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Some Relationships Between Fuzzy Sets, Mathematical Morphology, Rough Sets, F-Transforms, and Formal Concept Analysis

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In this paper we extend some previously established links between the derivation operators used in formal concept analysis and some mathematical morphology operators to fuzzy concept analysis. We also propose to use mathematical morphology to navigate in a fuzzy concept lattice and perform operations on it. Links with other lattice-based formalisms such as rough sets and F-transforms are also established. This paper proposes a discussion and new results on such links and their potential interest.

Keywords: Formal concept analysis; fuzzy formal concepts; mathematical morphology; residuated lattice; fuzzy sets; rough sets; F-transforms.

1. Introduction

While lattice frameworks for information processing are more and more developed, it is interesting and useful to establish links between different theories to make each one inherit from properties and operators from other ones. Here, based on our previous work on mathematical morphology and formal concept analysis, we establish further links, by considering also other settings, such as fuzzy sets, possibility theory, rough sets, and F-transforms. In all these settings, the underlying algebraic structure is a lattice. While some links have already been exhibited, the new contribution is to include a morphological and fuzzy flavor. Another objective

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			

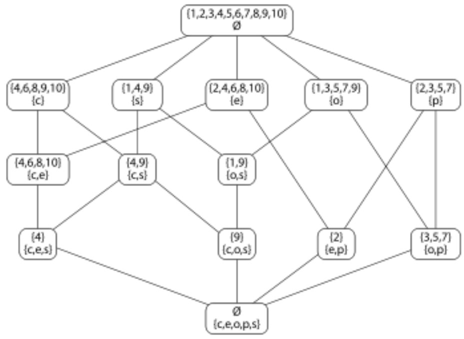

Fig. 1. A simple example of a context and its concept lattice from Wikipedia. Objects are integers from 1 to 10, and attributes are composite (c) (i.e. non prime integer strictly greater than 1), even (e), odd (o), prime (p) and square (s).

and contribution of this paper is to propose morphological operators working on concept lattices, in particular on fuzzy concept lattices, to navigate among formal concepts and perform operations on them, based on the notions of neighborhood and distances. Indeed, building morphological operators from neighborhoods and from distances is a classical approach in mathematical morphology,[1] and this idea is used here in a new context with respect to existing works. The interest is to provide explicit operators, not defined implicitly only via the adjunction property or from the commutativity with respect to the supremum and infimum. These operators constitute new tools for navigating in concept lattices, and for performing various reasoning tasks. Morphological reasoning is useful for instance for fusion, merging, revision, abduction, as already developed for different types of logics.[2–7]

As a running example, we consider in this paper a set of objects which are integers between 1 and 10, and some of their properties, as displayed in Fig. 1.

In Sec. 2, we recall some definitions and notations useful in the paper, related to formal concept analysis, mathematical morphology, and fuzzy sets. Our first contribution is detailed in Sec. 3, by exhibiting links between derivation operators, dilations and erosions from mathematical morphology, and the four operations of possibility theory, in the crisp and fuzzy cases. The rough set setting is addressed in Sec. 4, with links with F-transforms. Our second contribution concerns operations acting actually on formal concepts (Sec. 5). We propose morphological operators in a fuzzy concept lattice, based on decompositions, neighborhoods and distances. Related works and our contribution with respect to them are discussed in each appropriate section.

2. Preliminaries

2.1. *Formal concept analysis*

Let us introduce the main definitions and notations in formal concept analysis (FCA)[8] that will be useful in this paper. A formal context is a triplet $\mathbb{K} = (G, M, I)$, where G is the set of objects, M the set of attributes, and $I \subseteq G \times M$ a relation between objects and attributes ($(g, m) \in I$ means that the object g has the attribute m). A formal concept of the context $\mathbb{K}$ is a pair (X, Y), with $X \subseteq G$ and $Y \subseteq M$, such that (X, Y) is maximal with the property $X \times Y \subseteq I$. The set X is called the extent and the set Y is called the intent of the formal concept (X, Y). For any formal concept a, we denote its extent by $e(a)$ and its intent by $i(a)$, i.e. $a = (e(a), i(a))$.

The set of all formal concepts of a given context can be hierarchically ordered by inclusion of their extent (or equivalently by inclusion of their intent):

$$(X_1, Y_1) \preceq_C (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow Y_2 \subseteq Y_1).$$

This order, that reflects the subconcept-superconcept relation, induces a complete lattice which is called the concept lattice of the context (G, M, I), denoted $\mathbb{C}(\mathbb{K})$, or simply $\mathbb{C}$.

The lattice corresponding to the number example is shown in Fig. 1.

For $X \subseteq G$ and $Y \subseteq M$, the derivation operators α and β are defined as:

$$\alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\},$$

and

$$\beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}.$$

The pair (α, β) is a Galois connection between the partially ordered power sets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$ i.e.

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M), Y \subseteq \alpha(X) \Leftrightarrow X \subseteq \beta(Y).$$

Saying that (X, Y), with $X \subseteq G$ and $Y \subseteq M$, is a formal concept is equivalent to $\alpha(X) = Y$ and $\beta(Y) = X$.

In the example in Fig. 1, the pair $(\{3, 5, 7\}, \{o, p\})$ is a formal concept.

2.2. *Mathematical morphology*

Let us recall the algebraic framework of mathematical morphology (MM). Let $(\mathcal{L}, \preceq)$ and $(\mathcal{L}', \preceq')$ be two complete lattices (which do not need to be equal). All the following definitions and results are common to the general algebraic framework of mathematical morphology in complete lattices.[4,9–14] Note that different terminologies can be found in different lattice theory related contexts (refer to Ref. 15 for equivalence tables).

Definition 1. An operator $\delta \colon \mathcal{L} \to \mathcal{L}'$ is an *algebraic dilation* if it commutes with the supremum (sup-preserving mapping):

$$\forall (x_i) \in \mathcal{L},\ \delta(\vee_i x_i) = \vee'_i \delta(x_i)\,,$$

where $\vee$ denotes the supremum associated with $\preceq$ and $\vee'$ the one associated with $\preceq'$.

An operator $\varepsilon\colon \mathcal{L}' \to \mathcal{L}$ is an *algebraic erosion* if it commutes with the infimum (inf-preserving mapping):

$$\forall (x_i) \in \mathcal{L}',\ \varepsilon(\wedge'_i x_i) = \wedge_i \varepsilon(x_i)\,,$$

where $\wedge$ and $\wedge'$ denote the infimum associated with $\preceq$ and $\preceq'$, respectively.

This general definition allows defining mathematical morphology operators such as dilations and erosions in many types of settings, such as sets, functions, fuzzy sets, rough sets, graphs, hypergraphs, various logics, etc., based on their corresponding lattices.

Algebraic dilations δ and erosions ε are increasing operators; moreover δ preserves the smallest element and ε preserves the largest element.

A fundamental notion in this algebraic framework is the one of adjunction.

Definition 2. A pair of operators (ε, δ), $\delta\colon \mathcal{L} \to \mathcal{L}'$, $\varepsilon\colon \mathcal{L}' \to \mathcal{L}$, defines an *adjunction* if

$$\forall x \in \mathcal{L}, \forall y \in \mathcal{L}', \delta(x) \preceq' y \iff x \preceq \varepsilon(y).$$

The main properties, that will be used in the following, are summarized as follows.

Proposition 1. [*e.g. Refs.* 11, 12] *If a pair of operators* (ε, δ) *defines an adjunction, then the following results hold:*

- δ *preserves the smallest element and* ε *preserves the largest element;*
- δ *is a dilation and* ε *is an erosion (in the sense of Definition 1);*
- $\delta\varepsilon$ *is anti-extensive:* $\delta\varepsilon \preceq' Id_{\mathcal{L}'}$*, where* $Id_{\mathcal{L}'}$ *denotes the identity mapping on* $\mathcal{L}'$*, and* $\varepsilon\delta$ *is extensive:* $Id_{\mathcal{L}} \preceq \varepsilon\delta$*. The compositions* $\delta\varepsilon$ *and* $\varepsilon\delta$ *are called morphological opening and morphological closing, respectively;*
- $\varepsilon\delta\varepsilon = \varepsilon$*,* $\delta\varepsilon\delta = \delta$*,* $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$ *and* $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$*, i.e. morphological opening and closing are idempotent operators;*
- *if* $\mathcal{L} = \mathcal{L}'$ *then the following statements are equivalent:*
  (a) δ *is a closing (i.e. increasing, extensive and idempotent),*
  (b) ε *is an opening (i.e. increasing, anti-extensive and idempotent),*
  (c) $\delta\varepsilon = \varepsilon$*,*
  (d) $\varepsilon\delta = \delta$*.*

Let δ *and* ε *be two increasing operators such that* $\delta\varepsilon$ *is anti-extensive and* $\varepsilon\delta$ *is extensive. Then* (ε, δ) *is an adjunction.*

The following representation result also holds. If ε *is an increasing operator, it is an algebraic erosion if and only if there exists* δ *such that* (ε, δ) *is an adjunction. The operator* δ *is then an algebraic dilation and can be expressed as* $\delta(x) = \wedge'\{y \in \mathcal{L}' \mid x \preceq \varepsilon(y)\}$*. A similar representation result holds for erosion.*

Particular forms of dilations and erosions can be defined based on the notion of structuring element, which can be a neighborhood relation or any binary relation.[4,10] Examples will be given in the next sections.

Finally, operators that exchange the supremum and infimum are called anti-dilations and anti-erosions.

Definition 3. An operator $\delta^a\colon \mathcal{L} \to \mathcal{L}'$ is an *anti-dilation* if

$$\forall (x_i) \in \mathcal{L},\ \delta^a(\vee_i x_i) = \wedge'_i \delta^a(x_i)\,.$$

An operator $\varepsilon^a\colon \mathcal{L}' \to \mathcal{L}$ is an *anti-erosion* if

$$\forall (x_i) \in \mathcal{L}',\ \varepsilon^a(\wedge'_i x_i) = \vee_i \varepsilon^a(x_i)\,.$$

2.3. *Lattice of fuzzy sets*

For the fuzzy case, we will rely on a classical residuated lattice for fuzzy sets. Membership functions are taking values in L endowed with a lattice structure (typically $L = [0,1]$ but all what follows extends directly for more general L-fuzzy sets[16]), and the corresponding residuated lattice is denoted $(L, \leq, \wedge, \vee, *, \to)$, where $\wedge$ is the infimum, $\vee$ the supremum, and $*$ and $\to$ are adjoint conjunction and implication. In this paper, we use conjunctions defined as operators that are increasing in both arguments, commutative and associative, and admit 1 (or more generally the greatest element of the lattice L) as unit element, i.e. t-norms. Implications are defined as operators that are decreasing in the first argument, increasing in the second one, and satisfy $0 \to 0 = 0 \to 1 = 1 \to 1 = 1, 1 \to 0 = 0$ (or more general expressions by replacing 0 and 1 by the smallest and greatest elements of L, respectively). The adjunction property writes $c * a \leq b \Leftrightarrow c \leq a \to b$ and the implication defined by residuation from the conjunction is expressed as:

$$\forall (a,b) \in L^2,\ a \to b = \sup\{c \in L \mid c * a \leq b\}\,.$$

Non-commutative conjunctions can also be considered,[17] with two associated implications, leading to adjoint triplets, and accordingly multi-adjoint concept lattices in the framework of formal concept analysis.

The corresponding partial ordering on fuzzy sets is defined as:

$$\mu \preceq_F \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x)\,,$$

where μ and ν are two fuzzy sets (or equivalently their membership functions), defined on an underlying space $\mathcal{S}$. The residuated lattice of fuzzy sets is denoted by $(\mathcal{F}, \preceq_F, \wedge^F, \vee^F, *, \to)$, with $\wedge^F = \min$ and $\vee^F = \max$ (or inf and sup more generally).

In particular we will use fuzzy sets defined on $\mathcal{S} = G$, i.e. $\mathcal{F} = L^G$, and on $\mathcal{S} = M$, i.e. $\mathcal{F} = L^M$.

Algebraic morphological operators are defined on this lattice as in Definition 1, and definitions based on structuring elements also extend to the fuzzy case.[18–20]

3. Derivation Operators and Mathematical Morphology Operators

3.1. *Crisp setting*

As already briefly noticed e.g. in Ref. 4 and further detailed in Ref. 21, formal concept analysis and mathematical morphology both rely on complete lattice structures which share some similarities. In this section, we highlight some parallel properties of dilations and erosions on the one hand, and of derivation operators on the other hand, in the classical setting. The first important link is that (ε, δ) is an adjunction (sometimes called monotone Galois connection), while (α, β) is an antitone Galois connection. It is obvious that the two properties are equivalent if we reverse the order in one of the lattices. The same holds for all properties derived from adjunctions or Galois connections (cf. Proposition 1). The most important ones are summarized in Table 1.[a]

As mentioned above, terminology may slightly differ: increasing, idempotent and extensive operators are called closings in MM and closure operators in FCA, while increasing, idempotent and anti-extensive operators are called openings in MM and kernel operators in FCA. Similarly, in FCA literature it is more common to speak of closure systems, instead of Moore families.[b]

Table 1. Similarities between mathematical morphology and formal concept analysis.[21]

Adjunctions, dilations and erosions	Galois connection, derivation operators
$\delta\colon (\mathcal{L}, \preceq) \rightarrow (\mathcal{L}', \preceq')$, $\varepsilon\colon (\mathcal{L}', \preceq') \rightarrow (\mathcal{L}, \preceq)$	$\alpha\colon \mathcal{P}(G) \rightarrow \mathcal{P}(M)$, $\beta\colon \mathcal{P}(M) \rightarrow \mathcal{P}(G)$
$\delta(x) \preceq' y \iff x \preceq \varepsilon(y)$	$X \subseteq \beta(Y) \iff Y \subseteq \alpha(X)$
increasing operators	decreasing operators
$\varepsilon\delta\varepsilon = \varepsilon$, $\delta\varepsilon\delta = \delta$	$\alpha\beta\alpha = \alpha$, $\beta\alpha\beta = \beta$
$\varepsilon\delta$ = closing (closure operator), $\delta\varepsilon$ = opening (kernel operator)	$\alpha\beta$ and $\beta\alpha$ = both closure operators (closings)
$\mathrm{Inv}(\varepsilon\delta) = \varepsilon(\mathcal{L}')$, $\mathrm{Inv}(\delta\varepsilon) = \delta(\mathcal{L})$	$\mathrm{Inv}(\alpha\beta) = \alpha\big(\mathcal{P}(G)\big)$, $\mathrm{Inv}(\beta\alpha) = \beta\big(\mathcal{P}(M)\big)$
$\varepsilon(\mathcal{L}')$ is a Moore family, $\delta(\mathcal{L})$ is a dual Moore family	$\alpha\big(\mathcal{P}(G)\big)$ and $\beta\big(\mathcal{P}(M)\big)$ are Moore families (or closure systems)
δ is a dilation: $\delta(\vee x_i) = \vee'\big(\delta(x_i)\big)$	α is an anti-dilation: $\alpha(\cup X_i) = \cap\alpha(X_i)$
ε is an erosion: $\varepsilon(\wedge' y_i) = \wedge\big(\varepsilon(y_i)\big)$	β is an anti-dilation: $\beta(\cup Y_i) = \cap\beta(Y_i)$

In Ref. 21, we went beyond this simple translation of terminology from one theory to the other by proposing new morphological operators acting on concept lattices. These operators can then be used to reason on such lattices. This idea is further investigated in Sec. 5, and extended to the fuzzy case.

[a]In the table we denote by $\mathrm{Inv}(\varphi)$ the set of invariants of an operator φ (i.e. $x \in \mathrm{Inv}(\varphi)$ iff $\varphi(x) = x$).
[b]$M \subseteq \mathcal{L}$ is a Moore family if any element of $\mathcal{L}$ has a smallest upper bound in M.

Definition 4. Let us take as a structuring element centered at $m \in M$, or a neighborhood of m, the set of $g \in G$ such that $(g, m) \in I$ (and conversely the set of $m \in M$ such that $(g, m) \in I$ is a neighborhood of g). We define operators δ_I and ε_I^* from $\mathcal{P}(M)$ into $\mathcal{P}(G)$, and δ_I^* and ε_I from $\mathcal{P}(G)$ into $\mathcal{P}(M)$ as:

$$\begin{aligned}
&\forall X \in \mathcal{P}(G),\ \forall\ Y \in \mathcal{P}(M) \\
&\qquad \delta_I(Y) = \left\{g \in G \mid \exists m \in Y, (g, m) \in I\right\}, \\
&\qquad \varepsilon_I(X) = \left\{m \in M \mid \forall g \in G, (g, m) \in I \Rightarrow g \in X\right\}, \\
&\qquad \delta_I^*(X) = \left\{m \in M \mid \exists g \in X, (g, m) \in I\right\}, \\
&\qquad \varepsilon_I^*(Y) = \left\{g \in G \mid \forall m \in M, (g, m) \in I \Rightarrow m \in Y\right\}.
\end{aligned}$$

Proposition 2. *The pairs of operators $(\varepsilon_I, \delta_I)$ and $(\varepsilon_I^*, \delta_I^*)$ are adjunctions (and δ_I and δ_I^* are dilations, ε_I and ε_I^* are erosions). Moreover, the following duality relations hold: $\delta_I(M \setminus Y) = G \setminus \varepsilon_I^*(Y)$ and $\delta_I^*(G \setminus X) = M \setminus \varepsilon_I(X)$.*

Proof. Let us first assume that $\delta_I(Y) \subseteq X$. Let $m \in Y$. Then we have:

$$\forall g \in G, (g, m) \in I \Rightarrow g \in \delta_I(Y) \Rightarrow g \in X\,.$$

Hence $Y \subseteq \varepsilon_I(X)$. Similarly $Y \subseteq \varepsilon_I(X) \Rightarrow \delta_I(Y) \subseteq X$, and $(\varepsilon_I, \delta_I)$ is an adjunction. The proof for $(\varepsilon_I^*, \delta_I^*)$ is similar.

Let $g \in \delta_I(M \setminus Y)$. Then

$$\exists m \in M \setminus Y, (g, m) \in I\,,$$

which implies that $g \notin \varepsilon_I^*(Y)$, and we have

$$\delta_I(M \setminus Y) \subseteq G \setminus \varepsilon_I^*(Y)\,.$$

Similarly $G \setminus \varepsilon_I^*(Y) \subseteq \delta_I(M \setminus Y)$, hence $\delta_I(M \setminus Y) = G \setminus \varepsilon_I^*(Y)$. The duality between δ_I^* and ε_I is proved similarly. □

Proposition 3. *Using the FCA derivation operators on the context (G, M, I), the operators in Definition 4 can be expressed as:*

$$\begin{aligned}
\delta_I(Y) &= \bigcup_{m \in Y} \beta(\{m\})\,, \\
\varepsilon_I(X) &= \{m \in M \mid \beta(\{m\}) \subseteq X\}\,, \\
\delta_I^*(X) &= \bigcup_{g \in X} \alpha(\{g\})\,, \\
\varepsilon_I^*(Y) &= \{g \in G \mid \alpha(\{g\}) \subseteq Y\}\,.
\end{aligned}$$

Proof. The proof is direct from the expressions of the derivation operators applied on singletons:

$$\beta(\{m\}) = \{g \in G \mid (g, m) \in I\} \ \text{ and } \ \alpha(\{g\}) = \{m \in M \mid (g, m) \in I\}\,.$$ □

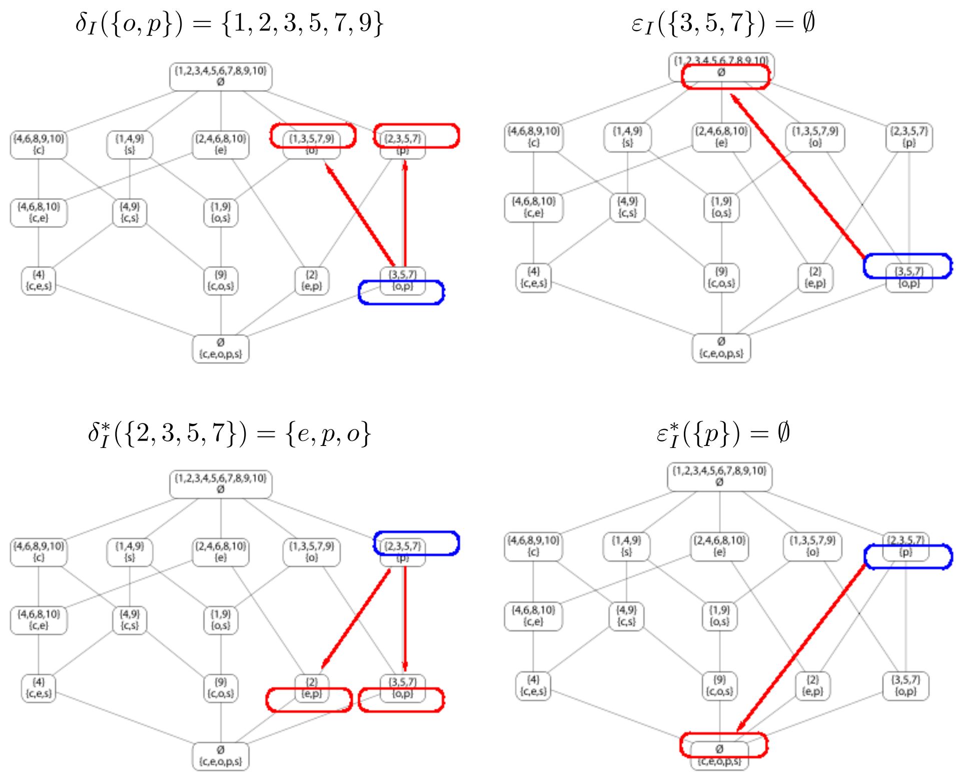

Fig. 2. Examples of erosions and dilations of subsets of G and M in the concept lattice in Fig. 1 using the operators in Definition 4.

An example illustrating Definition 4 on the concept lattice in Fig. 1 is given in Fig. 2.

We will show in the next section that using this morphological construction, we recover operators proposed in a possibilistic setting.

3.2. *Possibilistic setting*

Let us first remain with the crisp definition of the context, and consider the four set functions of the possibility theory, the interest of which has already been demonstrated for FCA in Refs. 22 and 23.[c] Using the notations of this paper, these functions are defined, for $X \in \mathcal{P}(G)$, as:[22]

potential possibility: $I_G^{\Pi}(X) = \{m \in M \mid \exists g \in X, (g,m) \in I\}$, which expresses the properties satisfied by at least one object in X;

[c]For instance, these four operators have been shown to be useful for characterizing independent sub-contexts, i.e. with no object in common and no property in common.[23]

actual necessity: $I^N_G(X) = \{m \in M \mid \forall g \in G, (g,m) \in I \Rightarrow g \in X\}$, which includes all properties such that any object satisfying one of them is necessarily in X;

actual possibility: $I^\Delta_G(X) = \{m \in M \mid \forall g \in X, (g,m) \in I\}$, which is the set of properties shared by all objects in X;

potential necessity: $I^\nabla_G(X) = \{m \in M \mid \{g \in G \mid (g,m) \in I\} \cup X \neq G\}$, which includes each property such that there exists an object in $\bar{X} = G \setminus X$ which does not satisfy it;

and similar expressions for $Y \in \mathcal{P}(M)$, denoted by $I^\Pi_M(Y)$, $I^N_M(Y)$, $I^\Delta_M(Y)$, and $I^\nabla_M(Y)$.

Note that similar notions (with sometimes other names) can be found in Refs. 24 and 25 for rough sets (hence the following results implicitly establish links between FCA, rough sets and mathematical morphology). These links will be further investigated in Sec. 4.

The following proposition (the proof of which being straightforward) exhibits links between these operators and morphological ones, from which their properties can be easily derived.

Proposition 4. *We have, for all $X \in \mathcal{P}(G)$:*

- $I^\Pi_G(X) = \delta^*_I(X) = \bigcup_{g \in X}\{m \in M \mid (g,m) \in I\}$, *i.e. it is a dilation from $\mathcal{P}(G)$ into $\mathcal{P}(M)$ (see Definition 4 and Proposition 2), and hence commutes with union, and is increasing;*
- $I^N_G(X) = \varepsilon_I(X)$, *i.e. an erosion from $\mathcal{P}(G)$ into $\mathcal{P}(M)$ (see Definition 4 and Proposition 2), which is dual of δ^*_I, commutes with the intersection, and is increasing;*
- $I^\Delta_G(X) = \alpha(X)$ *and it is an anti-dilation (see Table 1);*
- $I^\nabla_G(X)$ *is dual of $I^\Delta_G(X)$ and it is an anti-erosion, i.e. $I^\nabla_G(X \cap X') = I^\nabla_G(X) \cup I^\nabla_G(X')$;*

and similar results for operators acting on $Y \in \mathcal{P}(M)$: $I^\Pi_M(Y) = \delta_I(Y)$ and is a dilation, $I^N_M(Y) = \varepsilon^*_I(Y)$ and is an erosion, $I^\Delta_M(Y) = \beta(Y)$ and is an anti-dilation, and $I^\nabla_M(Y)$ is an anti-erosion.

3.3. *Fuzzy setting*

Let us now move to fuzzy contexts, i.e. X and Y are fuzzy subsets of G and M, and I is a fuzzy relation ($I(g,m)$ now denotes the degree to which the object g has the property m). The residuated lattice introduced in Sec. 2.3 is used, and degrees take values in any residuated lattice L. In the examples, we will use $L = [0,1]$ for the sake of simplicity, but the theorerical results hold for any residuated lattice.

In this section, we first recall existing definitions of fuzzy contexts and derivation operators. Then we exhibit their properties in terms of mathematical morphology, and we define fuzzy dilations and erosions on fuzzy intents and extents. Finally, we show the links with some other constructions.

The derivation operators have been generalized to the fuzzy case in Refs. 26, 27 (see Ref. 28 for a discussion of various approaches for fuzzy concept analysis), leading to fuzzy sets $\alpha(X)$ and $\beta(Y)$ defined as:

$$\alpha(X)(m) = \wedge_{g\in G}(X(g) \to I(g,m))\,, \tag{1}$$

$$\beta(Y)(g) = \wedge_{m\in M}(Y(m) \to I(g,m))\,. \tag{2}$$

Note that in the early work,[29] the implication was defined from a t-conorm and a complementation. We rely here on fuzzy implications related to a fuzzy conjunction by the adjunction property, i.e. residuated implications, such as in later works, which guarantees good properties, as detailed next.

As in the crisp case, a fuzzy formal concept is a pair of fuzzy sets (X, Y) such that $\alpha(X) = Y$ and $\beta(Y) = X$. From the classical partial ordering on fuzzy sets $\preceq_F$ (we use here the same notation for the ordering on L^G and on L^M), a partial ordering $\preceq_{FC}$ on fuzzy formal concepts is defined as:

$$(X_1, Y_1) \preceq_{FC} (X_2, Y_2) \Leftrightarrow X_1 \preceq_F X_2$$

and equivalently

$$(X_1, Y_1) \preceq_{FC} (X_2, Y_2) \Leftrightarrow Y_2 \preceq_F Y_1\,,$$

and this ordering induces a complete lattice structure on the fuzzy formal concepts, denoted $\mathbb{C}^F$. As shown in Ref. 27, the infimum and supremum of a family of fuzzy concepts $(X_t, Y_t)_{t\in T}$ are:

$$\wedge^{FC}{}_{t\in T}(X_t, Y_t) = \big(\wedge^F{}_{t\in T} X_t, \alpha\big(\beta(\vee^F{}_{t\in T} Y_t)\big)\big)\,, \tag{3}$$

$$\vee^{FC}{}_{t\in T}(X_t, Y_t) = \big(\beta\big(\alpha(\vee^F{}_{t\in T} X_t)\big), \wedge^F{}_{t\in T} Y_t\big)\,, \tag{4}$$

where $\wedge^F$ and $\vee^F$ are the classical intersection and union of fuzzy sets, defined as the pointwise infimum and supremum of the membership functions (Sec. 2.3).

Strict ordering relations $\prec_F$ and $\prec_{FC}$ are defined from $\preceq_F$ and $\preceq_{FC}$ as usual.

In this paper we consider only one pair of adjoint conjunction and implication. This can be extended by considering several pairs (or triplets in case of non commutative conjunction), where a mapping associates each object (respectively each attribute) to a specific pair, leading to the notion of multi-adjoint object-oriented (respectively property-oriented) concept lattices.[30] Such extensions are not further considered in this paper.

Let us consider a fuzzy version of the context depicted in Fig. 1, as shown in Table 2, where degrees different from 0 and 1 can refer for instance to incomplete, imprecise or uncertain knowledge, according to the semantics of the domain and type of imperfection. In this example, we can consider degrees as the gradual satisfaction of a property, represented by a (crisp) number in $[0, 1]$. In this context,

Table 2. An example of a fuzzy context.

$\mathbb{K}$	Composite	Even	Odd	Prime	Square
1	0.2	0	1	0.2	1
2	0.2	1	0	1	0
3	0.2	0	1	1	0
4	0.8	1	0	0	1
5	0.2	0	1	1	0
6	0.8	1	0	0	0
7	0.2	0	1	1	0
8	0.8	1	0	0	0
9	0.8	0	1	0	1
10	0.8	1	0	0	0

and for Lukasiewicz conjunction and implication,[d] an example of a fuzzy formal concept is:

$$X_1(1) = 0.4, X_1(2) = \cdots = X_1(8) = 0, X_1(9) = 0.9, X_1(10) = 0\,,$$

and

$$Y_1(c) = 0.8, Y_1(e) = 0.1, Y_1(o) = 1, Y_1(p) = 0.1, Y_1(s) = 1\,.$$

Another example is:

$$X_2(1) = 0.3, X_2(2) = 0, X_2(3) = 0.6, X_2(4) \cdots = X_2(8) = 0, X_2(9) = 0.9, X_2(10) = 0\,,$$

and

$$Y_2(c) = 0.9, Y_2(e) = 0.1, Y_2(o) = 0.4, Y_2(p) = 0.1, Y_2(s) = 1\,.$$

We have $\alpha(X_i) = Y_i, \beta(Y_i) = X_i$ for $i = 1, 2$.

Proposition 5. *The derivation operators α and β defined by Eqs.* (1) *and* (2) *are fuzzy anti-dilations.*

Proof. Let us consider any family of fuzzy subsets X_i of G, for an index set Ξ ($i \in \Xi$). We have, using the fact that the fuzzy implication $\rightarrow$ is decreasing with respect to the first argument (see Sec. 2.3):

$$\begin{aligned}\forall m \in M,\ \alpha(\vee^F_{i\in\Xi}(X_i))(m) &= \wedge_{g\in G}((\vee^F{}_{i\in\Xi}(X_i))(g) \rightarrow I(g,m)) \\ &= \wedge_{g\in G}(\vee_{i\in\Xi}(X_i(g)) \rightarrow I(g,m)) \\ &= \wedge_{i\in\Xi}(\wedge_{g\in G}(X_i(g) \rightarrow I(g,m))) \\ &= \wedge_{i\in\Xi}(\alpha(X_i)(m)) = (\wedge^F{}_{i\in\Xi}\alpha(X_i))(m)\,,\end{aligned}$$

hence α is a fuzzy anti-dilation. The proof for β is similar. □

[d] The Lukasiewicz conjunction is defined as $a * b = \max(0, a + b - 1)$ and the implication by $a \rightarrow b = \min(1, 1 - a + b)$.

Definition 5. The extension of the morphological operators introduced in Definition 4 to the fuzzy case derives from fuzzy mathematical morphology:[18,19]

$$
\begin{aligned}
&\forall X \in L^G, \forall Y \in L^M, \forall g \in G, \forall m \in M,\\
&\delta_I(Y)(g) = \vee_{m\in M}(Y(m) * I(g,m)),\\
&\varepsilon_I(X)(m) = \wedge_{g\in G}(I(g,m) \to X(g)),\\
&\delta_I^*(X)(m) = \vee_{g\in G}(X(g) * I(g,m)),\\
&\varepsilon_I^*(Y)(g) = \wedge_{m\in M}(I(g,m) \to Y(m)).
\end{aligned}
$$

Note that we use here a direct extension of definitions and results from Refs. 18, 19, by considering that the fuzzy structuring element is any fuzzy binary relation, without any assumption that there is an underlying metric space. Also, the two lattices defining the domains of definition and image of the morphological operators do not need to be identical, while preserving the same properties.

These definitions are illustrated for $(X_i, Y_i), i = 1, 2$, given as examples above in the following tables (0 membership values are omitted):

	1	2	3	4	5	6	7	8	9	10
X_1	0.4								0.9	
$\delta_I(Y_1)$	1	0.1	1	1	1	0.6	1	0.6	1	0.6
$\varepsilon_I^*(Y_1)$	0.9	0.1	0.1	0.1	0.1	0.1	0.1	0.1	1	0.1
X_2	0.3		0.6						0.9	
$\delta_I(Y_2)$	1	0.1	0.4	1	0.4	0.7	0.4	0.7	1	0.7
$\varepsilon_I^*(Y_2)$	0.4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.4	0.1

	c	e	o	p	s
Y_1	0.8	0.1	1	0.1	1
$\delta_I^*(X_1)$	0.7		0.9		0.9
$\varepsilon_I(X_1)$	0.2				
Y_2	0.9	0.1	0.4	0.1	1
$\delta_I^*(X_2)$	0.7		0.9	0.6	0.9
$\varepsilon_I(X_2)$	0.2				

Let us comment on these results, by detailing $\delta_I(Y_1)$: since o belongs to Y_1 with degree 1, the dilation contains all objects g having the "odd" property, i.e. 1, 3, 5, 7, 9. Similarly, s belongs to Y_1 with degree 1, so the dilation additionally contains object 4 with degree 1. On the other hand, c belongs to Y_1 with a degree 0.8, and hence objects 6, 8, 10 belong to the dilation with a lower degree. Finally e and p belong to Y_1 with a degree 0.1, hence object 2 belongs to the dilation with a very low degree. These results fit well the intuition. Similar interpretations can be provided for the other examples in the table.

Proposition 6. *The four set functions of the possibility theory extend to the fuzzy case, and we have in particular, for any fuzzy subset X of G:*

- $I_G^{\Pi}(X) = \delta_I^*(X)$ *and is a dilation,*
- $I_G^N(X) = \varepsilon_I(X)$ *and is an erosion,*
- $I_G^{\Delta}(X) = \alpha(X)$ *and is an anti-dilation,*
- $I_G^{\nabla}(X)$ *is an anti-erosion.*

Similarly for the operators acting on Y (fuzzy subset of M), we have, for any fuzzy subset Y of M:

- $I_M^{\Pi}(Y) = \delta_I(Y)$ *and is a dilation,*
- $I_M^N(Y) = \varepsilon_I^*(Y)$ *and is an erosion,*
- $I_M^{\Delta}(Y) = \beta(Y)$ *and is an anti-dilation,*
- $I_M^{\nabla}(Y)$ *is an anti-erosion.*

The proof is straightforward, as in the crisp case.

Some of these results have been obtained independently, sometimes in particular cases only, in Refs. 31, 32, with illustrations on crisp spatially invariant structuring elements (however this limits the scope of the applications), or also in Ref. 33.

As for the Galois connection or adjunction properties, two different views can be adopted.

In the first view, the crisp definition of these properties is kept, with the classical inclusion $\preceq_F$ between fuzzy sets (see Sec. 2.3).

Proposition 7. *The fuzzy versions of $(\varepsilon_I, \delta_I)$ and $(\varepsilon_I^*, \delta_I^*)$ (Definition 5) are adjunctions for adjoint connectives $*$ and $\rightarrow$.*

Proof. The proof follows the same lines as classical proofs of adjunction in mathematical morphology. Let X and Y be any two fuzzy subsets of G and M, respectively. We have, using the adjunction property between $*$ and $\rightarrow$:

$$\begin{aligned}
\delta_I(Y) \preceq_F X &\Leftrightarrow \forall g \in G,\ \delta_I(Y)(g) \leq X(g) \\
&\Leftrightarrow \forall g \in G,\ \vee_{m\in M}(Y(m) * I(g,m)) \leq X(g) \\
&\Leftrightarrow \forall g \in G, \forall m \in M,\ Y(m) * I(g,m) \leq X(g) \\
&\Leftrightarrow \forall m \in M, \forall g \in G,\ Y(m) \leq I(g,m) \rightarrow X(g) \\
&\Leftrightarrow \forall m \in M,\ Y(m) \leq \wedge_{g\in G}(I(g,m) \rightarrow X(g)) \\
&\Leftrightarrow Y \preceq_F \varepsilon_I(X)
\end{aligned}$$

and thus $(\varepsilon_I, \delta_I)$ is an adjunction. The proof for $(\varepsilon_I^*, \delta_I^*)$ is similar. □

Proposition 8. *The compositions*

- $I_M^{\Pi} I_G^N = \delta_I \varepsilon_I$ *and* $I_G^{\Pi} I_M^N = \delta_I^* \varepsilon_I^*$ *are algebraic openings,*

- $I_M^N I_G^\Pi = \varepsilon_I^* \delta_I^*$ *and* $I_G^N I_M^\Pi = \varepsilon_I \delta_I$ *are algebraic closings,*

for a continuous conjunction, if and only if $*$ *and* $\rightarrow$ *are adjoint.*

If duality with respect to the complementation is also required, then the Lukasiewicz operators (up to a bijection applied on the membership values) should be chosen.

Proof. The result is derived from the link between the set functions of the possibility theory and fuzzy dilations and erosions on the one hand (see Proposition 6), and from previous results establishing the conditions under which fuzzy dilations and erosions are adjoint or dual[19] on the other hand. In particular if $*$ and $\rightarrow$ are adjoint, then from the adjunctions identified in Proposition 7, the compositions are openings or closings. Conversely, since δ_I and ε_I are increasing since they are a dilation and an erosion (Proposition 6), the fact that their composition is anti-extensive (for opening), respectively extensive (for closing), implies that they form an adjunction (Proposition 1), and hence $*$ and $\rightarrow$ are adjoint as well. The same reasoning applies for δ_I^* and ε_I^*. □

These results also hold in the more general framework of multi-adjoint concept lattices, as shown in Ref. 34.

Note that the property of closing or opening is derived classically from adjunction in mathematical morphology in the general case, and conversely the increasingness of the operators, and the extensivity or anti-extensivity of their combination implies adjunction. In Refs. 34, 35, the closing property is obtained for the compositions of $I_G^\Delta = \alpha$ and $I_M^\Delta = \beta$ (which form an adjunction) for a property of the implication only (since the conjunction is not involved in these operators), expressed as $a \leq (a \rightarrow b) \rightarrow b$, for all a and b.

Since the adjunction property ($a * b \leq c \Leftrightarrow a \leq b \rightarrow c$) implies $a \leq (a \rightarrow b) \rightarrow b$ (assuming a commutative conjunction), this property holds a fortiori for adjoint $*$ and $\rightarrow$.

Similar results can be obtained for the anti-erosions I_G^∇ and I_M^∇.

In the second view, fuzzy notions of Galois connection and adjunction can be defined from a degree of inclusion S between fuzzy subsets of G or M, as in Ref. 26, thus establishing links with this work:

$$S(X, X') = \wedge_{g \in G}(X(g) \rightarrow X'(g)), \quad S(Y, Y') = \wedge_{m \in M}(Y(m) \rightarrow Y'(m)), \tag{5}$$

where $\rightarrow$ is a residuated implication (see Sec. 2.3).

As defined in Ref. 26 (see also Refs. 27, 36), (α, β) is a fuzzy Galois connection if

$$\begin{aligned} &S(X, X') \leq S(\alpha(X'), \alpha(X)), \\ &S(Y, Y') \leq S(\beta(Y'), \beta(Y)), \\ &S(X, \beta\alpha(X)) = S(Y, \alpha\beta(Y)) = 1 \end{aligned}$$

(or equivalently $S(X, \beta(Y)) = S(Y, \alpha(X))$), which holds for α and β defined from the residuated implication $\to$.

We now prove similar results for morphological operators:

Proposition 9. *For the fuzzy morphological operators of Definition 5 and adjoint connectives $*$ and $\to$ we have:*

$$\begin{aligned} \forall X, X' \in \mathcal{P}(G), \forall Y, Y' \in \mathcal{P}(M)\,, \\ S(\delta_I(\varepsilon_I(X)), X) &= S(X, \varepsilon_I^*(\delta_I^*(X))) = 1\,, \\ S(Y, \varepsilon_I(\delta_I(Y))) &= S(\delta_I^*(\varepsilon_I^*(Y)), Y) = 1\,, \\ S(X, \varepsilon_I^*(Y)) &= S(\delta_I^*(X), Y)\,, \\ S(Y, \varepsilon_I(X)) &= S(\delta_I(Y), X)\,, \\ S(X, X') &\leq S(\varepsilon_I(X), \varepsilon_I(X'))\,, \\ S(X, X') &\leq S(\delta_I^*(X), \delta_I^*(X'))\,, \\ S(Y, Y') &\leq S(\varepsilon_I^*(Y), \varepsilon_I^*(Y'))\,, \\ S(Y, Y') &\leq S(\delta_I(Y), \delta_I(Y'))\,. \end{aligned}$$

A direct consequence is that $(\varepsilon_I, \delta_I)$ and $(\varepsilon_I^*, \delta_I^*)$ are fuzzy adjunctions in the sense of Ref. 26.

Proof. The proof uses extensively the adjunction property and classical properties of adjoint connectives, supremum and infimum. In particular, the implication is increasing with respect to the second argument and decreasing with respect to the first one, and, for adjoint $*$ and $\to$, we have:

$$\forall (a, b) \in L^2, a \leq (a \to b) \to b \text{ and } a * (a \to b) \leq b\,.$$

We also have $X \preceq_F X' \Rightarrow S(X, X') = 1$.

Let us first show that $S(\delta_I(\varepsilon_I(X)), X) = 1$. We have:

$$\forall g \in G, \forall m \in M,\ \wedge_{g' \in G}(I(g', m) \to X(g')) \leq I(g, m) \to X(g)$$

then, using the definition of erosion and the adjunction property:

$$\varepsilon_I(X)(m) * I(g, m) \leq X(g)\,.$$

Taking the sup over m, we get:

$$\forall g \in G,\ \vee_{m \in M}(\varepsilon_I(X)(m) * I(g, m)) \leq X(g)$$

i.e. $\delta_I \varepsilon_I(X) \preceq_F X$, hence $S(\delta_I(\varepsilon_I(X)), X) = 1$.

Similarly we have $S(X, \varepsilon_I^*(\delta_I^*(X))) = S(Y, \varepsilon_I(\delta_I(Y))) = S(\delta_I^*(\varepsilon_I^*(Y)), Y) = 1$.

To show that $S(X, \varepsilon_I^*(Y)) \leq S(\delta_I^*(X), Y)$, we use the definition of S in Eq. (5), the adjunction property of $(*, \to)$, the increasingness of the supremum, the commutativity of $*$. Since the derivation is very similar as for the proofs in Ref. 26 (e.g. the one of Lemma 3), it is not detailed here. Similarly, we have $S(\delta_I^*(X), Y) \leq S(X, \varepsilon_I^*(Y))$, and we get $S(X, \varepsilon_I^*(Y)) = S(\delta_I^*(X), Y)$.

The other equalities are proved in a similar way.

Since the combination $\delta_I \varepsilon_I$ is anti-extensive, and S is decreasing with respect to its first argument, we have $S(X, X') \leq S(\delta_I \varepsilon_I(X), X')$. Since $S(\delta_I \varepsilon_I(X), X') = S(\varepsilon_I(X), \varepsilon_I(X'))$, we get $S(X, X') \leq S(\varepsilon_I(X), \varepsilon_I(X'))$, which proves the first inequality.

All other inequalities are proved in the same way. □

4. Links with Rough Sets and F-Transforms

In this section, we continue our investigation on the links between several algebraic frameworks for data analysis by establishing relationships between mathematical morphology, rough sets and F-transforms.

Several works have highlighted the links between rough sets, when formulated between two universes, and formal concept analysis (see e.g. Ref. 25 and the references therein), and the involved operators can be interpreted as modal operators in modal logics.[37] Based on previous work on mathematical morphology and rough sets,[38] and on mathematical morphology for modal logics,[2] our contribution in this section consists in further analyzing links between rough sets and FCA with a mathematical morphology point of view. These links hold in both the crisp and fuzzy cases. F-transforms[39] are additionally considered by providing a morphological interpretation of direct and inverse transforms, and of their links with rough sets, which have been recently suggested in the fuzzy case.[40]

Let X be a subset of a universe U, and R a binary relation on elements of U. Lower and upper approximations $\underline{R}X, \overline{R}X$ of the rough sets theory are morphological erosions and dilations, considering R as a structuring element.[38] A similar interpretation holds in the fuzzy case. Now, relationships with FCA require two universes, G and M, or L^G and L^M in the fuzzy case. By setting $R = I$, the links between rough sets, FCA, the four operators of possibility theory, modal logics and mathematical morphology are immediate. We consider directly the fuzzy case, the crisp one being then only a particular case.

Definition 6. Let $*$ and $\rightarrow$ be a conjunction and an implication, respectively. Let us define the operators $\underline{R}^G$ and $\overline{R}^G$ from L^G into L^M, and $\underline{R}^M$ and $\overline{R}^M$ from L^M into L^G as follows:

$$\forall X \in L^G, \forall m \in M, \underline{R}^G(X)(m) = \wedge_{g \in G}(R(g, m) \rightarrow X(g)), \tag{6}$$

$$\forall X \in L^G, \forall m \in M, \overline{R}^G(X)(m) = \vee_{g \in G}(R(g, m) * X(g)), \tag{7}$$

$$\forall Y \in L^M, \forall g \in G, \underline{R}^M(Y)(g) = \wedge_{m \in M}(R(g, m) \rightarrow Y(m)), \tag{8}$$

$$\forall Y \in L^M, \forall g \in G, \overline{R}^M(Y)(g) = \vee_{m \in M}(R(g, m) * Y(m)). \tag{9}$$

We have the following equivalences with the operators in Definition 5 and with the four possibilistic functions, the proof of which being straightforward.

Proposition 10. *Let (L^G, L^M, I) be a fuzzy formal context, where I is a binary relation over $G \times M$ taking values in L. Let $R = I$. Then we have:*

$$\begin{aligned}
&\forall X \in L^G, \underline{R}^G(X) = \varepsilon_I(X) = I_G^N(X)\,,\\
&\forall X \in L^G, \overline{R}^G(X) = \delta_I^*(X) = I_G^\Pi(X)\,,\\
&\forall Y \in L^M, \underline{R}^M(Y) = \varepsilon_I^*(Y) = I_M^N(Y)\,,\\
&\forall Y \in L^M, \overline{R}^M(Y) = \delta_I(Y) = I_M^\Pi(Y)\,.
\end{aligned}$$

Similar links exist straightforwardly between morphological operators and the sufficiency operators of Ref. 37 (see also Ref. 25).

Rough sets with two universes can be interpreted as follows: a set or fuzzy set of objects X is defined in an approximate way by some sets of properties verified by the elements (objects) of X. The lower approximation defines X by the properties such that each object g that satisfies one of these properties is in X. The upper approximation defines X as the set of properties satisfied by at least one of the objects in X. Similarly, a set or fuzzy set of properties Y is defined approximately by sets of objects satisfying properties in Y.

Let us now move to the framework of F-transforms.[39] We first establish some morphological properties of the direct and inverse transforms. Let f be a function defined on a universe U ($U = [0, 1]$ in Ref. 39) and taking values in $L = [0, 1]$. Let $\{A_k, k \in \{1, \dots, n\}\}$ be a fuzzy partition of U, such that

$$\forall x \in U, \exists k \in \{1, \dots, n\} \mid A_k(x) > 0\,.$$

The direct transforms are functions from L^U (the set of functions from U into L) into L^n defined as:[39]

$$\begin{aligned}
F^\uparrow(f) &= \{F_1^\uparrow(f), \dots, F_n^\uparrow(f)\}\\
F^\downarrow(f) &= \{F_1^\downarrow(f), \dots, F_n^\downarrow(f)\}
\end{aligned}$$

with, for each $k \in \{1, \dots, n\}$:

$$F_k^\uparrow(f) = \vee_{x \in U}(A_k(x) * f(x))\,, \tag{10}$$

the connective $*$ being a conjunction, and:

$$F_k^\downarrow(f) = \wedge_{x \in U}(A_k(x) \to f(x))\,, \tag{11}$$

the connective $\to$ being the adjoint implication of $*$.

The inverse transforms are functions from L^n into L^U defined as:[39]

$$\forall \varphi = (\varphi_1, \dots \varphi_n) \in L^n, \forall x \in U,\ f^\uparrow(\varphi)(x) = \wedge_{k=1}^n(A_k(x) \to \varphi_k)\,, \tag{12}$$

$$\forall \varphi = (\varphi_1, \dots \varphi_n) \in L^n, \forall x \in U,\ f^\downarrow(\varphi)(x) = \vee_{k=1}^n(A_k(x) * \varphi_k)\,. \tag{13}$$

Note that we use here slightly more general notations than in the original work,[39] where $f^\uparrow$ was defined for $\varphi_k = F_k^\uparrow(f)$ and $f^\downarrow$ for $\varphi_k = F^\downarrow(f)$.

Proposition 11. *Let us consider the two complete lattices* $\mathcal{L} = (L^U, \preceq_F)$ *and* $\mathcal{L}' = (L^n, \leq^n)$ *where* $\leq^n$ *denotes the component-wise (Pareto) order. The following morphological properties hold:*

- $F^\uparrow$ *commutes with the supremum and is therefore a dilation from* $\mathcal{L}$ *into* $\mathcal{L}'$*,*
- $F^\downarrow$ *commutes with the infimum and is therefore an erosion from* $\mathcal{L}$ *into* $\mathcal{L}'$*,*
- $f^\uparrow$ *commutes with the infimum and is therefore an erosion from* $\mathcal{L}'$ *into* $\mathcal{L}$*,*
- $f^\downarrow$ *commutes with the supremum and is therefore a dilation from* $\mathcal{L}'$ *into* $\mathcal{L}$*,*
- *the four operators are increasing,*
- *the pairs* $(f^\downarrow, F^\downarrow)$ *and* $(f^\uparrow, F^\uparrow)$ *are adjunctions, for* $*$ *and* $\rightarrow$ *being adjoint conjunction and implication,*
- $f^\uparrow(F^\uparrow)(f)$ *is a closing of* f *and* $f^\downarrow(F^\downarrow)(f)$ *is an opening of* f*, and in particular we have* $f^\downarrow(F^\downarrow)(f) \preceq_F f \preceq_F f^\uparrow(F^\uparrow)(f)$.

Proof. We have, for each $k \in \{1, \dots, n\}$, $F_k^\uparrow(f \vee g) = F_k^\uparrow(f) \vee F_k^\uparrow(g)$ since any conjunction $*$ is distributive with respect to $\vee$. Hence $F^\uparrow$ commutes with the supremum, and is therefore a dilation. Similarly, $f^\downarrow$ commutes with the supremum, and $F^\downarrow$ and $f^\uparrow$ commute with the infimum. It follows directly that these operators are increasing. Note that this monotony property is derived differently in Ref. 39.

Let us show that $(f^\downarrow, F^\downarrow)$ is an adjunction for adjoint connectives $*$ and $\rightarrow$. Let $\varphi = (\varphi_1, \dots, \varphi_n) \in L^n$ and $g \in L^U$:

$$\begin{aligned} f^\downarrow(\varphi) \preceq_F g &\Leftrightarrow \forall x \in U,\ \vee_{k=1}^n (A_k(x) * \varphi_k) \leq g(x) \\ &\Leftrightarrow \forall x \in U, \forall k \in \{1, \dots, n\},\ A_k(x) * \varphi_k \leq g(x) \\ &\Leftrightarrow \forall x \in U, \forall k \in \{1, \dots, n\},\ \varphi_k \leq A_k(x) \rightarrow g(x) \\ &\Leftrightarrow \forall k \in \{1, \dots, n\},\ \varphi_k \leq \wedge_{x \in U}(A_k(x) \rightarrow g(x)) \\ &\Leftrightarrow \varphi \leq^n F^\downarrow(g) \end{aligned}$$

which proves the adjunction property. A similar reasoning allows proving that $(f^\uparrow, F^\uparrow)$ is an adjunction.

The last item is derived from the adjunction properties. □

It has been shown in Ref. 40 that for a binary relation R defined as $R = \cup_k R_k$ for

$$R_k(x, y) = \begin{cases} A_k(y) & \text{if } x \in Core(A_k)\,, \\ 1 & \text{if } x = y\,, \\ 0 & \text{otherwise} \end{cases}$$

then the direct transforms are equivalent to upper and lower approximations of the rough sets theory for the relation R, and $f^\uparrow(F^\uparrow)(f) = \underline{R}(\overline{R}(f))$, $f^\downarrow(F^\downarrow)(f) =$

$\overline{R}(\underline{R}(f))$, which are hence closing and opening of f, respectively, according to Ref. 38. These relations between rough sets and F-transforms are then another way to recover the last results of Proposition 11.

Now to go one step further and establish links with FCA, two universes are again needed. Let us consider a fuzzy context $\mathbb{K}$ as previously. We propose to identify U with G and $\{1, \ldots, n\}$ with M if $|M| = n$, thus corresponding to a simple numbering of the properties in M (note that the converse could do as well). Then φ is a fuzzy subset of M and f a fuzzy subset of G. For the relation, we propose to identify $A_k(x)$ with $I(g, m_k)$, with $g = x$ and k is the index of property m_k. According to the representation in Table 2, $A_k(\cdot)$ is a column of $\mathbb{K}$ and $A_{\cdot}(g)$ a line of this table, with $m_1 = c, m_2 = e, m_3 = o, m_4 = p, m_5 = s$. For instance, for $k = 4$, $m_k = p$ and $A_k(1) = 0.2, A_k(2) = A_k(3) = A_k(5) = A_k(7) = 1, A_k(4) = A_k(6) = A_k(8) = A_k(9) = A_k(10) = 0$, and for $g = 1$, $A_1(g) = A_4(g) = 0.2, A_2(g) = 0, A_3(g) = A_5(g) = 1$.

Then we derive directly the following result, completing the links between the different formalisms.

Proposition 12.

- $F^{\uparrow} = \delta_I^*$,
- $F^{\downarrow} = \varepsilon_I$,
- $f^{\uparrow} = \varepsilon_I^*$,
- $f^{\downarrow} = \delta_I$.

5. Mathematical Morphology on Formal Concepts

Beyond the links established in Sec. 3, we propose in this section to build morphological operators acting on concept lattices. As in any complete lattice, dilations and erosions are defined as operations that commute with the supremum and the infimum, respectively.

In Ref. 21, we developed this idea in the crisp case, and proposed two types of operators. The first one is based on the notion of structuring element, defined as an elementary neighborhood of elements of G or as a binary relation between elements of G. We defined such a neighborhood as a ball of radius 1 of some distance function on G derived from a distance on $\mathbb{C}$. Such a distance can be built for instance from valuations on the lattice (see e.g. Refs. 9, 41 for details on valuations, their properties and derived metrics). In the second approach we defined morphological operators directly from a distance on $\mathbb{C}$.

We do not detail this previous work here, and move directly to the fuzzy case. We propose in this section extensions of morphological operators to fuzzy operators acting on a lattice of fuzzy concepts. As explained in the introduction, this is a common way in mathematical morphology to define concrete operators, which has proved useful in many applications, in various domains, such as image processing, data analysis, fuzzy sets, or logical reasoning.

In this section, we take $L = [0,1]$, and we further assume that G and M are finite. Several results are however more general.

5.1. *Distances from valuations*

A fuzzy concept $a \in \mathbb{C}^F$ will be denoted by $a = (e(a), i(a))$, with $e(a) \in L^G$ and $i(a) \in L^M$. The cardinality of a fuzzy set X of G is considered here as a crisp number: $|X| = \sum_{g\in G} X(g)$ (and similarly for a fuzzy subset of M). This cardinality is increasing, i.e.

$$\forall X_1, X_2,\ X_1 \preceq_F X_2 \Rightarrow |X_1| \leq |X_2|\,,$$

and satisfies

$$\forall X_1, X_2,\ |X_1 \vee^F X_2| = |X_1| + |X_2| - |X_1 \wedge^F X_2|\,.$$

Indeed, $|X_1| + |X_2| - |X_1 \wedge^F X_2| = \sum_{g\in G}(X_1(g) + X_2(g) - \min(X_1(g), X_2(g))) = \sum_{g\in G} \max(X_1(g), X_2(g)) = |X_1 \vee^F X_2|$.

The proposed construction relies on the following result,[41,42] that holds in any lattice, and that we write here with the notations of this paper.

Theorem 1.[41,42] *Let ω be a real-valued function on a concept lattice $(\mathbb{C}^F, \preceq_{FC})$. Then the function defined as:*

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ d_\omega(a_1, a_2) = 2\omega(a_1 \wedge^{FC} a_2) - \omega(a_1) - \omega(a_2)\,, \tag{14}$$

where $\wedge^{FC}$ is the infimum of fuzzy concepts (Eq. (3)), is a pseudo-metric if and only if ω is decreasing and is an upper valuation, i.e. satisfying the submodular property:

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ \omega(a_1) + \omega(a_2) \geq \omega(a_1 \wedge^{FC} a_2) + \omega(a_1 \vee^{FC} a_2)\,, \tag{15}$$

where $\vee^{FC}$ is the supremum of fuzzy concepts (Eq. (4)).

The function defined as:

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ d_\omega(a_1, a_2) = \omega(a_1) + \omega(a_2) - 2\omega(a_1 \vee^{FC} a_2) \tag{16}$$

is a pseudo-metric if and only if ω is decreasing and is a lower valuation, i.e. satisfying the supermodular property:

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ \omega(a_1) + \omega(a_2) \leq \omega(a_1 \wedge^{FC} a_2) + \omega(a_1 \vee^{FC} a_2)\,. \tag{17}$$

Based on this general result, metrics are obtained by defining suitable valuations on $(\mathbb{C}^F, \preceq_{FC})$. In what follows we introduce some examples of such valuations.

Proposition 13. *On $(\mathbb{C}^F, \preceq_{FC})$, the real-valued function defined as:*

$$\forall a \in \mathbb{C},\ \omega_G(a) = |G| - |e(a)| \tag{18}$$

is a strictly decreasing upper valuation.

Proof. Let $a_1 = (X_1, Y_1)$ and $a_2 = (X_2, Y_2)$ be two formal concepts. The strict decreasingness of ω_G follows from the fact that: $(X_1, Y_1) \prec_{FC} (X_2, Y_2)$ implies $X_1 \prec_F X_2$, hence $|G| - |X_1| > |G| - |X_2|$.

Let us now prove that ω_G is an upper valuation, i.e. it satisfies the submodular property. From Eqs. (3) and (4) we have:

$$\omega_G(a_1 \wedge^{FC} a_2) + \omega_G(a_1 \vee^{FC} a_2) = 2|G| - |X_1 \wedge^F X_2| - \left|\beta\left(\alpha\left(X_1 \vee^F X_2\right)\right)\right|.$$

Then:

$$\begin{aligned}
&\omega_G(a_1) + \omega_G(a_2) - \omega_G(a_1 \wedge^{FC} a_2) - \omega_G(a_1 \vee^{FC} a_2) \\
&\quad = |X_1 \wedge^F X_2| - |X_1| - |X_2| + \left|\beta\big(\alpha(X_1 \vee^F X_2)\big)\right| \\
&\quad = \left|\beta\big(\alpha(X_1 \vee^F X_2)\big)\right| - |X_1 \vee^F X_2| \\
&\quad \geq 0
\end{aligned}$$

since the closure operator $\beta\alpha$ is extensive ($X \preceq_F \beta(\alpha(X))$). This completes the proof. □

Proposition 14. *The function defined as:*

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ d_{\omega_G}(a_1, a_2) = 2\omega_G(a_1 \wedge^{FC} a_2) - \omega_G(a_1) - \omega_G(a_2)$$

is a metric on $(\mathbb{C}^F, \preceq_{FC})$, *and*

$$d_{\omega_G}(a_1, a_2) = |e(a_1) \vee^F e(a_2)| - |e(a_1) \wedge^F e(a_2)|.$$

Proof. From Theorem 1 and Proposition 13, d_{ω_G} is a pseudo-metric. Let $a_1 = (X_1, Y_1), a_2 = (X_2, Y_2)$ be formal concepts in $\mathbb{C}^F$. Then $d_{\omega_G}(a_1, a_2)$ can be written as:

$$\begin{aligned}
d_{\omega_G}(a_1, a_2) &= |X_1| + |X_2| - 2|X_1 \wedge^F X_2| \\
&= |X_1 \vee^F X_2| - |X_1 \wedge^F X_2| \\
&= |e(a_1) \vee^F e(a_2)| - |e(a_1) \wedge^F e(a_2)|
\end{aligned}$$

and it is then a metric on $\mathbb{C}^F$ since $|X_1 \vee^F X_2| - |X_1 \wedge^F X_2| = 0$ implies $X_1 = X_2$ and hence $a_1 = a_2$ (since they are formal concepts). □

Proposition 15. *The real-valued function defined on* $(\mathbb{C}^F, \preceq_{FC})$ *as:*

$$\forall a \in \mathbb{C}^F,\ \omega_M(a) = |i(a)| \tag{19}$$

is a strictly decreasing lower valuation.

Proof. The proof is similar as for ω_G. □

Proposition 16. *The function defined as:*

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,\ d_{\omega_M}(a_1, a_2) = \omega_M(a_1) + \omega_M(a_2) - 2\omega_M(a_1 \vee^{FC} a_2)$$

is a metric on $(\mathbb{C}^F, \preceq_{FC})$, *and*

$$d_{\omega_M}(a_1, a_2) = |i(a_1) \vee^F i(a_2)| - |i(a_1) \wedge^F i(a_2)|.$$

Proof. As for $d_{\omega_G}(a_1, a_2)$, $d_{\omega_M}(a_1, a_2)$ is a pseudo-metric from Theorem 1 and Proposition 15. By denoting $Y_i = i(a_i)$, and using the fact that $i(a_1 \vee^{FC} a_2) = i(a_1) \wedge^F i(a_2)$ (Eq. (4)), $d_{\omega_M}(a_1, a_2)$ can be written as $|Y_1| + |Y_2| - 2|Y_1 \wedge^F Y_2| = |Y_1 \vee^F Y_2| - |Y_1 \wedge^F Y_2|$. It is then a metric on $\mathbb{C}^F$ since $|Y_1 \vee^F Y_2| - |Y_1 \wedge^F Y_2| = 0$ implies $a_1 = a_2$. □

As an illustration of these definitions, the distance between the two fuzzy concepts (X_1, Y_1) and (X_2, Y_2) given as examples in Sec. 3.3 is equal to 0.7 (for both d_{ω_G} and d_{ω_M}).

In the particular case where fuzzy sets are crisp, these results become equivalent to the ones in Ref. 21.

5.2. *Distances from filters and ideals*

The distances introduced in previous work[21] based on filters and ideals also directly extend to the fuzzy case.

Let us recall that the ideal and filter associated with $a \in \mathbb{C}^F$ are defined respectively as:

$$I_a = \{b \in \mathbb{C}^F \mid b \preceq_{FC} a\},$$
$$F_a = \{b \in \mathbb{C}^F \mid a \preceq_{FC} b\},$$

these definitions being direct extensions of the ones in the crisp case. Note that they provide crisp subsets of $\mathbb{C}^F$. It is easy to show that $I_{a \vee^{FC} b} = I_a \cup I_b$, $I_{a \wedge^{FC} b} = I_a \cap I_b$, and similar results for filters.

Proposition 17. *Let us denote by $\omega_I(a) = |I_a|$ the cardinality of the ideal generated by an element a of $\mathbb{C}^F$. The function ω_I is increasing and supermodular (lower valuation). Then one can define a pseudo-metric as:*

$$d_{\omega_I}(a_1, a_2) = \omega_I(a_1) + \omega_I(a_2) - 2\omega_I(a_1 \wedge^{FC} a_2).$$

Proof. It is straightforward to see that ω_I is increasing. From the properties of ideals, we have:

$$\begin{aligned} \omega_I(a \wedge^{FC} b) + \omega_I(a \vee^{FC} b) &= |I_{a \wedge^{FC} b}| + |I_{a \vee^{FC} b}| \\ &= |I_a \cap I_b| + |I_a \cup I_b| \\ &= |I_a \cap I_b| + |I_a| + |I_b| - |I_a \cap I_b| \\ &= |I_a| + |I_b| = \omega_I(a) + \omega_I(b). \end{aligned}$$

This shows that ω_I is a lower valuation. Hence $-\omega_I$ is a decreasing upper valuation, and from Theorem 1 applied on $-\omega_I$, d_{ω_I} is a pseudo-metric. □

Proposition 18. *Let us denote by $\omega_F(a) = |F_a|$ the cardinality of the filter generated by an element a of $\mathbb{C}^F$. The function ω_F is decreasing and supermodular* *(lower valuation). Then one can define a pseudo-metric as:*

$$d_{\omega_F}(a_1, a_2) = \omega_F(a_1) + \omega_F(a_2) - 2\omega_F(a_1 \vee^{FC} a_2)\,.$$

Proof. The proof is similar to the one for d_{ω_I}, by exploiting this time the properties of filters. □

By generalizing the previous valuations, one can define the following ones:[43] consider a non-negative real-valued function f on $\mathbb{C}^F$, then the function defined as:

$$\omega_f(a) = \sum_{b \preceq_{FC} a} f(b)$$

is an increasing lower valuation, and

$$\omega^f(a) = \sum_{a \preceq_{FC} b} f(b)$$

is a decreasing lower valuation.

5.3. *Morphological operators on L^G, L^M and $\mathbb{C}^F$*

In the following, we denote by d^F any metric defined on $\mathbb{C}^F$. It induces a pseudo-metric on G or M by applying it on the object concepts or attribute concepts. For any $g \in G$, $p(g) = (\beta\alpha(\{g\}), \alpha(\{g\}))$ is the fuzzy object concept of g, as a direct extension of the notion of object concept in the crisp case. Then we can define for instance for any $g_1, g_2 \in G$:

$$d(g_1, g_2) = d^F(p(g_1), p(g_2))\,.$$

A similar construction can be performed based on attribute concepts.

Definition 7. Dilations and erosions on $(L^G, \preceq)$ are defined, for all $X \in L^G$, as:

$$\forall g \in G, \delta_b(X)(g) = \vee_{g' \in b(g)} X(g')\,,$$
$$\forall g \in G, \varepsilon_b(X)(g) = \wedge_{g' \in b(g)} X(g')\,,$$

where b is a structuring element defined as:

$$b(g) = \{g' \in G \mid d(g, g') \leq 1\}\,.$$

Dilations and erosions of size n are defined by using the set $\{g' \in G \mid d(g, g') \leq n\}$ as structuring element.

Proposition 19. *δ_b is extensive and ε_b is anti-extensive.*

Proof. The result follows directly from the fact that $g \in b(g)$. □

Similar definitions and results apply on L^M.

5.4. *Using generators*

Another construction can be derived from the following definitions and results, using generators.

Definition 8. An *elementary fuzzy set* X_g, associated with a fuzzy set X is defined as:

$$X_g(g) = X(g) \text{ and } \forall g' \in G \setminus \{g\}, X_g(g') = 0\,.$$

Proposition 20. *Any fuzzy set X is sup-generated by the associated X_g, i.e.*

$$X = \vee^F{}_{g\in G} X_g\,.$$

More generally, the set $\{X_g^\lambda, g \in G, \lambda \in L\}$, such that

$$X_g^\lambda(g) = \lambda \text{ and } \forall g' \neq g, X_g^\lambda(g') = 0$$

sup-generates the lattice of fuzzy sets. A similar result for fuzzy attributes is given in Ref. 44 in the framework of multi-adjoint concept lattices.

Definition 9. The *fuzzy object concept* of an elementary fuzzy set X_g is defined as $\tilde{p}(X_g) = (\beta\alpha(X_g), \alpha(X_g))$.

Proposition 21. $\mathbb{C}^F$ *is sup-generated by fuzzy object concepts associated with elementary fuzzy sets, i.e.:*

$$\forall a = (X,Y) \in \mathbb{C}^F, (X,Y) = \vee^{FC}{}_{g\in G}\tilde{p}(X_g)\,,$$

where $\vee^{FC}$ *is given by Eq.* (4).

Proof. The proof is derived from the expression of the supremum in Eq. (4), from Definition 9 and from the fact that $(X,Y) \in \mathbb{C}^F$:

$$\begin{aligned}
\vee^{FC}{}_{g\in G}\tilde{p}(X_g) &= \vee^{FC}{}_{g\in G}(\beta\alpha(X_g), \alpha(X_g)) \\
&= (\beta\alpha(\vee^F{}_{g\in G}\beta\alpha(X_g)), \wedge^F{}_{g\in G}\alpha(X_g)) \\
&= (\beta(\wedge^F{}_{g\in G}\alpha\beta\alpha(X_g)), \alpha(\vee^F{}_{g\in G}X_g)) \\
&= (\beta(\wedge^F{}_{g\in G}\alpha(X_g)), \alpha(X)) \\
&= (\beta\alpha(\vee^F{}_{g\in G}X_g), Y) \\
&= (\beta\alpha(X), Y) = (X,Y)\,.
\end{aligned}$$

□

Note that this result could also be derived from the results in Ref. 44, and linking elementary fuzzy attributes and meet irreducible elements (as also used in the next subsection in the present work).

Definition 10. From any dilation $\tilde{\delta}$ on the generating fuzzy object concepts (image of elementary fuzzy sets by $\tilde{p}$), a dilation on $\mathbb{C}^F$ is derived as:

$$\forall a = (X,Y) \in \mathbb{C}^F, \delta(a) = \vee^{FC}{}_{g\in G}\tilde{\delta}(\tilde{p}(X_g))\,.$$

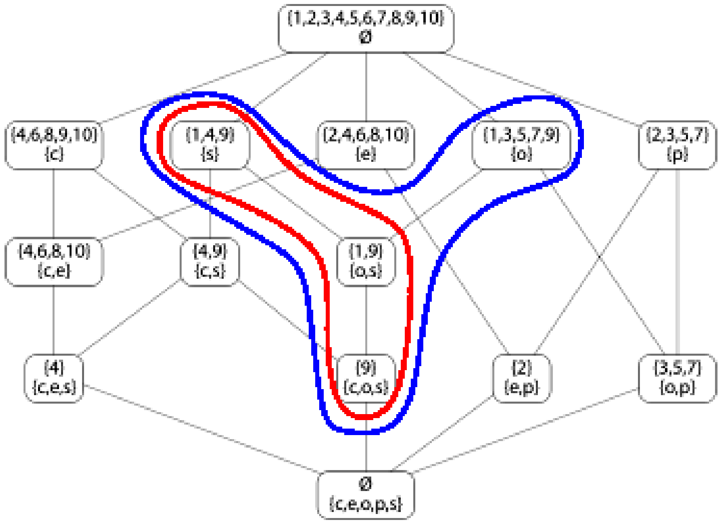

Fig. 3. Dilation of $\{a\} = \{(\{1, 9\}, \{o, s\})\}$ using as a structuring element a ball of d_{ω_G} (red) and of d_{ω_M} (blue).

A similar construction applies for erosion, from the inf-generated property of concepts, extended to the fuzzy case, and the commutativity of erosion with the infimum.

Let us consider again the example in Fig. 1 (in the crisp case for this simple illustration), and the concept $a = (\{1, 9\}, \{o, s\})$. We have:

$$d_{\omega_G}(a, a_1) = d_{\omega_G}(a, a_3) = 1 \text{ and } d_{\omega_M}(a, a_1) = d_{\omega_M}(a, a_2) = d_{\omega_M}(a, a_3) = 1\,,$$

where $a_1 = (\{1, 4, 9\}, \{s\}), a_2 = (\{1, 3, 5, 7, 9\}, \{o\}), a_3 = (\{9\}, \{c, o, s\})$. Hence:

$$\delta^1_G(\{a\}) = \{a, a_1, a_3\} \text{ and } \delta^1_M(\{a\}) = \{a, a_1, a_2, a_3\}\,.$$

This is illustrated in Fig. 3.

5.5. *Using irreducible decompositions*

Another classical decomposition in ordered sets is the join irreducible decomposition. Let us denote by $\mathcal{J}(\mathbb{C}^F)$ the set of join irreducible elements of $\mathbb{C}^F$, i.e.

$$\mathcal{J}(\mathbb{C}^F) = \{a \in \mathbb{C}^F \setminus \{\bot\} \mid \forall (b, c) \in \mathbb{C}^F \times \mathbb{C}^F, a = b \vee^{FC} c \Rightarrow a = b \text{ or } a = c\}\,.$$

The smallest element $\bot$ is excluded from the set of irreducible elements. We have:[45]

$$\forall a \in \mathbb{C}^F \setminus \{\bot\}, a = \vee^{FC}\{b \in \mathcal{J}(\mathbb{C}^F) \mid b \preceq_{FC} a\}\,.$$

We denote by $\mathcal{J}(a)$ the set of elements b involved in this decomposition. A minimality constraint can also be added, as suggested in Ref. 21. Note that $\bot$ is equal to the supremum of an empty family in a complete lattice, so this expression would hold also for $\bot$ (and in that case $\mathcal{J}(a)$ would be empty). The same consideration applies for the greatest element $\top$ in the meet irreductible decomposition below. These limit cases are not further considered in the following.

Such decompositions have been used in more general contexts as well, such as object- and property-oriented concept lattices.[46,47]

Definition 11. From a distance d in $\mathbb{C}^F$, we derive a dilation (of size n) from $\mathcal{J}(\mathbb{C}^F)$ into $\mathbb{C}^F$ as:

$$\forall b \in \mathcal{J}(\mathbb{C}^F),\ \delta_J(b) = \vee^{FC}\{b \in \mathbb{C}^F \mid d(a,b) \leq n\}\,,$$

and a dilation on $\mathbb{C}^F$ as:

$$\forall a \in \mathbb{C}^F,\ \delta(a) = \vee^{FC}\{\delta_J(b) \mid b \in \mathcal{J}(\mathbb{C}^F) \text{ and } b \preceq_{FC} a\}\,.$$

Alternatively, the supremum in this definition can be restricted to the set of b forming a minimal decomposition of a.

A similar construction for erosion can be performed from the meet irreducible decomposition of any element of $\mathbb{C}^F$. Let $\mathcal{M}(\mathbb{C}^F)$ denote the set of meet irreducible elements (defined in a similar way as $\mathcal{J}(\mathbb{C}^F)$). The decomposition of any $a \in \mathbb{C}^F \setminus \{\top\}$ into meet irreducible elements is defined as:

$$a = \wedge^{FC}\{b \in \mathcal{M}(\mathbb{C}^F) \mid a \preceq_{FC} b\}\,.$$

Let $\mathcal{M}(a)$ be the set of elements b involved in this decomposition.

Definition 12. From a distance d in $\mathbb{C}^F$, we derive an erosion (of size n) from $\mathcal{M}(\mathbb{C}^F)$ into $\mathbb{C}^F$ as:

$$\forall b \in \mathcal{M}(\mathbb{C}^F),\ \varepsilon_M(b) = \wedge^{FC}\{b \in \mathbb{C}^F \mid d(a,b) \leq n\}\,,$$

and an erosion on $\mathbb{C}^F$ as:

$$\forall a \in \mathbb{C}^F,\ \varepsilon(a) = \wedge^{FC}\{\varepsilon_M(b) \mid b \in \mathcal{M}(\mathbb{C}^F) \text{ and } a \preceq_{FC} b\}\,.$$

Let us now consider particular distances, which are said $\vee$-compatible, and derive dilations from them, as an extension of the construction done in Ref. 21 for the crisp case.

Definition 13. A distance is $\vee$-*compatible*, and denoted by $d^\vee$, if for any n in $\mathbb{R}^+$ and any family $(a_i)_{i\in\Xi}$ of elements of $\mathbb{C}^F$, for an index set Ξ:

$$\{b \in \mathbb{C}^F \mid d^\vee(\vee^{FC}_{i\in\Xi} a_i, b) \leq n\} = \cup_{i\in\Xi}\{b \in \mathbb{C}^F \mid d^\vee(a_i,b) \leq n\}\,.$$

Proposition 22. *Let d be any distance on the fuzzy concept lattice $(\mathbb{C}^F, \preceq)$, and $\mathcal{J}$ the join-irreducible decomposition operator on $(\mathbb{C}^F, \preceq)$. Then the following function:*

$$\forall (a,b) \in {\mathbb{C}^F}^2, d^\vee(a,b) = \inf_{a_i \in \mathcal{J}(a)} d(a_i, b)$$

is $\vee$-compatible.

Proof. Let $a = \vee^{FC}\{a_i \mid a_i \in \mathcal{J}(a)\}$ (i.e. a_i is irreducible for each i). We have:

$$\{b \in \mathbb{C}^F \mid d^\vee(a,b) \leq n\} = \{b \in \mathbb{C}^F \mid \inf_{a_i \in \mathcal{J}(a)} d(a_i, b) \leq n\}$$
$$= \cup_{\{i \mid a_i \in \mathcal{J}(a)\}}\{b \in \mathbb{C}^F \mid d(a_i, b) \leq n\}\,.$$

Let now $a = \vee^{FC}{}_{i\in\Xi} a_i$ for any family of a_i indexed by Ξ (not necessarily irreducible). Each a_i can be decomposed as $a_i = \vee^{FC}\{a_{ij} \mid a_{ij} \in \mathcal{J}(a_i)\}$, and $a = \vee^{FC}{}_{i\in\Xi} \vee^{FC} \{a_{ij} \mid a_{ij} \in \mathcal{J}(a_i)\}$. Let us note $\Xi_i = \{j \mid a_{ij} \in \mathcal{J}(a_i)\}$. Then we have:

$$\{b \in \mathbb{C}^F \mid d^\vee(\vee^{FC}_{i\in\Xi} a_i, b) \leq n\} = \{b \in \mathbb{C}^F \mid \inf_{a_{ij} \in \cup_{i\in\Xi}\mathcal{J}(a_i)} d(a_{ij}, b) \leq n\}$$
$$= \cup_{i\in\Xi, j\in\Xi_i}\{b \in \mathbb{C}^F \mid d(a_{ij}, b) \leq n\}$$
$$= \cup_{i\in\Xi}\{b \in \mathbb{C}^F \mid \inf_{j\in\Xi_i} d(a_{ij}, b) \leq n\}$$
$$= \cup_{i\in\Xi}\{b \in \mathbb{C}^F \mid d^\vee(a_i, b) \leq n\}\,.$$

□

Proposition 23. *Let $d^\vee$ be a $\vee$-compatible distance on $\mathbb{C}^F$. For any n in $\mathbb{N}$, the operator defined as:*

$$\forall a \in \mathbb{C}^F, \delta(a) = \vee^{FC}\{b \in \mathbb{C}^F \mid d^\vee(a,b) \leq n\}$$

is a dilation.

Proof. Let us prove that δ commutes with the supremum. Since $d^\vee$ is $\vee$-compatible, we have:

$$\forall (a_1, a_2) \in \mathbb{C}^F \times \mathbb{C}^F,$$
$$\delta(a_1 \vee^{FC} a_2) = \vee^{FC}\{b \in \mathbb{C}^F \mid d^\vee(a_1 \vee^{FC} a_2, b) \leq n\}$$
$$= \vee^{FC}(\{b \in \mathbb{C}^F \mid d^\vee(a_1, b) \leq n\} \cup \{b \in \mathbb{C}^F \mid d^\vee(a_2, b) \leq n\})$$
$$= (\vee^{FC}(\{b \in \mathbb{C}^F \mid d^\vee(a_1, b) \leq n\})$$
$$\vee^{FC} (\vee^{FC}\{b \in \mathbb{C}^F \mid d^\vee(a_2, b) \leq n\})$$
$$= \delta(a_1) \vee^{FC} \delta(a_2)\,.$$

This result extends to any family of a_i, hence δ commutes with the supremum. Note that it uses the fact that, since $\preceq_{FC}$ is a partial ordering, we have, for any subsets A and B of $\mathbb{C}^F$, $(\vee^{FC} A) \vee^{FC} (\vee^{FC} B) = \vee^{FC}(A \cup B)$. □

Similarly, $\wedge$-compatible distances can be defined, i.e. such that for any n in $\mathbb{R}^+$ and any family $(a_i)_{i\in\Xi}$ of elements of $\mathbb{C}^F$:

$$\{b \in \mathbb{C}^F \mid d^\wedge(\wedge^{FC}_{i\in\Xi} a_i, b) \leq n\} = \cap_{i\in\Xi}\{b \in \mathbb{C}^F \mid d^\wedge(a_i, b) \leq n\}\,,$$

based on meet irreducible decompositions, from which erosions can be derived.

As underlying distance d on $\mathbb{C}$, the distances defined above such as d_{ω_G} and d_{ω_M} can be used, for instance.

Let us illustrate these definitions in the crisp case, on the example in Fig. 1. In this lattice, the irreducible elements are:

$$(\{4,6,8,10\},\{c,e\}) = \tilde{p}(X_6) = \tilde{p}(X_8) = \tilde{p}(X_{10})\,,$$
$$(\{1,9\},\{o,s\}) = \tilde{p}(X_1) = a\,,$$
$$(\{4\},\{c,e,s\}) = \tilde{p}(X_4) = a_4\,,$$
$$(\{9\},\{c,o,s\}) = \tilde{p}(X_9) = a_3\,,$$
$$(\{2\},\{e,p\}) = \tilde{p}(X_2)\,,$$
$$(\{3,5,7\},\{o,p\}) = \tilde{p}(X_3) = \tilde{p}(X_5) = \tilde{p}(X_7)\,.$$

Note that in this example irreducible elements are exactly elementary object concepts. Moreover, if the context is made non redundant by clarification (by removing the identical lines and columns on this particular example, here objects 5, 7, 8, 10 in Fig. 1) then irreducible elements and elementary object concepts are in one-to-one correspondence.

Let us decompose $a_1 = (\{1,4,9\},\{s\})$:

$$a_1 = a_4 \vee^{FC} a \vee^{FC} a_3\,.$$

Let us take d_{ω_G} as distance on $\mathbb{C}$, and the associated dilation δ_G on irreducible

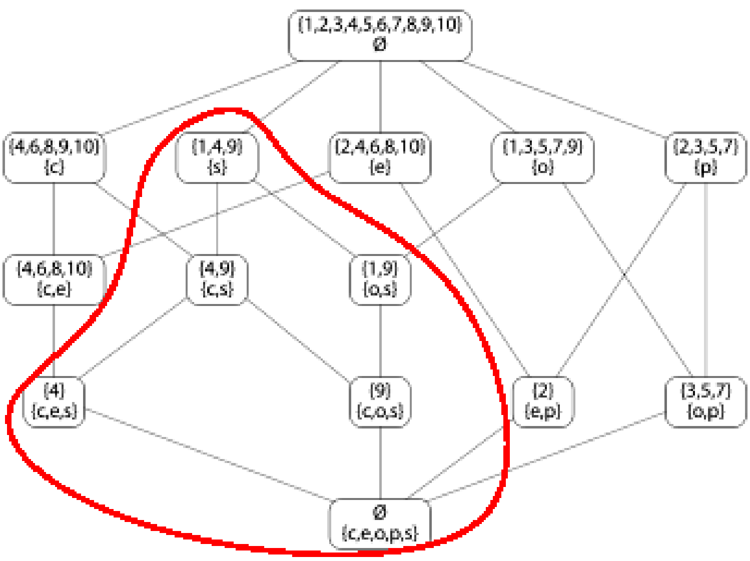

Fig. 4. Dilation of $\{a_1\} = \{(\{1,4,9\},\{s\})\}$ using as a structuring element a ball of d_{ω_G} for each irreducible element of its decomposition.

elements. We have:

$$\delta_G(\{a_4\}) = \{a_4, a_5 = (\{4,9\}, \{c,e,s\}), \bot\}$$
$$\delta_G(\{a\}) = \{a, a_1, a_3\}$$
$$\delta_G(\{a_3\}) = \{a_3, a_5, a, a_0\}$$

and thus $\delta(\{a_1\}) = \{\bot, a, a_1, a_3, a_4, a_5\}$. Note that this is not equivalent to computing directly $\delta_G(\{a_1\})$, thus really providing a new definition. This is illustrated in Fig. 4.

While this figure illustrates one particular dilation, different from the direct dilation of $\{a_1\}$, several others could be defined and illustrated similarly. For instance, other distances could be chosen (from which the dilation would be derived), acting either on G, on M, or depending on both G and M. This open choice offers flexibility for adapting the operators according to the application at hand.

6. Conclusion

The contribution of this paper is twofold. First, we exhibited links between formal concept analysis and mathematical morphology in different frameworks (sets, fuzzy sets, rough sets, F-transforms), paving the way for further discussions on these links and on the properties that each framework could inherit from the others. Note that these formal links should be exploited by keeping in mind that the semantics of the different formalisms may have to be differentiated. An interesting outcome of these links is that they allow to have unified representations between formalisms with originally different aims (e.g. explaining vs. inferring, describing vs. predicting, etc.). These links are summarized in Fig. 5.

Secondly we proposed operational ways to transform formal concepts using mathematical morphology in these frameworks. These operations could now be

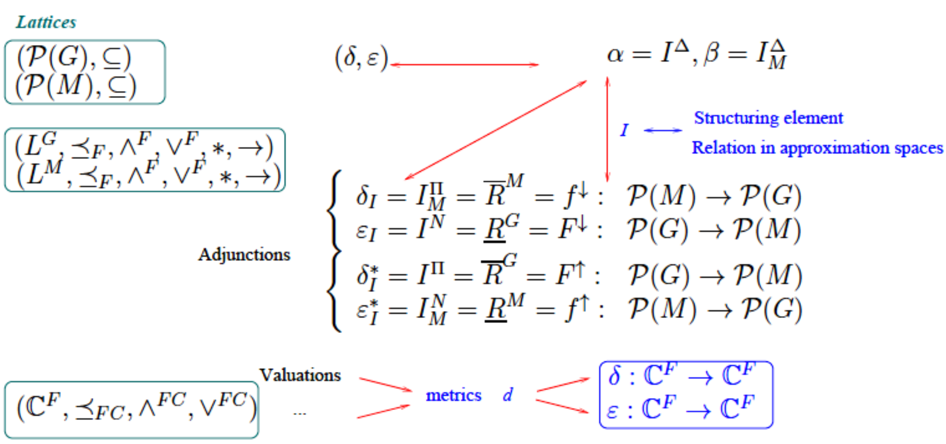

Fig. 5. Summary of the contributions and established links.

explored to reason on concept lattices, for navigating among concepts, etc. (for instance for abduction in the crisp case[7] or in a fuzzy setting[6]).

Algorithmic questions should also be addressed, for instance to build a fuzzy concept lattice. In the proposed extensions to the fuzzy case, cardinality and distance values are crisp numbers. Another possible extension could be to define them as fuzzy numbers, however at the cost of an increased complexity. Other links, for instance with graphs and hypergraphs, will be explored in our future work, based on mathematical morphology operators defined on such structures, in particular transforming a set of vertices into a set of edges or hyperdeges, or the converse.[48,49]

Acknowledgements

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References

1. J. Serra, *Image Analysis and Mathematical Morphology* (Academic Press, New York, 1982).
2. I. Bloch, Modal logics based on mathematical morphology for spatial reasoning, *Journal of Applied Non Classical Logics* **12**(3–4) (2002) 399–424.
3. I. Bloch, R. Pino-Pérez, and C. Uzcategui, A unified treatment of knowledge dynamics, in *International Conference on the Principles of Knowledge Representation and Reasoning* (*KR2004*), Canada, 2004, pp. 329–337.
4. I. Bloch, H. Heijmans, and C. Ronse, Mathematical morphology, in *Handbook of Spatial Logics*, eds. M. Aiello, I. Pratt-Hartman, and J. van Benthem, Chapter 13, Springer, 2007, pp. 857–947.
5. N. Gorogiannis and A. Hunter, Merging first-order knowledge using dilation operators, in *Fifth International Symposium on Foundations of Information and Knowledge Systems* (*FoIKS'08*), Vol. LNCS 4932,January 2008, pp. 132–150.
6. J. Atif, I. Bloch, F. Distel, and C. Hudelot, A fuzzy extension of explanatory relations based on mathematical morphology, in *8th Conference of the European Society for Fuzzy Logic and Technology* (*EUSFLAT 2013*), Milano, Italy, September 2013, pp. 244–351.
7. J. Atif, C. Hudelot, and I. Bloch, Explanatory reasoning for image understanding using formal concept analysis and description logics, *IEEE Transactions on Systems, Man and Cybernetics: Systems* **44**(5) (2014) 552–570.
8. B. Ganter, R. Wille, and C. Franzke, *Formal Concept Analysis: Mathematical Foundations* (Springer-Verlag New York, Inc., 1997).
9. G. Birkhoff, *Lattice Theory*, Vol. 25, 3rd edn. (American Mathematical Society, 1979).
10. J. Serra (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances* (Academic Press, London, 1988).
11. H. J. A. M. Heijmans and C. Ronse, The algebraic basis of mathematical morphology — Part I: Dilations and erosions, *Computer Vision, Graphics and Image Processing* **50** (1990) 245–295.
12. C. Ronse and H. J. A. M. Heijmans, The algebraic basis of mathematical morphology — Part II: Openings and closings, *Computer Vision, Graphics and Image Processing* **54** (1991) 74–97.
13. H. J. A. M. Heijmans, *Morphological Image Operators* (Academic Press, Boston, 1994).

14. L. Najman and H. Talbot, *Mathematical Morphology: From Theory to Applications* (ISTE-Wiley, June 2010).
15. C. Ronse, Adjunctions on the lattices of partitions and of partial partitions, *Applicable Algebra in Engineering, Communication and Computing* **21**(5) (2010) 343–396.
16. J. A. Goguen, L-fuzzy sets, *Journal of Mathematical Analysis and Applications* **18**(1) (1967) 145–174.
17. J. Medina, M. Ojeda-Aciego and J. Ruiz-Calvino, Formal concept analysis via multi-adjoint concept lattices, *Fuzzy Sets and Systems* **160**(2) (2009) 130–144.
18. I. Bloch and H. Maître, Fuzzy mathematical morphologies: A comparative study, *Pattern Recognition* **28**(9) (1995) 1341–1387.
19. I. Bloch, Duality versus adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations, *Fuzzy Sets and Systems* **160** (2009) 1858–1867.
20. M. Nachtegael and E. E. Kerre, Classical and fuzzy approaches towards mathematical morphology, in *Fuzzy Techniques in Image Processing*, eds. E. E. Kerre and M. Nachtegael, Studies in Fuzziness and Soft Computing, Chap. 1, Physica-Verlag, Springer, 2000, pp. 3–57.
21. J. Atif, I. Bloch, F. Distel, and C. Hudelot, Mathematical morphology operators over concept lattices, in *International Conference on Formal Concept Analysis*, Vol. LNAI 7880, Dresden, Germany, May 2013, pp. 28–43.
22. D. Dubois, F. Dupin de Saint-Cyr, and H. Prade, A possibility-theoretic view of formal concept analysis, *Fundamenta Informaticae* **75**(1) (2007) 195–213.
23. D. Dubois and H. Prade, Possibility theory and formal concept analysis: Characterizing independent sub-contexts, *Fuzzy Sets and Systems* **196** (2012) 4–16.
24. I. Düntsch and E. Orlowska, Mixing modal and sufficiency operators, *Bulletin of the Section of Logic, Polish Academy of Sciences* **28**(2) (1999) 99–106.
25. Y. Yao, A comparative study of formal concept analysis and rough set theory in data analysis, in *Rough Sets and Current Trends in Computing*, Vol. LNCS 3066, 2004, pp. 59–68.
26. R. Belohlavek, Fuzzy Galois connections, *Mathematical Logic Quarterly* **45**(4) (1999) 497–504.
27. R. Belohlavek, Concept lattices and order in fuzzy logic, *Annals of Pure and Applied Logic* **128**(1) (2004) 277–298.
28. R. Belohlavek and V. Vychodil, What is a fuzzy concept lattice? in *International Conference on Concept Lattices and Their Applications* (*CLA*), 2005, pp. 34–45.
29. A. B. Juandeaburre and R. Fuentes-González, The study of the L-fuzzy concept lattice, *Mathware & Soft Computing* **3** (1994) 209–218.
30. J. Medina, Multi-adjoint property-oriented and object-oriented concept lattices, *Information Sciences* **190** (2012) 95–106.
31. C. Alcalde, A. Burusco, J. C. Díaz, R. Fuentes-González, and J. Medina-Moreno, Fuzzy property-oriented concept lattices in morphological image and signal processing, in *International Work-Conference on Artificial Neural Networks*, Vol. LNCS 7903, 2013, pp. 246–253.
32. C. Alcalde, A. Burusco, and R. Fuentes-González, Application of the L-fuzzy concept analysis in the morphological image and signal processing, *Annals of Mathematics and Artificial Intelligence* **72**(1–2) (2014) 115–128.
33. J. C. Diaz, N. Madrid, J. Medina, and M. Ojeda-Aciego, New links between mathematical morphology and fuzzy property-oriented concept lattices, in *IEEE International Conference on Fuzzy Systems, FUZZ-IEEE*, 2014, pp. 599–603.

34. J. C. Díaz-Moreno, J. Medina, and M. Ojeda-Aciego, On basic conditions to generate multi-adjoint concept lattices via Galois connections, *International Journal of General Systems* **43**(2) (2014) 149–161.
35. Y. Djouadi and H. Prade, Possibility-theoretic extension of derivation operators in formal concept analysis over fuzzy lattices, *Fuzzy Optimization and Decision Making* **10**(4) (2011) 287–309.
36. G. Georgescu and A. Popescu, Non-dual fuzzy connections, *Archive for Mathematical Logic* **43**(8) (2004) 1009–1039.
37. I. Düntsch and G. Gediga, Modal-style operators in qualitative data analysis, in *IEEE International Conference on Data Mining* (*ICDM 2002*), 2002, pp. 155–162.
38. I. Bloch, On links between mathematical morphology and rough sets, *Pattern Recognition* **33**(9) (2000) 1487–1496.
39. I. Perfilieva, Fuzzy transforms: Theory and applications, *Fuzzy Sets and Systems* **157**(8) (2006) 993–1023.
40. I. Perfilieva, A. P. Singh, and S. P. Tiwari, On the relationship among F-transform, fuzzy rough set and fuzzy topology, in *2015 Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology* (*IFSA-EUSFLAT-15*), 2015.
41. B. Monjardet, Metrics on partially ordered sets — a survey, *Discrete mathematics* **35**(1) (1981) 173–184.
42. B. Leclerc, Lattice valuations, medians and majorities, *Discrete Mathematics* **111**(1) (1993) 345–356.
43. C. Orum and C. A. Joslyn, Valuations and metrics on partially ordered sets, *arXiv preprint arXiv:0903.2679*, 2009.
44. M. E. Cornejo, J. Medina, and E. Ramírez-Poussa, Attribute reduction in multi-adjoint concept lattices, *Information Sciences* **294** (2015) 41–56.
45. N. Caspard, B. Leclerc, and B. Monjardet, *Finite Ordered Sets: Concepts, Results and Uses* (Cambridge University Press, 2012).
46. X. Wang and W. Zhang, Relations of attribute reduction between object and property oriented concept lattices, *Knowledge-Based Systems* **21**(5) (2008) 398–403.
47. J. Medina, Relating attribute reduction in formal, object-oriented and property-oriented concept lattices, *Computers & Mathematics with Applications* **64**(6) (2012) 1992–2002.
48. I. Bloch and A. Bretto, Mathematical morphology on hypergraphs, application to similarity and positive kernel, *Computer Vision and Image Understanding* **117**(4) (2013) 342–354.
49. J. Cousty, L. Najman, F. Dias, and J. Serra, Morphological filtering on graphs, *Computer Vision and Image Understanding* **117**(4) (2013) 370–385.