *and every* $\eta \in [\![M]\!]_\Sigma$, $B_\eta \cap \check{B}_\eta \neq \emptyset$,
    (c) $(D_B(E_B(\varphi)) \Rightarrow \varphi) \equiv (\varphi \Rightarrow E_B(D_B(\varphi))) \equiv \top$.

Proof: (1) These first two properties are obvious to check.

(2) Let $M \models_\Sigma \varphi$. Let $\eta \in [\![M]\!]_\Sigma$ and let $\eta' \in B_\eta$. By hypothesis, $M \models_\Sigma^{\eta'} \varphi$, and then $M \models_\Sigma^{\eta} E_B(\varphi)$.

(3) Let $M \models_\Sigma \varphi$. Let $\eta \in [\![M]\!]_\Sigma$. As $\eta \in B_\eta$, we directly deduce that $M \models_\Sigma D_B(\varphi)$.Let $M \models_\Sigma E_B(\varphi)$. Let $\eta \in [\![M]\!]_\Sigma$. As $\eta \in B_\eta$, by hypothesis we have that $M \models_\Sigma^{\eta} \varphi$ whence we can conclude.

(4) Let $M \in Mod(D_B(\varphi \vee \psi))$. This means that for every $\eta \in [\![M]\!]_\Sigma$, there exists $\eta' \in B_\eta$ such that $M \models_\Sigma^{\eta'} (\varphi \vee \psi)$, and then $M \models_\Sigma^{\eta'} \varphi$ or $M \models_\Sigma^{\eta'} \psi$. From this, we can directly conclude that $M \models_\Sigma^{\eta} D_B(\varphi)$ or $M \models_\Sigma^{\eta} D_B(\psi)$, i.e. $M \models_\Sigma^{\eta} D_B(\varphi) \vee D_B(\psi)$.
Let $M \in Mod(D_B(\varphi) \vee D_B(\psi))$. This means that for every $\eta \in [\![M]\!]_\Sigma$, there exists $\eta' \in B_\eta$ such that $M \models_\Sigma^{\eta'} \varphi$ or $M \models_\Sigma^{\eta'} \psi$, and then $M \models_\Sigma^{\eta'} (\varphi \vee \psi)$. From this, we can directly conclude that $M \models_\Sigma^{\eta} D_B(\varphi \vee \psi)$.
The proof to show that $Mod(E_B(\varphi \wedge \psi)) = Mod(E_B(\varphi) \wedge E_B(\psi))$ is (relatively) similar.
Let $M \models_\Sigma D_B(\varphi \wedge \psi)$. Let $\eta \in [\![M]\!]_\Sigma$. By hypothesis, there exists $\eta' \in B_\eta$ such that $M \models_\Sigma^{\eta'} (\varphi \wedge \psi)$, and then $M \models_\Sigma^{\eta'} \varphi$ and $M \models_\Sigma^{\eta'} \psi$. Therefore, we can write that $M \models_\Sigma^{\eta} D_B(\varphi)$ and $M \models_\Sigma^{\eta} D_B(\psi)$, whence we directly have that $M \models_\Sigma^{\eta} (D_B(\varphi) \wedge D_B(\psi))$. The converse inequality if however not true and examples where the inequality is strict can be exhibited, similar to the ones in the set theoretical setting.
Let $M \models_\Sigma E_B(\varphi) \vee E_B(\psi)$. Let $\eta \in [\![M]\!]_\Sigma$ and let $\eta' \in B_\eta$. By hypothesis, we necessarily have that $M \models_\Sigma^{\eta'} \varphi$ or $M \models_\Sigma^{\eta'} \psi$. Otherwise, we would have neither $M \models_\Sigma^{\eta} E_B(\varphi)$ nor $M \models_\Sigma^{\eta} E_B(\psi)$ which would be a contradiction. Hence $M \models_\Sigma^{\eta'} \varphi \vee \psi$, and $M \models_\Sigma^{\eta} E_B(\varphi \vee \psi)$. Again counter-examples showing that the converse is not true are easy to exhibit.

(5)

$$
\begin{aligned}
M \models_\Sigma E_B(\varphi) &\Leftrightarrow \forall \eta \in [\![M]\!]_\Sigma, M \models_\Sigma^{\eta} E_B(\varphi) \\
&\Leftrightarrow \forall \eta \in [\![M]\!]_\Sigma, \forall \eta' \in B_\eta, M \models_\Sigma^{\eta'} \varphi \\
&\Leftrightarrow \forall \eta \in [\![M]\!]_\Sigma, \forall \eta' \in B_\eta, M \not\models_\Sigma^{\eta'} \neg\varphi \\
&\Leftrightarrow \forall \eta \in [\![M]\!]_\Sigma, M \not\models_\Sigma^{\eta} D_{\check{B}}(\neg\varphi) \\
&\Leftrightarrow \forall \eta \in [\![M]\!]_\Sigma, M \models_\Sigma^{\eta} \neg D_{\check{B}}(\neg\varphi)
\end{aligned}
$$

(6) (a) Let $M \models_\Sigma E_B(\varphi \Rightarrow \psi)$. Let $\eta \in [\![M]\!]_\Sigma$ such that $M \models_\Sigma^{\eta} E_B(\varphi)$. Let $\eta' \in B_\eta$. By hypothesis, $M \models_\Sigma^{\eta'} \varphi$, and then, as $M \models_\Sigma E_B(\varphi \Rightarrow \psi)$, we also have that $M \models_\Sigma^{\eta'} (\varphi \Rightarrow \psi)$, and $M \models_\Sigma^{\eta'} \psi$.

(b) Let $\eta \in [\![M]\!]_\Sigma$ such that $M \models_\Sigma^{\eta} E_B(\varphi)$. Let $\eta' \in B_\eta \cap \check{B}_\eta$ (by hypothesis this intersection is not empty). Then we have that $M \models_\Sigma^{\eta'} \varphi$ since $\eta' \in B_\eta$, and then $M \models_\Sigma^{\eta} D_B(\varphi)$ since $\eta' \in \check{B}_\eta$.

(c) These properties come from the extensivity of closing and from the extensivity of opening, which hold for $\preceq_M$ (see Corollary 4.1).

■

These properties are for instance satisfied by the examples in Section 4.3.3, which provide illustrations of these results.

4.4. Dual logical operators as algebraic dilation and erosion

In this section, we provide an algebraic view of dual dilation and erosion, without referring to any structuring element over the set $[\![M]\!]_\Sigma$.

4.4.1. Definition

Definition 4.3 (Algebraic erosion and dilation): Let E and D be two logical operators for $\mathcal{I}$ satisfying the following properties:

- **Duality:** $\forall M \in Mod(\Sigma), \forall \varphi \in Sen(\Sigma), E(\varphi) \equiv_M \neg D(\neg\varphi)$
- **Stability:** $\forall M \in Mod(\Sigma), \forall \varphi, \psi \in Sen(\Sigma), \varphi \equiv_M \psi \Longrightarrow \begin{cases} E(\varphi) \equiv_M E(\psi), \text{ and} \\ D(\varphi) \equiv_M D(\psi) \end{cases}$

We will say that E and D are **algebraic erosion and dilation** if they satisfy the following equations: $\forall M \in Mod(\Sigma)$,

(1) for any family $(\varphi_i), \varphi_i \in Sen(\Sigma), D(\vee_i \varphi_i) \equiv_M \vee_i D(\varphi_i)$;
(2) $\forall \varphi \in Sen(\Sigma), [\![M]\!]_\Sigma(\varphi) = \emptyset \Longrightarrow D(\varphi) \equiv_M \varphi$;
(3) for any family $(\varphi_i), \varphi_i \in Sen(\Sigma), E(\wedge_i \varphi_i) \equiv_M \wedge_i E(\varphi_i)$;
(4) $\forall \varphi \in Sen(\Sigma), [\![M]\!]_\Sigma(\varphi) = [\![M]\!]_\Sigma \Longrightarrow E(\varphi) \equiv_M \varphi$.

The logical operators E and D are said algebraic because by the stability property we can define two operators also denoted E and D, over $Sen_{inf}(\Sigma)_{/\equiv}$ (and then also over $Sen(\Sigma)_{/\equiv}$) as follows:

$$E([\varphi]_{\equiv_M}) = [E(\varphi)]_{\equiv_M}$$
$$D([\varphi]_{\equiv_M}) = [D(\varphi)]_{\equiv_M}$$

It is easy to show that D commutes with the supremum and preserves the least element $\bot$, and E commutes with the infimum and preserves the greatest element $\top$. Hence, E and D are algebraic erosion and dilation over the complete lattice $(Sen_{inf}(\Sigma)_{/\equiv}, \preceq_M)$.

By standard results of mathematical morphology, we have the following properties:

Proposition 4.4:

- ***Monotonicity of D:*** *if* $[\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$, *then* $D([\varphi]_{\equiv_M}) \preceq_M D([\psi]_{\equiv_M})$;
- ***Monotonicity of E:*** *if* $[\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$, *then* $E([\varphi]_{\equiv_M}) \preceq_M E([\psi]_{\equiv_M})$.

Unlike dilation and erosion defined through structuring elements, the dual logical operators E and D defined as algebraic erosion and dilation do not form necessarily an adjunction (see Section 4.4.2 for an example) which is expressed, when it holds, as follows:

$$\forall \varphi, \psi \in Sen(\Sigma), [D(\varphi)]_{\equiv_M} \preceq_M [\psi]_{\equiv_M} \Longleftrightarrow [\varphi]_{\equiv_M} \preceq_M [E(\psi)]_{\equiv_M}$$

When adjunction holds between E and D, by standard results in mathematical morphology, the following properties are satisfied:

- $[\varphi]_{\equiv_M} \preceq_M [E(D(\varphi))]_{\equiv_M}$ (extensivity of ED);
- $[D(E(\varphi))]_{\equiv_M} \preceq_M [\varphi]_{\equiv_M}$ (anti-extensivity of DE);
- $E(D(E(\varphi))) \equiv_M E(\varphi)$;
- $D(E(D(\varphi))) \equiv_M D(\varphi)$;
- $E(D(E(D(\varphi)))) \equiv_M E(D(\varphi))$;
- $D(E(D(E(\varphi)))) \equiv_M D(E(\varphi))$.

Some properties are preserved independently of a model M.

Theorem 4.5: *The following properties are satisfied by dilation and erosion of formulas*:

- ***Duality:*** $D(\varphi) \equiv \neg E(\neg\varphi)$.
- ***Commutativity:*** $D(\varphi_1 \vee \varphi_2) \equiv D(\varphi_1) \vee D(\varphi_2)$ *and* $E(\varphi_1 \wedge \varphi_2) \equiv E(\varphi_1) \wedge E(\varphi_2)$, *and similar expressions for any families of formulas.*
- ***Monotonicity:*** *if* $\varphi \models \psi$, *then* $D(\varphi) \models D(\psi)$ *and* $E(\varphi) \models E(\psi)$.
- ***Preservation:*** $D(\bot) \equiv \bot$ *and* $E(\top) \equiv \top$.

Proof: Duality, commutativity and preservation are direct consequences of the fact that $(\forall M \in Mod(\Sigma), \varphi \equiv_M \psi) \Longrightarrow \varphi \equiv \psi$. To prove monotonicity, let us suppose that $\varphi \models \psi$. Therefore, for every $M \in Mod(\varphi)$ we have that $M \models_\Sigma \varphi$ and $M \models_\Sigma \psi$, and then for every $\eta \in [\![M]\!]_\Sigma$ we have $M \models_\Sigma^\eta \varphi$ and $M \models_\Sigma^\eta \psi$ i.e. $[\![M]\!]_\Sigma(\varphi) = [\![M]\!]_\Sigma(\psi) = [\![M]\!]_\Sigma$, whence we conclude $\varphi \equiv_M \psi$. As D is monotonous for $\equiv_M$, we then have that $D(\varphi) \equiv_M D(\psi)$. Hence, for every $\eta \in [\![M]\!]_\Sigma$, we have that $M \models_\Sigma^\eta D(\psi)$, and then $M \in Mod(D(\psi))$. The reasoning for E is similar. ■

4.4.2. Example: modalities for topos-models

When the modalities $\Box$and $\diamond$ are interpreted topologically, they cannot be expressed as erosion and dilation based on a structuring element. The reason is the heterogeneity of elements used to express $M \models_\Sigma^\eta \Box\varphi$ where we quantify existentially over open sets and universally over elements in open sets. We might be tempted to define the modality $\Box$ by an erosion E_B followed by a dilation D_B (i.e. a morphological opening) where B would be the structuring element defined as: $\forall \eta \in [\![M]\!]_\Sigma, B_\eta = \bigcup\{O \in \tau \mid \eta \in O\}$ where $M = (X, \tau, \nu)$ is a topos-model. The problem is that in this case we would quantify universally on open sets and not existentially. However, we have seen that $[\![M]\!]_\Sigma(\Box\varphi)$ and $[\![M]\!]_\Sigma(\Diamond\varphi)$ define topological interior and closure of $[\![M]\!]_\Sigma(\varphi)$. It is well known that interior and closure commute with finite intersection and finite union, respectively. Hence, $\Box$and $\diamond$ are not algebraic erosion and dilation, even if formulas are extended to infinite conjunction and disjunction. They are algebraic erosion and dilation when topos-models are Alexandroff topologies (Alexandroff, 1937; Alexandroff

& Hopf, 1935). Now, this is not restrictive because we only handle finite conjunctions and disjunctions in the logic **TMPL**. Moreover, $\Box$and $\diamond$ possess the other good properties of erosion and dilation which are useful to us in this paper. They are dual. Moreover, $\Box$is anti-extensive (and dually $\diamond$ is extensive) for $\preceq_M$. Indeed, let $\eta \in [\![M]\!]_\Sigma(\Box\varphi)$ be a state. This means that there exists an open set $O \in \tau$ such that $\eta \in O$ and for every $\eta' \in O$, $\eta' \in [\![M]\!]_\Sigma(\varphi)$. Hence, we necessarily have that $\eta \in [\![M]\!]_\Sigma(\varphi)$. We can also easily show that $\varphi \equiv \Box\varphi$.[10] By contrast, adjunction does not hold in general except under the (necessary and sufficient) condition that the underlying topology of topos-models satisfies that the closed sets defining formulas are precisely the open sets.

Proposition 4.6: *Let $M = (X, \tau, \nu)$ be a topos-model over a signature Σ. Then, we have: $\forall \varphi, \psi \in Sen(\Sigma), [\Diamond\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M} \iff [\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$ if and only if for every $\varphi \in Sen(\Sigma)$, $[\![M]\!]_\Sigma(\varphi)$ is a closed set of X is equivalent to $[\![M]\!]_\Sigma(\varphi)$ is an open set of X.*

Proof: $\Rightarrow$: Let us assume that $\forall \varphi, \psi \in Sen(\Sigma), [\Diamond\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M} \iff [\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$. Let φ be a Σ-formula such that $[\![M]\!]_\Sigma(\varphi)$ is a closed set. We then have that $[\![M]\!]_\Sigma(\Diamond\varphi) = [\![M]\!]_\Sigma(\varphi)$, and therefore $[\Diamond\varphi]_{\equiv_M} \preceq_M [\varphi]_{\equiv_M}$. By applying the equivalence $[\Diamond\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M} \iff [\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$ to $\psi = \varphi$, we obtain that $[\varphi]_{\equiv_M} \preceq_M [\Box\varphi]_{\equiv_M}$. As $\Box$ is anti-extensive, we can then conclude that $[\![M]\!]_\Sigma(\varphi) = [\![M]\!]_\Sigma(\Box\varphi)$, and then $[\![M]\!]_\Sigma(\varphi)$ is open. Dually, applying this to the complement set allows us to conclude that all open sets of X are closed.

$\Leftarrow$: Let us assume that the closed sets of X defining a formula are precisely the open sets of X. Let φ and ψ be two formulas such that $[\Diamond\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$. By monotonicity of $\Box$, we have that $[\Box\Diamond\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$. Now, by definition of $\diamond$, $[\![M]\!]_\Sigma(\Diamond\varphi)$ is open, and then closed by hypothesis. Hence, we have that $[\![M]\!]_\Sigma(\Box\Diamond\varphi) = [\![M]\!]_\Sigma(\Diamond\varphi)$. But, as $\diamond$ is extensive, we have that $[\varphi]_{\equiv_M} \preceq_M [\Diamond\varphi]_{\equiv_M}$, and we can conclude that $[\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$.

Conversely, if $[\varphi]_{\equiv_M} \preceq_M [\Box\psi]_{\equiv_M}$, then by monotonicity of $\diamond$, we have that $[\Diamond\varphi]_{\equiv_M} \preceq_M [\Diamond\Box\psi]_{\equiv_M}$. But, $[\![M]\!]_\Sigma(\Box\psi)$ is open and then by hypothesis closed. Hence, we have that $[\![M]\!]_\Sigma(\Diamond\Box\psi) = [\![M]\!]_\Sigma(\Box\psi)$. By anti-extensivity of $\Box$, we can directly conclude that $[\Diamond\varphi]_{\equiv_M} \preceq_M [\psi]_{\equiv_M}$. ∎

4.5. A sound and complete entailment system

In this section, we define the syntactic approach to truth for stratified institutions equipped with dual operators. This consists in establishing consequence relations $\vdash$, called *proofs*, between set of formulas and formulas. The syntactic approach of truth is then complementary to the semantic one represented by the semantic consequence $\models$. When we have that $\vdash \subseteq \models$, the syntactic approach is said *sound* and when we have the opposite inclusion, it is said *complete*. To obtain the result of completeness, we need to consider that formulas are built inductively from 'basic' formulas by applying iteratively Boolean connectives and a I-indexed family of dual operators E^i and D^i (resp. E^i_B and $D^i_{\check{B}}$ when erosion and dilation are defined based on a structuring element B) for $i \in I$. In Sections 3.1, 4.3 and 4.4, we have already given an abstract definition of Boolean connectives and of dual operators E^i and D^i. It remains then to give an abstract definition of basic formulas.

Definition 4.7 (Basic formulas): A set of formulas $Bc \subseteq Sen(\Sigma)$ is **basic** if there exists a Σ-model $M_{Bc} \in Mod(\Sigma)$ and a state $\eta \in [\![M_{Bc}]\!]_\Sigma$ such that for every $M \in Mod(\Sigma)$ and every $\eta' \in [\![M]\!]_\Sigma$, $M \models_\Sigma^{\eta'} Bc$ if and only if there exists a morphism $\mu_{\eta'} : M_{Bc} \to M$ such that $[\![\mu_{\eta'}]\!]_\Sigma(\eta) = \eta'$.

M_{Bc} and η are called **basic model** and **basic state** for Bc, respectively.

The notion of basic formulas has been first defined in Diaconescu (2008), Gaina and Petria (2010) but in institutions, and then for sentences (i.e. closed formulas). Here, to take into account open formulas, the definition of basic formulas involves states.

Proposition 4.8: *Any set of atomic formulas in **PL**, **FOL**, **MPL**, **TMPL** and **MMPL** is basic.*

Proof: **PL.** Let P be a propositional signature. Let $Bc \subseteq P$. Let M_{Bc} be the model that associates 1 to any $p \in Bc$ and 0 to any $p \in P \setminus Bc$. The choice of $\eta \in [\![M_{Bc}]\!]_P$ is obvious because $[\![M_{Bc}]\!]_P = \mathbb{1}$ (cf. Example 2.6).

Let $M \in Mod(P)$ such that $M \models_P Bc$. This means that for every $p \in Bc$, $M(p) = 1$ whence we can conclude that $M_{Bc} \leq M$ where $\leq$ is the partial ordering on models in $Mod(P)$. Conversely, let us suppose a morphism $\mu : M_{Bc} \to M$ (obviously, by the definition of models in **PL**, we have that $[\![\mu]\!]_P(\mathbb{1}) = \mathbb{1}$). By hypothesis, we have that $M_{Bc} \leq M$ whence we can directly conclude that for every $p \in Bc$, $M(p) = 1$.

FOL. Let $\Sigma = (S, F, P)$ be a signature. Let Bc be a set of atomic formulas over a set of variables X. Let us denote M_{Bc} the Σ-model defined by:

- $\forall s \in S, M_{Bc_s} = T_F(X)_s$;
- $\forall f : s_1 \times \ldots \times s_n \to s \in F, f^{M_{Bc}} : (t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n)$;
- $\forall p : s_1 \times \ldots \times s_n \in P, p^{M_{Bc}} = \{(t_1, \ldots, t_n) \mid p(t_1, \ldots, t_n) \in Bc\}$.

Let us set η the variable interpretation defined as $x \mapsto x$.

Let $M \in Mod(\Sigma)$ be a model and $\nu : X \to M$ be an interpretation such that $M \models_\Sigma^\nu Bc$. Therefore, we can define $\mu_\nu : \begin{cases} x \mapsto \nu(x) \\ f(t_1, \ldots, t_n) \mapsto f^M(\mu_\nu(t_1), \\ \ldots, \mu_\nu(t_n)) \end{cases}$
which is a morphism. Obviously, we have that $[\![\mu_\nu]\!]_\Sigma(\eta) = \nu$.

Conversely, let us suppose a morphism $\mu : M_{Bc} \to M$ such that $[\![\mu]\!]_\Sigma(\eta) = \nu$. Let $p(t_1, \ldots, t_n) \in Bc$. As $[\![\mu]\!]_\Sigma(\eta) = \nu$, for every $t \in T_F(X)$, we have that $\mu(t) = \nu(t)$, and then, as μ is a morphism, we can conclude that $(\nu(t_1), \ldots, \nu(t_n)) \in p^M$.

MPL. Let P be a propositional signature. Let Bc be a subset of P. Let M_{Bc} be the model defined by:

- $I = \mathbb{1}$ (any singleton);
- $W^{\mathbb{1}} = Bc$;
- $R = \emptyset$.

Obviously, $\eta = \mathbb{1}$. Let $M = (I', W', R')$ be a P-model and let $i' \in I'$ be a state such that $M \models_P^{i'} Bc$. Let us define the morphism $\mu_{i'} : \mathbb{1} \mapsto i'$. Obviously, we have that $[\![\mu_{i'}]\!]_P(\mathbb{1}) = i'$.

Conversely, let us suppose a morphism $\mu : M_{Bc} \to M$ such that $[\![\mu]\!]_P(\mathbb{1}) = i'$. As $W^{\mathbb{1}} \subseteq W'^{i'}$, we directly have that $M \models_P^{i'} Bc$.

It is standard in modal logic to restrict the class of models to satisfy supplementary axioms. For instance, to satisfy $\Box\varphi \Rightarrow \varphi$, models have to be reflexive (i.e. the accessibility relation is reflexive). In this case, the basic model M_{Bc} is defined as previously except that $R = \{(\mathbb{1}, \mathbb{1})\}$.

TMPL. Let P be a propositional signature. Let $Bc \subseteq P$. Let us denote M_{Bc} the P-model defined by:

- $X = \{Bc\}$;
- $\tau = \{\emptyset, \{Bc\}\}$ (the topology is both discrete and trivial);
- $\nu : p \mapsto \begin{cases} \{Bc\} & \text{if } p \in Bc \\ \emptyset & \text{otherwise} \end{cases}$

Let us set $\eta = Bc$. Let $M = (X', \tau, \nu')$ be a P-model and $x \in X'$ such that $M \models^x_P Bc$. Then, let us define the mapping $\mu_x : Bc \mapsto x$. Let us show that μ_x is a morphism. First, let us show that it is continuous. Let $O \in \tau'$ be an open set. Two possibilities can occur:

(1) $x \in O$. In this case, $\mu_x^{-1}(O) = \{Bc\}$;
(2) $x \notin O$. In this case, $\mu_x^{-1}(O) = \emptyset$.

In both cases, $\mu_x^{-1}(O)$ is an open set, and then μ_x is continuous. Let $p \in P$. Here, two cases have to be considered:

(1) $p \in Bc$. As $M \models^x_P Bc$, we have that $x \in \nu'(p)$, and then $\mu_x(\nu(p)) \subseteq \nu'(p)$;
(2) $p \notin Bc$. By definition of M_{Bc}, $\nu(p) = \emptyset$, and then $\mu_x(\nu(p)) = \emptyset$.

Conversely, let us suppose a morphism $\mu : M_{Bc} \to M$ such that $[\![\mu]\!]_P(Bc) = x$. Let $p \in Bc$. As μ is a morphism, we have that $\mu(Bc) = x \in \nu'(p)$, and then $M \models^x_P Bc$.

MMPL. The construction of the model M_{Bc} for the logic **MMPL** is similar to that for **TMPL**, as from any metric space a topology can be induced.

■

Then, let us set the framework for this section.

Framework: we consider a stratified institution $\mathcal{I}$ whose functor *Sen* has a subfunctor $Sen^{base} : Sig \to Set$ (i.e. $Sen^{base}(\Sigma) \subseteq Sen(\Sigma)$) such that for every signature $\Sigma \in Sig$:

- $Sen^{base}(\Sigma)$ is basic, and
- $Sen(\Sigma)$ is inductively defined from $Sen^{base}(\Sigma)$ by applying Boolean connectives in $\{\wedge, \vee, \Rightarrow, \neg\}$ and a I-indexed family of dual operators E^i and D^i (resp. E^i_B and $D^i_{\check{B}}$ when erosion and dilation are defined over a structuring element B) such that for each $i \in I$, E^i and D^i are anti-extensive and extensive, respectively, and for all $\varphi \in Sen(\Sigma)$, $\varphi \models E^i(\varphi)$. [11]

For all the examples of stratified institutions developed in this paper, we define the functor Sen^{base} as the mapping which associates to any signature $\Sigma \in Sig$ the set of atomic formulas. In **PL**, the family of dual operators is indexed by the emptyset. In **FOL**, the family of dual operators is indexed by a set of variables X. Hence, in **FOL**, E^x and D^x are respectively $\forall x$ and $\exists x$. In **MPL**, **TMPL** and **MMPL**, the family is indexed by any singleton as we only consider the couple of dual operators $\Box$and $\diamond$.

We have seen for all the examples where the dual operators E^i and D^i are erosion and dilation based on a structuring element B that they are anti-extensive and extensive if for every model $M \in Mod(\Sigma)$ and for every state $\eta \in [\![M]\!]_\Sigma$, we have $\eta \in B_\eta$. Hence, **PL** and **FOL**, as well as **MPL** when the category of models is restricted to reflexive models, meet all the requirements of our framework. This is the same for **TMPL** (and hence for **MMPL**) as $\Box$and $\diamond$ define topological interior and closure which are known to be anti-extensive and extensive (see Section 4.4.2).

Finally, from Property 2 in Theorem 4.2, the property $\varphi \models E^i(\varphi)$ is always satisfied when dual operators E^i and D^i are defined using a structuring element B, as in **FOL** and **MPL**. For **TMPL** (and then **MMPL**), we have also seen in Section 4.4.2 that this last property holds.

Definition 4.9 (Tautology instance): We call **tautology instance** any formula $\varphi \in Sen(\Sigma)$ such that there exists a propositional tautology ψ (i.e. ψ is a tautology in the logic **PL**) whose propositional variables are among $\{p_1, \dots, p_n\}$ and n formulas $\varphi_i \in Sen(\Sigma)$ such that φ is obtained by replacing in ψ all the occurrences of p_i by φ_i for $i \in \{1, \dots, n\}$.

What justifies such a definition is the following result:

Proposition 4.10: *Let ψ be a propositional tautology whose propositional variables are among $\{p_1, \dots, p_n\}$. Let $\varphi_1, \dots, \varphi_n \in Sen(\Sigma)$ be n formulas. Then, the formula φ in $Sen(\Sigma)$ obtained by replacing in ψ all the occurences of p_i by φ_i for $i \in \{1, \dots, n\}$ is a tautology, i.e. for every $M \in Mod(\Sigma)$, $[\![M]\!]_\Sigma(\varphi) = [\![M]\!]_\Sigma$.*

Proof: Let $M \in Mod(\Sigma)$ be a model. Let $\eta \in [\![M]\!]_\Sigma$ be a state. Let us define the propositional model ν in **PL** by:

$$\nu : p_i \mapsto \begin{cases} 1 & \text{if } M \models^\eta_\Sigma \varphi_i \\ 0 & \text{otherwise} \end{cases}$$

By hypothesis, we have that $\nu \models \psi$, and then we can conclude that $M \models^\eta_\Sigma \varphi$. ∎

The proof of completeness that we present here follows Henkin's method (Henkin, 1949). This method relies on the proof that every consistent set of formulas has a model. This relies on the deduction theorem which is known to fail for modal logics except under some conditions (see Hakli & Negri, 2012). Here, we give a condition based on the notion of 'invariant formula' that we define just below and which ensures the deduction theorem. This condition differs from that given in Hakli and Negri (2012) in the sense that it is not about a restriction of the application of the inference rule *Necessity* (see below). As we will see later in this section, our condition will prove to be similar for **MPL** and **TMPL** (and then **MMPL**) to change the definition of $\Gamma \vdash_\Sigma \varphi$ into: $\Gamma \vdash_\Sigma \varphi$ iff there exists a finite susbset $\{\varphi_1, \dots, \varphi_n\} \subseteq \Gamma$ such that $\vdash_\Sigma \varphi_1 \wedge \dots \wedge \varphi_n \Rightarrow \varphi$ (so-called *local derivation*).

Definition 4.11 (Invariant formula): Let $\varphi \in Sen(\Sigma)$. φ is said **invariant** if: $\forall i \in I, \forall M \in Mod(\Sigma), \varphi \preceq_M E^i(\varphi)$.

When E^i and D^i are erosion and dilation based on a structuring element B, it is easy to see that every formula $\varphi \in Sen(\Sigma)$ such that, for every $M \in Mod(\Sigma)$, $[\![M]\!]_\Sigma(\varphi)$ is equal to either $[\![M]\!]_\Sigma$ or $\emptyset$ is an invariant formula. Hence, in **FOL**, all closed formulas (i.e. without free (unbound) variables) are invariant, and in **MPL**, tautologies and antilogies are invariant formulas. It is easy to see that when an invariant formula is a tautology or an antilogy, then so is its negation.

In **TMPL** (and then **MMPL**), all tautologies and antilogies are also invariant formulas.[12]

Definition 4.12 (Formula instance): Let $\varphi, \varphi' \in Sen(\Sigma)$. The formula φ' is an **instance of** φ for $i \in I$ (I is the index set of the family of the dual operators E^i and D^i) if for every $M \in Mod(\Sigma)$, $[\![M]\!]_\Sigma(E^i(\varphi)) \subseteq [\![M]\!]_\Sigma(\varphi')$.

Formula instance generalises in stratified institution the concept of substitutions which are standard in first-order logics. Indeed, in **FOL**, given a formula φ, we have for every variable $x \in X$ that $\forall x.\varphi \Rightarrow \varphi(x/t)$ is a tautology where $t \in T_F(X)$ and $\varphi(x/t)$ is the formula obtained from φ by substituting every free occurence of x by the term t. Of course, by the hypothesis that each E^i is anti-extensive, φ is always an instance of itself for $i \in I$.

We then consider the following Hilbert-system for the stratified institution $\mathcal{I}$.

- **Axioms:**
  - *Tautologies:* all tautology instances;
  - *Duality:* $E^i(\varphi) \Leftrightarrow \neg D^i(\neg\varphi)$;
  - *Distribution:* $E^i(\varphi \Rightarrow \psi) \Rightarrow E^i(\varphi) \Rightarrow E^i(\psi)$ (this axiom is called the Kripke schema);
  - *Instantiation:* $E^i(\varphi) \Rightarrow \varphi'$ when φ' is an instance of φ for $i \in I$;
  - *Invariability:* $\varphi \Rightarrow E^i(\varphi)$ when φ is an invariant formula.
- **Inference rules:**
  - *Modus Ponens:* $\frac{\varphi \Rightarrow \psi \quad \varphi}{\psi}$;
  - *Necessity:* $\frac{\varphi}{E^i(\varphi)}$.

In modal logic, the inference rules and axioms given above define the system T. The systems $S4$, B and $S5$ can be obtained by adding respectively the axioms written in our framework as follows:

- $E^i(\varphi) \Rightarrow E^i(E^i(\varphi))$ ($S4$),
- $\varphi \Rightarrow E^i(D^i(\varphi))$ (B),
- $D^i(\varphi) \Rightarrow E^i(D^i(\varphi))$ ($S5$),

In contrast, by imposing the anti-extensivity property, the systems K and D of the modal logic are not taken into account here.

Definition 4.13 (Derivation): A formula $\varphi \in Sen(\Sigma)$ is **derivable** from a set of assumptions $\Gamma \subseteq Sen(\Sigma)$, written $\Gamma \vdash_\Sigma \varphi$, if $\varphi \in \Gamma$, or is one of the axioms, or follows from derivable formulas through applications of the inference rules.

Hence, the proof system for $\mathcal{I}$ can be defined by the four following inference rules:

$$\frac{\varphi \in \Gamma}{\Gamma \vdash_\Sigma \varphi} \qquad \frac{\varphi\text{:Axiom}}{\Gamma \vdash_\Sigma \varphi}$$
$$\frac{\Gamma \vdash_\Sigma \varphi \Delta \vdash_\Sigma \varphi \Rightarrow \psi}{\Gamma \cup \Delta \vdash_\Sigma \psi} \qquad \frac{\Gamma \vdash_\Sigma \varphi}{\Gamma \vdash_\Sigma E^i(\varphi)}$$

These inference rules give rise to an entailment system (Meseguer, 1989), i.e. a *Sig*-indexed family of binary relations $\vdash_\Sigma \subseteq \mathcal{P}(Sen(\Sigma)) \times Sen(\Sigma)$. Standardly, the *Sig*-indexed family $\{\vdash_\Sigma\}_{\Sigma \in Sig}$ satisfies the following properties:

Transitivity if $\Gamma \vdash_\Sigma \Gamma'$ and $\Gamma' \vdash_\Sigma \Gamma''$, then $\Gamma \vdash_\Sigma \Gamma''$;
Monotonicity if $\Gamma \vdash_\Sigma \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_\Sigma \varphi$;
Compactness if $\Gamma \vdash_\Sigma \varphi$, then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_\Sigma \varphi$;
Translation $\Gamma \vdash_\Sigma \varphi$, then $\forall \sigma : \Sigma \to \Sigma', \sigma(\Gamma') \vdash_{\Sigma'} \sigma(\varphi)$.

Transitivity, Monotonicity and Translation are obvious to show. Compactness results from the fact that derivations are finite. Then, from an infinite set of hypothesis Γ, we can always make a derivation from a finite number of hypothesis. It is sufficient to consider the set of hypotheses Γ_0 which have been used in the derivation. As derivations are finite, Γ_0 is necessarily a finite set.

This system is enough to infer other properties of E^i and D^i such as the commutativity of E^i (resp. D^i) with infimum (resp. supremum).

Theorem 4.14: *The proof system defined above is sound, i.e. if* $\Gamma \vdash_\Sigma \varphi$, *then* $\Gamma \models_\Sigma \varphi$.

Proof: Directly results from the assumptions and the properties of dilation and erosion (see Sections 4.3.4 and 4.4). ■

Finally, thanks to the condition of 'invariability' for formulas, we get the deduction theorem.

Proposition 4.15 (Deduction theorem): *Let* $\Gamma \subseteq Sen(\Sigma)$ *be a set of assumptions. If* φ *is an invariant formula, then we have* $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$ *if and only if* $\Gamma \vdash_\Sigma \varphi \Rightarrow \psi$.

Proof: The necessary condition is obvious and can be easily obtained by Modus Ponens. The sufficient condition is proved by induction on the given proof. The more difficult case is that where the last inference rule is Necessity. We then have that $\Gamma \cup \{\varphi\} \vdash_\Sigma E^i(\psi)$. This means that $\Gamma \cup \{\varphi\} \vdash_\Sigma \psi$ previously in the proof, and then by the induction hypothesis we have that $\Gamma \vdash_\Sigma \varphi \Rightarrow \psi$. By Necessity, Distribution and Modus Ponens, we have that $\Gamma \vdash_\Sigma E^i(\varphi) \Rightarrow E^i(\psi)$. By the invariant axiom and the fact that φ is an invariant formula, $\Gamma \vdash_\Sigma \varphi \Rightarrow E^i(\varphi)$, and then by transitivity, we can conclude that $\Gamma \vdash_\Sigma \varphi \Rightarrow E^i(\psi)$. ■

The following corollary justifies proof by reduction ad absurbum.

Corollary 4.16: *For every $\Gamma \subseteq Sen(\Sigma)$ and $\varphi \in Sen(\Sigma)$ such that $\neg\varphi$ is an invariant formula, we have that $\Gamma \vdash_\Sigma \varphi$ if and only if $\Gamma \cup \{\neg\varphi\}$ is inconsistent* (*i.e. for every formula $\psi \in Sen(\Sigma)$, $\Gamma \cup \{\neg\varphi\} \vdash_\Sigma \psi$ and $\Gamma \cup \{\neg\varphi\} \vdash_\Sigma \neg\psi$*).

Proof: The '$\Rightarrow$' part is obvious. Let us prove the '$\Leftarrow$' part. Let us suppose that $\Gamma \cup \{\neg\varphi\}$ is inconsistent. This then means that we have both $\Gamma \cup \{\neg\varphi\} \vdash_\Sigma \varphi$ and $\Gamma \cup \{\neg\varphi\} \vdash_\Sigma \neg\varphi$. As $\neg\varphi$ is an invariant formula by Proposition 4.15 we can write that $\Gamma \vdash_\Sigma \neg\varphi \Rightarrow \varphi$. The formula $(\neg\varphi \Rightarrow \varphi) \Rightarrow \varphi$ is a tautology axiom, and then by Modus Ponens we have that $\Gamma \vdash_\Sigma \varphi$. ■

Definition 4.17 (Maximal Consistence): A set of formulas $\Gamma \subseteq Sen(\Sigma)$ is **maximally consistent** if it is consistent and there is no consistent set of formulas properly containing Γ (i.e. for each formula $\varphi \in Sen(\Sigma)$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both).

Proposition 4.18: *Let $\Gamma \subseteq Sen(\Sigma)$ be a consistent set of formulas. There exists a maximally consistent set of formulas $\overline{\Gamma} \subseteq Sen(\Sigma)$ that contains Γ.*

Proof: Let $S = \{\Gamma' \subseteq Sen(\Sigma) \mid \Gamma'$ is consistent and $\Gamma \subseteq \Gamma'\}$. The poset $(S, \subseteq)$ is inductive. Therefore, by Zorn's lemma, S has a maximal element $\overline{\Gamma}$. By definition of S, $\overline{\Gamma}$ is consistent and contains Γ. Moreover, it is maximal. Otherwise, there exists a formula $\varphi \in Sen(\Sigma)$ such that $\varphi \notin \overline{\Gamma}$. As $\overline{\Gamma}$ is maximal, this means that $\overline{\Gamma} \cup \{\varphi\}$ is inconsistent, and then $\overline{\Gamma} \cup \{\neg\varphi\}$ is. As $\overline{\Gamma}$ is maximal, we can conclude that $\neg\varphi \in \overline{\Gamma}$. ■

Proposition 4.18 is a quite direct generalisation to stratified institutions of Lindenbaum's Lemma. To obtain our result of completeness, we need to impose the following condition:

Assumption. For every basic set of formulas $Bc \subseteq Sen^{base}(\Sigma)$, there exists a basic model $M_{Bc} \in Mod(\Sigma)$ and a basic state $\eta \in [\![M_{Bc}]\!]_\Sigma$ for Bc such that for every $i \in I$ (I is the index-set of the dual operators E^i and D^i) and every $\varphi \in Sen(\Sigma)$, there exists a subset $Inst_i(\varphi)$ of instances of φ for i satisfying :

(1) for every $\varphi' \in Inst_i(\varphi)$, $|\varphi'| \leq |\varphi|$ where $|\varphi|$ and $|\varphi'|$ are the numbers of Boolean connectives and dual operators in φ and φ', and
(2) $(\forall \varphi' \in Inst_i(\varphi), M_{Bc} \models^\eta_\Sigma \varphi') \Longrightarrow M_{Bc} \models^\eta_\Sigma E^i(\varphi)$.

Proposition 4.19: *All the couples (M_{Bc}, η) defined in the proof of Proposition 4.8 for **PL**, **FOL**, **MPL**, **TMPL** and **MMPL** satisfy such an assumption.*

Proof: The proof for **PL** is obvious because the set of dual operators is empty (except the conjunction and disjunction which are assumed in the definition of the logic). For **MPL**, **TMPL** and **MMPL**, as $\Box$ is anti-extensive, for every $\varphi \in Sen(\Sigma)$, we can set $Inst_{\mathbb{1}}(\varphi) = \{\varphi\}$ (let us recall that the index set for dual operators is here represented by the singleton with the unique element $\mathbb{1}$). The first condition of the assumption is obviously satisfied. Finally, as $\Box$ is anti-extensive, the accessibility relation is reflexive, and then if $M_{Bc} \models^{\mathbb{1}}_\Sigma \varphi$ in **MPL** (resp. $M_{Bc} \models^{Bc}_\Sigma \varphi$ in **TMPL** and **MMPL**), then we necessary have that $M_{Bc} \models^{\mathbb{1}}_\Sigma \Box\varphi$ in **MPL** (resp. $M_{Bc} \models^{Bc}_\Sigma \Box\varphi$ in **TMPL** and **MMPL**).

In **FOL**, given a variable $x \in X$, let us set $Inst_x(\varphi) = \{\varphi(x/t) \mid t \in T_F(X)\}$. Obviously, the first condition of the assumption is satisfied. Finally, if we suppose that $M_{Bc} \models^{Id}_{\Sigma} \varphi(x/t)$ for every $t \in T_F(X)$, then we have for each $\sigma : X \to T_F(X)$ such that for every $y \neq x \in X$, $\sigma(y) = y$ and $\sigma(x) = t$ that $M_{Bc} \models^{\sigma}_{\Sigma} \varphi$, whence we can conclude that $M_{Bc} \models^{Id}_{\Sigma} \forall x.\varphi$. ■

Proposition 4.20: *Let the assumption be satisfied. Then, for every maximal consistent set of formulas $\Gamma \subseteq Sen(\Sigma)$, there exists a Σ-model M and a state $\eta \in [\![M]\!]_{\Sigma}$ such that $\Gamma = \{\varphi \mid M \models^{\eta}_{\Sigma} \varphi\}$.*

Proof: Let us denote $Bc = \Gamma \cap Sen^{base}(\Sigma)$. By definition of basic set of formulas, there exists a basic model M_{Bc} and a state η for Bc that satisfy the assumption. Then, let us show by induction on the size of φ that:

$$\Gamma \vdash \varphi \Longleftrightarrow M_{Bc} \models^{\eta}_{\Sigma} \varphi$$

The cases of basic formulas and Boolean connectives are easily provable. Then, let φ be of the form $E^i(\psi)$.

($\Rightarrow$) Let us suppose that $\Gamma \vdash E^i(\psi)$. By Modus Ponens with $\Gamma \vdash_{\Sigma} E^i(\psi) \Rightarrow \psi'$ (Instantiation) where $\psi' \in Inst_x(\psi)$, we then have that $\Gamma \vdash \psi'$. By the first condition of the assumption, we can apply the induction hypothesis on every $\psi' \in Inst_x(\psi)$, and then we have that $M_{Bc} \models^{\eta}_{\Sigma} \psi'$, whence by the seconde condition of the assumption, we can conclude that $M_{Bc} \models^{\eta}_{\Sigma} E^i(\psi)$.

($\Leftarrow$) Let us suppose that $M_{Bc} \models^{\eta}_{\Sigma} E^i(\psi)$. By anti-extensivity of E^i, we then have that $M_{Bc} \models^{\eta}_{\Sigma} \psi$. By the induction hypothesis, we have that $\Gamma \vdash \psi$, and then by Necessity, $\Gamma \vdash E^i(\psi)$. ■

Theorem 4.21 (Completeness): *Let the assumption be satisfied. Then, for every $\Gamma \subseteq Sen(\Sigma)$ and every $\varphi \in Sen(\Sigma)$ such that $\neg\varphi$ is an invariant formula, we have that:*

$$\Gamma \models \varphi \Longrightarrow \Gamma \vdash \varphi$$

Proof: If $\Gamma \nvdash \varphi$, then $\Gamma \cup \{\neg\varphi\}$ is consistent. By Proposition 4.18, there exists a maximal consistent set of formulas Γ' that extends Γ, and then by Proposition 4.20, there exists a model M and a state $\eta \in [\![M]\!]_{\Sigma}$ such that $M \models^{\eta}_{\Sigma} \neg\varphi$, i.e. $M \not\models^{\eta}_{\Sigma} \varphi$. ■

Corollary 4.22: *The inference rules for **PL** is complete for any formulas. They are complete in **FOL** for every closed formulas, and in **MPL** and **TMPL** for tautologies (and then so is for **MMPL**)*

We find the standard results of completeness, among other to **MPL** and **TMPL** (and then **MMPL**) where it is known that the completeness result holds for the local derivation (which amounts to demonstrate tautologies). More precisely, for **MPL**, we have shown the completeness for the proof system known under the name *T* and its extensions *S*4, *B* and *S*5. On the contrary, as the anti-extensivity and extensivity properties of E^i and D^i are imposed (and then the accessibility relations are necessarily reflexive), the abstract proof given here cannot be instantiated to show the completeness result for the systems K and D. For these two systems, we cannot use the model M_{Bc} defined for the logic **MPL** in the proof of Proposition 4.8 to prove their incompleteness. We have to consider the canonical model for which the set of states is the whole set of sets of maximally consistent formulas. The problem is that such a model has no equivalent for **PL** and **FOL**. An open problem would be to see if there exists a general proof based on Henkin's method which works both for logics with dual operators which are extensive and anti-extensive, and for logics with dual operators which are not.

Similar proofs of completeness have already been obtained in the framework of institutions but only for first-order logics (Gaina & Petria, 2010; Petria, 2007). In Petria (2007), the author follows Henkin's method to prove his first-order completeness result while in Gaina and Petria (2010), the authors use forcing methods to extend their first completeness result to infinitary first-order logics.

Here, we have extended these first results by unifying, in the framework of stratified institutions, a completeness proof which works both for **FOL** and the modal logics such as T, $S4$, B and $S5$, **TMPL** and **MMPL**.

5. Towards applications in qualitative spatial reasoning

When dealing with qualitative spatial reasoning, spatial relationships are usually classified into topological, metric or directional relations (Aiello et al., 2007; Kuipers, 2000). In this section, we briefly show how such relations can be expressed in our framework.

5.1. Topological relationships

Topological approaches to qualitative spatial reasoning usually describe relationships between spatial regions. Two models have emerged to formalise topological spatial relations between spatial entities: RCC-8 (Randell et al., 1992) and 9-intersection (Egenhofer, 1991; Egenhofer & Franzosa, 1991).

5.1.1. RCC-8

RRC-8 is a first-order theory based on a primitive connectedness relation **C**. From this binary relation **C**, many other binary relations can be defined, among which 8 were identified as being of particular importance, via the definition of a parthood predicate **P** defined from **C**:

1. **DC**(X, Y) means that X is disconnected from Y;
2. **EC**(X, Y) means that X is externally connected to Y;
3. **PO**(X, Y) means that X partially overlaps Y;
4. **TPP**(X, Y) (resp. **TPPi**(X, Y)) means that X (resp. Y) is a tangential proper part of Y (resp. X);
5. **NTPP**(X, Y) (resp. **NTPPi**(X, Y)) means that X (resp. Y) is a non-tangential proper part of Y (resp. X);
6. **EQ**(X, Y) means that X is identical to Y.

Here, given a stratified institution $\mathcal{I} = (Sig, Sen, Mod, [\![\_]\!], \models)$ and a model $M \in Mod(\Sigma)$, the elements in $[\![M]\!]_\Sigma$ are spatial entities, and then formulas are combinations of such entities. The model RCC-8 is a first-order theory which allows one to quantify on spatial entities. Here, this would amount to quantify on states which is not allowed by the langage. Following Aiello and van Benthem (2002), we introduce the modality U and its dual A [13] whose semantics is as follows:

- $M \models^{\eta}_{\Sigma} U\varphi$ iff $\forall \eta' \in [\![M]\!]_{\Sigma}, M \models^{\eta'}_{\Sigma} \varphi$
- $M \models^{\eta}_{\Sigma} A\varphi$ iff $\exists \eta' \in [\![M]\!]_{\Sigma}, M \models^{\eta'}_{\Sigma} \varphi$

Using these primitives connectors, following Bloch (2002), it is easy to define, independently of any stratified institution, simple relations such as inclusion, exclusion and intersection by using standard Boolean connectives in $\{\wedge, \vee, \Rightarrow, \neg\}$ and the modalities U and A. Hence, the binary relations **C**, **DC**, **PO** and **EQ** can be expressed in our framework as follows, where φ and ψ are formulas that denote, respectively, the regions X and Y:

- $\mathbf{C}(X, Y)$: $A(\varphi \wedge \psi)$;
- $\mathbf{DC}(X, Y)$: $U(\neg\varphi \vee \neg\psi)$;
- $\mathbf{PO}(X, Y)$: $A(\varphi \wedge \psi)$, $A(\varphi \wedge \neg\psi)$, and $A(\neg\varphi \wedge \psi)$;
- $\mathbf{EQ}(X, Y)$: $\varphi \Leftrightarrow \psi$.

The other relations can benefit from the morphological operators. For this, we suppose that the stratified institution $\mathcal{I}$ is equipped with two dual logical operators E and D defined as an erosion and a dilation on the lattice $(Sen(\Sigma)_{/\equiv_M}, \preceq_M)$ for every signature Σ and every Σ-model M such that E and D are anti-extensive and extensive, respectively, for the binary relation $\preceq_M$. To define adjacency (or external connection) $\mathbf{EC}(X, Y)$ between two regions X and Y, we can then consider that these regions do not intersect but as soon as one of them is dilated, it has a non-empty intersection with the other one. This can be expressed as:

- $\mathbf{EC}(X, Y)$: $\neg(\varphi \wedge \psi)$ and $A(D(\varphi) \wedge \psi)$ and $A(\varphi \wedge D(\psi))$.

Now, the fact that a region X is a *tangential proper part of* a region Y (i.e. $\mathbf{TPP}(X, Y)$) can be expressed by the fact that X is included in Y but the dilation of X is not, i.e.:

- $\mathbf{TPP}(X, Y)$: $\varphi \Rightarrow \psi$ and $A(D(\varphi) \wedge \neg\psi)$.

Similarly, the fact that a region X is a *non-tangential proper part of* a region Y (i.e. $\mathbf{NTPP}(X, Y)$) can be expressed as:

- $\mathbf{NTPP}(X, Y)$: $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow E(\psi)$ (or equivalently, $D(\varphi) \Rightarrow \psi$).

5.1.2. 9-intersection

The 9-intersection model transforms the topological relationships between two spatial entities X and Y into a point-set topology problem. That is, the topological relations between two objects X and Y are defined in terms of the intersection of boundary, interior and exterior of X and Y. Hence, the 9-intersection model captures the topological relation between two spatial entities X and Y

based on the intersections of the three topological parts of X and those of Y. These 3×3 types of intersections are concisely represented by the 9-intersection matrix:

$$\begin{pmatrix} \delta X \cap \delta Y & \delta X \cap Y^o & \delta X \cap Y^- \\ X^o \cap \delta Y & X^o \cap Y^o & X^o \cap Y^- \\ X^- \cap \delta Y & X^- \cap Y^o & X^- \cap Y^- \end{pmatrix}$$

where $\_^o$, $\_^-$ and $\delta\_$ denote the interior, the exterior and the boundary, respectively.

For any stratified institution whose models are topos-model, these 3×3 types of intersections can be easily defined. Indeed, if we suppose that the two regions X and Y are denoted by the two formulas φ and ψ, then

- their interior are $\Box\varphi$ and $\Box\psi$,
- the exterior are $\neg\Diamond\varphi$ and $\neg\Diamond\psi$, and
- their boundary are $\varphi \wedge \neg\Box\varphi$ and $\psi \wedge \neg\Box\psi$, and in our framework $\Box$and $\diamond$ are algebraic erosion and dilation, respectively.

5.2. Distances and directional relative position

Here, we assume a stratified institution $\mathcal{I}$ such that

- either the category of states is the category of metric spaces *Met* and in this case $\mathcal{I}$ is equipped with two logical operators E and D defined as erosion and dilation on the lattice $(Sen(\Sigma)_{/\equiv_M}, \preceq_M)$ for every signature Σ and every Σ-model M such that E and D are anti-extensive and extensive, respectively, for the binary relation $\preceq_M$;
- or $\mathcal{I}$ is equipped with two logical operators E and D defined as an erosion and dilation based on an elementary symmetrical structuring element B.

In this last case, we can define a distance d that can take different forms depending on the considered spatial domain, as follows:

- $\forall \eta, d(\eta, \eta) = 0$;
- $\forall \eta, \eta', \eta \neq \eta', d(\eta, \eta') = 1$ iff $\eta' \in B_\eta$,
- $\forall \eta, \eta', d(\eta, \eta') = \inf_{\pi(\eta,\eta')} l(\pi)$, where $\pi(\eta, \eta')$ is a path from η to η', i.e. a sequence $\eta_0 = \eta, \eta_1, \ldots \eta_n = \eta'$ such that $\forall i = 0, \ldots n-1, d(\eta_i, \eta_{i+1}) = 1$, and $l(\pi)$ is the length of the path (i.e. for $\pi = \eta_0, \eta_1, \ldots \eta_n), l(\pi) = n = \sum_{i=0}^{n-1} d(\eta_i, \eta_{i+1})$).

By construction, d defines a metric.

In both cases, we can define a distance to a formula for every model $M \in Mod(\Sigma)$ as done in the Euclidean space for a distance from a point to a compact set:

$$d(\eta, \varphi) = \inf_{M \models_\Sigma^{\eta'} \varphi} d(\eta, \eta').$$

Given two formulas φ and φ', their minimum $d_{\min}$ and Hausdorff d_H distances can be derived as:

$$d_{\min}(\varphi, \varphi') = \inf_{M \models^{\eta}_{\Sigma} \varphi} d(\eta, \varphi'),$$

$$d_H(\varphi, \varphi') = \max\left(\sup_{M \models^{\eta}_{\Sigma} \varphi'} d(\eta, \varphi), \sup_{M \models^{\eta'}_{\Sigma} \varphi'} d(\eta', \varphi) \right).$$

As in the Euclidean case, these two distances can be conveniently expressed in terms of mathematical morphology. Details for the logic **PL** are given in Bloch (2002). Similarly, we have here:

$$d_{\min}(\varphi, \varphi') \leq n \text{ iff } A(D^n(\varphi) \wedge \varphi'),$$

where D^0 is the identity mapping, $D^1 = D$ and $D^n = DD^{n-1}$ for $n > 1$, and:

$$d_H(\varphi, \varphi') \leq n \text{ iff } \varphi' \Rightarrow D^n_B(\varphi) \quad \text{and} \quad \varphi \Rightarrow D^n_B(\varphi').$$

As an example of the potential use of such links between distances and dilation in spatial reasoning, let us consider the example in Bloch (2002). If we are looking at an object represented by ψ in an area which is at a distance in an interval $[n_1, n_2]$ of a region represented by φ, this corresponds to a minimum distance greater than n_1 and to a Hausdorff distance less than n_2. Then we have to check the following relation:

$$\psi \Rightarrow \neg D^{n_1}(\varphi) \wedge D^{n_2}(\varphi).$$

This expresses in a symbolic way an imprecise knowledge about distances represented as an interval. If we consider a fuzzy interval, this extends directly by means of fuzzy dilation. These expressions show how we can convert distance information, which is usually defined in an analytical way, into algebraic expressions through mathematical morphology, and then into logical expressions through the proposed abstract dual operators based on dilation and erosion.

Directional relations can be defined in a similar way in the proposed framework, extending directly the **PL** case detailed in Bloch (2002). Here, D^d denotes the dilation corresponding to a directional information in the direction d. Then assessing whether φ' represents a region of space which is in direction d with respect to the region represented by φ amounts to check the following relation:

$$\varphi' \Rightarrow D^d(\varphi).$$

6. Conclusion

In this paper, we have shown that the abstract framework of stratified institutions allows for unified definitions of connectives, quantifiers and morphological operators. Morphological dilation and erosion are defined in this framework both algebraically as operators that commute with the supremum and infimum of the underlying lattices, and using structuring elements. The duality property is emphasised, as a common property of pairs of operators or modalities in several logics. The proposed abstract definitions and properties are then instantiated in different logics, such as propositional logic, first order logic, modal logics, fuzzy logics. Finally, they are used in qualitative spatial reasoning framework to define abstract topological, metric and directional relations. This is consistent with the common use of mathematical morphology to deal with spatial information.

Many perspectives are naturally occurring. First, the completeness result of this paper requires that the dual operators E^i and D^i are anti-extensive and extensive, respectively, which excludes the modal logics *D* and *K*. As mentioned in Section 4.5, it would be interesting to see whether there exists a general proof based on Henkin's method which works both for logics with dual operators which are extensive and anti-extensive, and for logics with dual operators which are not. Another interesting perspective would be to extend our general completeness result to the fuzzy setting. Finally, future work will aim at further exploring the spatial reasoning aspects. Moreover, theoretical results on complexity and tractability could be explored.

Notes

1. Standardly in category theory, Sig^{op} is the opposite of *Sig* by reversing morphisms.
2. S^+ is the set of all non-empty sequences of elements in S and $S^* = S^+ \cup \{\epsilon\}$ where ϵ denotes the empty sequence.
3. $T_F(X)_s$ is the term algebra of sort s built over F with sorted variables in a given set X.
4. We follow the definition given in van Benthem and Bezhanishvili (2007).
5. In many concrete categories of interest the converse is also true. However, this does not hold in general.
6. Roughly, a 2-category is a category E where for all objects e, e' in E, $Hom_E(e, e')$ is also a category. Morphisms between morphisms are then called 2-morphisms. The archetypical 2-category is *Cat* where objects are categories, morphisms are functors and 2-morphisms are natural transformations.
7. In the particular example of a set with an additive law $+$, the corresponding relation would be $(x, y) \in R_B$ iff $\exists b \in B, y = x + b$.
8. Let us recall that for simplicity in the notations we use $[\![M]\!]_\Sigma$ to denote both the object in the concrete category $\mathcal{C}$ and the underlying set associated by the faithful functor $\mathcal{U}$.
9. Here, we consider the set $\check{B}$ to define dilation because the accessibility relation is not necessarily symmetrical.
10. Let us note that the equivalence $\varphi \equiv E(\varphi)$ is satisfied by all logics for which the satisfaction of formulas of the form $E(\varphi)$ requires that, for all models, the relation between states is reflexive, such as **FOL**, **MPL** with reflexive model, **TMPL** and **MMPL**. On the other hand, we do not have for every $M \in Mod(\Sigma)$ that $[\varphi]_{\equiv_M} \preceq_M [E(\varphi)]_{\equiv_M}$.
11. In modal logic, the proof systems satisfying such a condition are said normal.
12. Note that the name 'invariant' was chosen since it also holds that $E^i(\varphi) \preceq_M \varphi$.
13. In Aiello and van Benthem (2002), authors use *E*. We prefer *A* in order to avoid confusion with the notation for erosion.

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References

Aiello, M., Pratt-Hartman, I., & van Benthem, J. (2007). *Handbook of spatial logics*. Dordrecht: Springer.

Aiello, M., & van Benthem, J. (2002). A modal walk through space. *Journal of Applied Non-Classical Logics*, *12*(3-4), 319–363.

Aiguier, M., Atif, J., Bloch, I., & Hudelot, C. (2018). Belief revision, minimal change and relaxation: A general framework based on satisfaction systems, and applications to description logics. *Artificial Intelligence*, *256*, 160–180.

Aiguier, M., Atif, J., Bloch, I., & Pino Pérez, R. (2018). Explanatory relations in arbitrary logics based on satisfaction systems, cutting and retraction. *International Journal of Approximate Reasoning*, *102*, 1–20.

Aiguier, M., & Diaconescu, R. (2007). Stratified institutions and elementary homomorphisms. *Information Processing Letters*, *103*(1), 5–13.

Alexandroff, P. (1937). Diskrete Raüme. *Matematicheskii Sbornik*, *2*, 501–518.

Alexandroff, P., & Hopf, H. (1935). *Topologie, erster band*. Berlin: Springer-Verlag.

Baets, B.-D. (1995). Idempotent closing and opening operations in fuzzy mathematical morphology. In *North American Fuzzy Information Processing Society (NAFIPS)* (pp. 228–233). IEEE Computer Society.

Barr, M., & Wells, C. (1990). *Category theory for computing science*. New York: Prentice-Hall.

Barwise, J. (1974). Axioms for abstract model theory. *Annnals of Mathematical Logic*, *7*, 221–265.

Bennett, B., & Duntsch, I. (2007). *Handbook of spatial logics*, chapter Axioms, Algebras and Topology (pp. 99–159). Springer-Verlag.

Bloch, I. (2002). Modal logics on mathematical morphology for qualitative spatial reasoning. *Journal of Applied Non-Classical Logics*, *12*(3-4), 399–423.

Bloch, I. (2005). Fuzzy spatial relationships for image processing and interpretation: A review. *Image and Vision Computing*, *23*(2), 89–110.

Bloch, I. (2006). Spatial reasoning under imprecision using fuzzy set theory, formal logics and mathematical morphology. *International Journal of Approximate Reasoning*, *41*(2), 77–95.

Bloch, I. (2009). Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations. *Fuzzy Sets and Systems*, *160*, 1858–1867.

Bloch, I. (2012, July). Mathematical morphology on bipolar fuzzy sets: General algebraic framework. *International Journal of Approximate Reasoning*, *53*, 1031–1061.

Bloch, I., Bretto, A., & Leborgne, A. (2015). Robust similarity between hypergraphs based on valuations and mathematical morphology operators. *Discrete Applied Mathematics*, *183*, 2–19.

Bloch, I., Heijmans, H., & Ronse, C. (2007). *Handbook of spatial logics*, chapter Mathematical Morphology (pp. 857–947). Springer-Verlag.

Bloch, I., & Lang, J. (2002). Towards mathematical morpho-logics. In B. Bouchon-Meunier, J. Gutierrez-Rios, L. Magdalena, & R. Yager (Eds.), *Technologies for constructing intelligent systems* (pp. 367–380). Springer-Verlag.

Bloch, I., & Maître, H. (1993). Constructing a fuzzy mathematical morphology: Alternative ways. In *International Conference on Fuzzy Systems (FUZZIEEE)* (pp. 1303–1308). IEEE Computer Society.

Bloch, I., & Maître, H. (1995). Fuzzy mathematical morphology: A comparative study. *Pattern Recognition*, *28*(9), 1341–1387.

Clementini, E., & Felice, O.-D. (1997). Approximate topological relations. *International Journal of Approximate Reasoning*, *16*, 173–204.

Cohn, A., Bennett, B., Gooday, B., & Gotts, N.-M. (1997). Representing and reasoning with qualitative spatial relations about regions. In O. Stock (Ed.), *Spatial and temporal reasoning* (pp. 97–134). Kluwer.

Cousty, Jean, Najman, Laurent, Dias, Fabio, & Serra, Jean (2013). Morphological filtering on graphs. *Computer Vision and Image Understanding*, *117*, 370–385.

Diaconescu, R. (2006). Proof systems for institutional logic. *Journal of Logic and Computation*, *16*(3), 339–357.
Diaconescu, R. (2008). *Institution-independent model theory*. Birkauser: Universal Logic.
Diaconescu, R. (2013). Institutional semantics for many-valued logics. *Fuzzy Sets and Systems*, *218*, 32–52.
Diaconescu, R. (2014). Graded consequence: An institution theoretic study. *Soft Computing*, *18*(7), 1247–1267.
Diaconescu, R. (2017). Implicit Kripke semantics and ultraproducts in stratified institutions. *Journal of Logic and Computation*, *27*(5), 1577–1606.
Dubois, D., & Prade, H. (2008). Special issue on bipolar representations of information and preference. *International Journal of Intelligent Systems*, *23*(8–10), 863–1152.
Egenhofer, M.-J. (1991). Reasoning about binary topological relations. In *Advances in spatial databases, second international symposium, SSD'91*, volume 525 of *Lecture Notes in Computer Science* (pp. 143–160). Springer.
Egenhofer, M.-J., & Franzosa, R. (1991). Point-set topological spatial relations. *International Journal of Geographical Information Systems*, *5*, 161–174.
Fiadeiro, J., & Sernadas, A. (1988). Structuring theories on consequence. In *Recent trends in algebraic development techniques*, volume 332 of *Lecture Notes in Computer Science* (pp. 44–72). Springer.
Gaina, D., & Petria, M. (2010). Completeness by forcing. *Journal of Logic and Computation*, *20*(6), 1165–1186.
Goguen, J. A. (1967). L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, *18*(1), 145–174.
Goguen, J. A., & Burstall, R.-M. (1992). Institutions: Abstract model theory for specification and programming. *Journal of the ACM*, *39*(1), 95–146.
Gorogiannis, N., & Hunter, A. (2008). Merging first-order knowledge using dilation operators. In *Fifth international symposium on foundations of information and knowledge systems, FoIKS'08*, volume LNCS 4932 (pp. 132–150).
Hakli, R., & Negri, S. (2012). Does the deduction theorem fail for modal logic? *Synthese*, *187*(3), 849–867.
Henkin, L. (1949). The completeness of the first-order functional calculus. *The Journal of Symbolic Logic*, *14*(3), 159–166.
Kuipers, B. (2000). The spatial semantic hierarchy. *Artifical Intelligence*, *119*, 191–233.
Ligozat, G. (2012). *Qualitative spatial and temporal reasoning*. Hoboken, NJ: Wiley.
MacLane, S. (1971). *Categories for the working mathematician*. New York: Springer-Verlag.
Madrid, N., Ojeda-Aciego, M., Medina, J., & Perfilieva, I. (2019). L-fuzzy relational mathematical morphology based on adjoint triples. *Information Sciences*, *474*, 75–89.
Meseguer, J. (1989). General logics. In *Logic colloq.'87* (pp. 275–329). Holland.
Meyer, F., & Stawiaski, J. (2009). Morphology on graphs and minimum spanning trees. In M. H. F. Wilkinson & J. B. T. M. Roerdink (Eds.), *International symposium on mathematical morphology ISMM 2009*, volume LNCS 5720 (pp. 161–170). Groningen.
Mossakowski, T., & Moratz, R. (2015). Relations between spatial calculi about directions and orientations. *Journal of Artificial Intelligence Research*, *54*, 277–308.
Nachtegael, M., & Kerre, E. E. (2000). Classical and fuzzy approaches towards mathematical morphology. In E. E. Kerre & M. Nachtegael (Eds.), *Fuzzy techniques in image processing*, Studies in Fuzziness and Soft Computing, chapter 1 (pp. 3–57). Physica-Verlag, Springer.
Najman, L., & Talbot, H. (2010, June). *Mathematical morphology: From theory to applications*. London: ISTE.
Petria, M. (2007). An institutional version of Godel's completeness theorem. In *Algebra and coalgebra in computer-science*, volume 4624 of *Lecture Notes in Computer Science* (pp. 409–425). Springer.
Randell, D., Cui, Z., & Cohn, A. (1992). A spatial logic based on regions and connection. In *International conference on principles of knowledge representation and reasoning (KR)* (pp. 165–176). AAAI Press.

Renz, J., & Nebel, B. (2007). *Handbook of spatial logics*, chapter *Qualitative spatial reasoning using constraint calculi* (pp. 161–215). Springer-Verlag.
Serra, J. (1982). *Image analysis and mathematical morphology*. Orlando, FL: Academic Press, Inc.
Sussner, P. (2016). Lattice fuzzy transforms from the perspective of mathematical morphology. *Fuzzy Sets and Systems*, *288*, 115–128.
Sussner, P., Nachtegael, M., Mélange, T., Deschrijver, G., Esmi, E., & Kerre, E. (2012). Interval-valued and intuitionistic fuzzy mathematical morphologies as special cases of L-fuzzy mathematical morphology. *Journal of Mathematical Imaging and Vision*, *43*(1), 50–71.
Tarlecki, A. (1999). *Algebraic foundations of systems specification*, chapter *Institutions: An abstract framework for formal specifications*. IFIP State-of-the-Art Reports. Springer-Verlag.
van Benthem, J., & Bezhanishvili, G. (2007). *Handbook of spatial logics*, chapter Modal Logics of Space (pp. 217–298). Springer-Verlag.
Ward, M., & Dilworth, R. P. (1938). Residuated lattices. *Proceedings of the National Academy of Sciences*, *24*(3), 162–164.