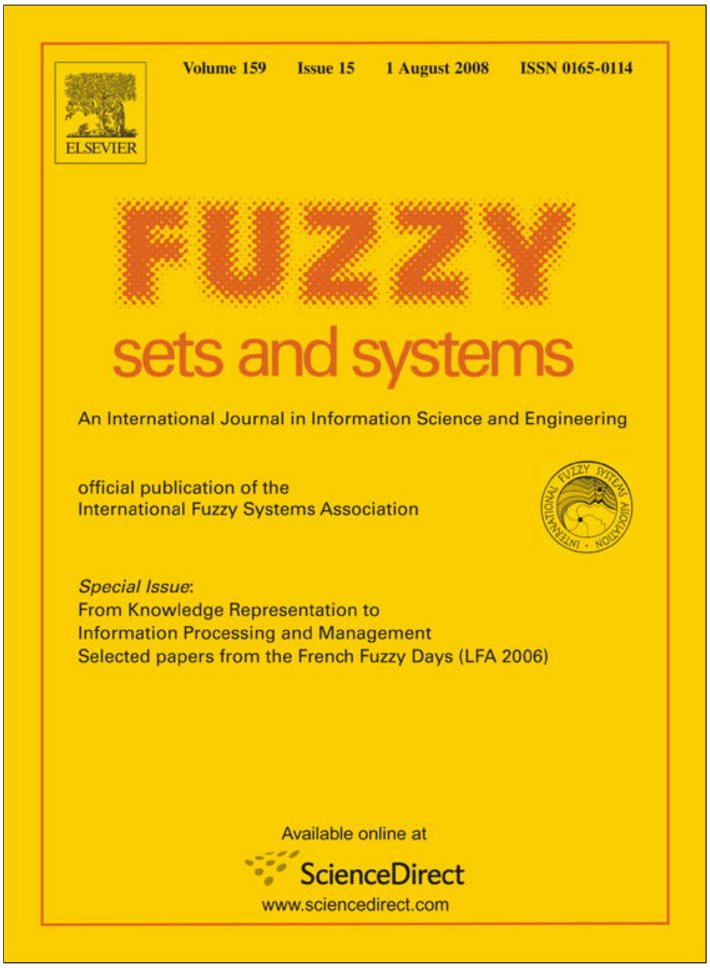
Volume 159 Issue 15 1 August 2008 ISSN 0165-0114
ELSEVIER
FUZZY
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An International Journal in Information Science and Engineering
official publication of the
International Fuzzy Systems Association
Special Issue:
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Information Processing and Management
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Fuzzy Sets and Systems 159 (2008) 1973 – 1990
FUZZY sets and systems
www.elsevier.com/locate/fss

Fuzzy skeleton by influence zones—Application to interpolation between fuzzy sets

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Available online 4 March 2008

Abstract

This paper introduces a definition of influence zones for fuzzy sets. The notion of fuzzy skeleton by influence zones (SKIZ), or fuzzy generalized Voronoï diagram, is derived. These definitions are based on fuzzy dilations and their interpretations in terms of distances. As another contribution, we show how this notion can be used to define a fuzzy median set, and a series of fuzzy sets interpolating between two fuzzy sets.

Keywords: Fuzzy skeleton by influence zones; Voronoï diagram; Fuzzy median set; Interpolation between fuzzy sets; Mathematical morphology

1. Introduction

The notions of Voronoï diagram, generalized Voronoï diagram or skeleton by influence zones (SKIZ) define regions of space which are closer to a region or object than to another one, and have important properties and applications [25,29]. Despite their interest, surprisingly enough they have not really been exploited in a fuzzy context until now. If knowledge or information is modeled using fuzzy sets, it is natural to see the influence zones of these sets as fuzzy sets too. The extension of these notions to the fuzzy case is therefore important, for applications such as partioning the space where fuzzy sets are defined, implementing the notion of separation, reasoning on fuzzy sets (fusion, interpolation, negotiations, spatial reasoning on fuzzy regions of space, etc.). These potential applications have motivated the work presented in this paper, which extends preliminary work presented in [4].

The first contribution is to propose definitions of notions of influence zones and SKIZ for fuzzy sets. Both influence zones and the SKIZ are then fuzzy sets, defined on the same space. The proposed definitions rely on formal expressions of the SKIZ in terms of distances and morphological dilations.

The second contribution is to exploit the notion of fuzzy SKIZ to define the median fuzzy set of two intersecting fuzzy sets. This median set can be interpreted as a fusion operator taking distance information into account, as a negotiation result between two sets of preferences or constraints, or as a compromise situation between two sets. The iterative application of the median set computation leads to the construction of a series of interpolating sets from one fuzzy set to another one. To our knowledge, this idea of interpolation between fuzzy sets is also novel.

No hypothesis on the definition space of the fuzzy sets is done. It can be continuous ($\mathbb{R}^n$ for instance) or discrete ($\mathbb{Z}^n$) and have any dimension. It will be denoted by $\mathcal{S}$. Similarly, no assumption on the semantics is done. The fuzzy

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doi:10.1016/j.fss.2008.02.016

sets defined on $\mathcal{S}$ can be spatial fuzzy sets, sets of preferences, of possibilities, etc. In the spatial domain, applications are similar to those developed on binary objects, but new applications using different semantics are possible as well, as those mentioned above.

The paper is organized as follows. The main definitions existing in the crisp case are recalled in Section 2, mainly issued from the domain of mathematical morphology. Two expressions of influence zones, in terms of morphological dilations on the one hand and in terms of distances on the other hand, constitute the basis for the proposed definitions in the fuzzy case (Section 3). Median fuzzy sets are then introduced in Section 4, as well as the interpolation between fuzzy sets.

2. Existing work: influence zones and SKIZ in the crisp case

Let us first recall how influence zones and SKIZ are classically defined in the crisp case. More details can be found e.g. in [25,29]. Notions of influence zones and SKIZ have been developed in particular in the image processing community, and the domain $\mathcal{S}$ in which sets are defined then represents the spatial domain. But the definitions are more general and do not make any assumption about the semantics of $\mathcal{S}$. The only assumption is that $\mathcal{S}$ is endowed with a distance d and a connectivity.

Definition 1. Let X be a subset of $\mathcal{S}$ composed of several connected components: $X = \bigcup_i X_i$, with $X_i \cap X_j = \emptyset$ for $i \neq j$. The influence zone of X_i, denoted as $IZ(X_i)$, is defined as the set of points which are strictly closer to X_i than to X_j for $j \neq i$, according to a distance d defined on $\mathcal{S}$ (usually the Euclidean distance or a discrete version of it on digital spaces):

$$IZ(X_i) = \{x \in \mathcal{S} / d(x, X_i) < d(x, X \setminus X_i)\}. \tag{1}$$

In $\mathbb{R}^n$, the components X_i are supposed to be compact sets, and the distance from a point to a compact set is classically defined as

$$d(x, X_i) = \inf_{y \in X_i} d(x, y),$$

where $d(x, y)$ denotes the Euclidean distance in $\mathbb{R}^n$. In the discrete case ($\mathcal{S} = \mathbb{Z}^n$ for instance), the subsets X_i are connected components in the sense of the discrete connectivity defined on the digital space.

From this notion, the definition of SKIZ is derived. [1]

Definition 2. The SKIZ of X, denoted as $SKIZ(X)$, is the set of points which belong to none of the influence zones, i.e. which are equidistant of at least two components X_i:

$$SKIZ(X) = \left(\bigcup_i IZ(X_i) \right)^c. \tag{2}$$

This definition extends the one of Voronoï diagram [18,23], where seed points are replaced by extended spatial entities. The SKIZ is also called generalized Voronoï diagram.

Note that the SKIZ is a subset of the morphological skeleton of X^c (i.e. the set of centers of maximal balls included in X^c where X^c denotes the complement of X in $\mathcal{S}$) [25,29]. It is not necessarily connected and contains in general less branches than the skeleton of X^c (this may be exploited in a number of applications).

An example of SKIZ for three objects defined in a 2D space is shown in Fig. 1. The lines of the SKIZ separate the influence zones associated with each object or connected component.

An important and interesting property of this definition based on distance (Eq. (1)) is that it can be expressed in terms of morphological operations as well. This feature will be the basis for one of the definitions we propose in the fuzzy case.

[1] In the discrete case, it may happen that the complement of the influence zones is empty due to parity problems, but we ignore such situations in this paper.

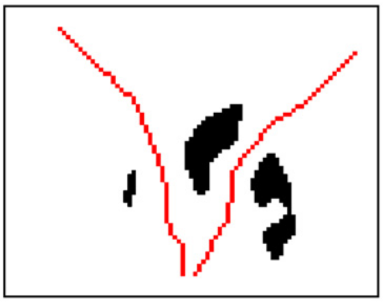

Fig. 1. Three objects (dark components) and their SKIZ, limiting the influence zones.

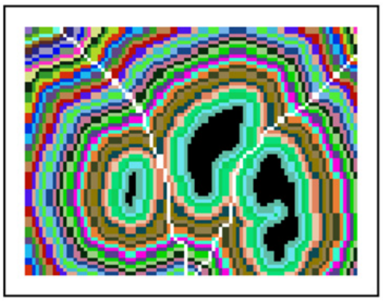

Fig. 2. Distance function to the connected components (the value at each point corresponds to the distance to the closest point of $\cup X_i$), and watersheds of this function, equal to the SKIZ.

Proposition 1 (*Beucher [1]*). *Let us denote by δ_λ the dilation by a ball of radius λ, and ε_λ the erosion by a ball of radius λ (see* [25] *for the main morphological concepts). Then the influence zones can be expressed as*

$$IZ(X_i) = \bigcup_\lambda \left(\delta_\lambda(X_i) \cap \varepsilon_\lambda \left(\left(\left(\bigcup_{j \neq i} X_j \right)^{\mathrm{c}} \right) \right) \right)$$
$$= \bigcup_\lambda \left(\delta_\lambda(X_i) \cap \left(\delta_\lambda \left(\bigcup_{j \neq i} X_j \right) \right)^{\mathrm{c}} \right)$$
$$= \bigcup_\lambda \left(\delta_\lambda(X_i) \setminus \delta_\lambda \left(\bigcup_{j \neq i} X_j \right) \right). \tag{3}$$

Another link between SKIZ and distance can be expressed by involving the watersheds.

Proposition 2 (*Soille [29]*). *The SKIZ is exactly the watersheds of the distance function computed at each point of X^{c}. By denoting $d(y, X)$ this function, for $y \in X^{\mathrm{c}}$, and by WS the watersheds, we have*

$$SKIZ(X) = WS(d(y, X), y \in X^{\mathrm{c}}). \tag{4}$$

This property is illustrated in Fig. 2.

3. Fuzzy influence zones and fuzzy SKIZ

While several notions involved in the SKIZ definition have been generalized to fuzzy sets (such as distances, dilations, erosions) influence zones and SKIZ have, to the best of our knowledge, never been defined in the case of fuzzy sets. This is the aim of this section. For sake of clarity, we will assume two fuzzy sets, with membership functions μ_1 and μ_2 defined on $\mathcal{S}$. The extension to an arbitrary number of fuzzy sets is then straightforward.

3.1. Fuzzy structuring element and fuzzy dilation and erosion

The morphological operations involved in the crisp case (Section 2) are performed using a structuring element which is a ball of a distance. In $\mathbb{R}^n$, the Euclidean distance is generally considered. In a digital space, such as $\mathbb{Z}^n$, a discrete distance is defined, based on an underlying discrete connectivity. The ball of radius 1 of this distance is then constituted by the center point and its neighbors according to the choice of the connectivity. More generally, the structuring element can be defined from a binary relation on $\mathcal{S}$, that is assumed to be symmetrical in this paper (which is consistent if it is a ball of a distance). In the fuzzy case, the same crisp structuring elements can be used. We can also base the operations on a fuzzy structuring element, which can represent local imprecision or a fuzzy binary relation. As will be seen later, this choice may lead to somewhat stronger properties. We denote the structuring element by its membership function v. All what follows applies for crisp and for fuzzy structuring elements.

In $\mathbb{R}^n$ or $\mathbb{Z}^n$, $v(x)$ represents the degree to which x belongs to v and $v(y-x)$ the degree to which y belongs to the translation of v at point x. If v is derived from a fuzzy binary relation, $v(y-x)$ denotes the degree to which y is in relation to x.

Let us denote by $\delta_v(\mu)$ and $\varepsilon_v(\mu)$ the dilation and erosion of the fuzzy set μ by the structuring element v. Here, dual definitions of these operations are chosen [7], i.e. verifying $\varepsilon_v(\mu^c) = (\delta_v(\mu))^c$, since this property is important, as seen in Eq. (3). They are expressed as

$$\forall x \in \mathcal{S}, \quad \delta_v(\mu)(x) = \sup_{y \in \mathcal{S}} \top[\mu(y), v(y-x)], \tag{5}$$

$$\forall x \in \mathcal{S}, \quad \varepsilon_v(\mu)(x) = \inf_{y \in \mathcal{S}} \bot[\mu(y), c(v(y-x))], \tag{6}$$

where $\top$ is a t-norm and $\bot$ the t-conorm dual of $\top$ with respect to a complementation c (which automatically guarantees the duality between δ and ε). In this paper, the following classical complementation is used:

$$\forall t \in [0,1], \quad c(t) = 1 - t. \tag{7}$$

Other definitions of fuzzy mathematical morphology have been proposed (e.g. [10–12,22,27]), based on different operators. Links with the ones used here are developed in [3,7].

Important properties of the definitions given in Eqs. (5) and (6), that will be intensively used in the following, are:

- fuzzy dilation and erosion are equivalent to the classical dilation and erosion in case both μ and v are crisp;
- $v(0) = 1 \Rightarrow \mu \leqslant \delta_v(\mu)$ and $\varepsilon_v(\mu) \leqslant \mu$, where 0 denotes the origin of $\mathcal{S}$ (if v represents a binary relation, it means that this relation is reflexive);
- fuzzy dilation and erosion are increasing with respect to μ, dilation is increasing with respect to v while erosion is decreasing;
- fuzzy dilation commutes with the supremum and fuzzy erosion with the infimum;
- duality: $\varepsilon_v(\mu^c) = (\delta_v(\mu))^c$;
- iterativity property: successive dilations (respectively, erosions) are equivalent to one dilation (respectively, erosion) with a structuring element equal to the dilation of all structuring elements.

3.2. Definition based on fuzzy dilations

Let us first consider the expression of influence zone using morphological dilations (Eq. (3)). This expression can be extended to fuzzy sets by using fuzzy intersection and union, and fuzzy mathematical morphology.

Definition 3. For a given structuring element v, we define the influence zone of μ_1 as

$$\begin{aligned} IZ_{\mathrm{dil}}(\mu_1) &= \bigcup_\lambda (\delta_{\lambda v}(\mu_1) \cap \varepsilon_{\lambda v}(\mu_2^{\mathrm{c}})) \\ &= \bigcup_\lambda (\delta_{\lambda v}(\mu_1) \cap (\delta_{\lambda v}(\mu_2))^{\mathrm{c}}) \\ &= \bigcup_\lambda (\delta_{\lambda v}(\mu_1) \setminus \delta_{\lambda v}(\mu_2)). \end{aligned} \tag{8}$$

The influence zone for μ_2 is defined in a similar way. The extension to any number of fuzzy sets μ_i is straightforward:

$$IZ_{\mathrm{dil}}(\mu_i) = \bigcup_\lambda \left(\delta_{\lambda v}(\mu_i) \cap \varepsilon_{\lambda v} \left(\left(\bigcup_{j \neq i} \mu_j \right)^{\mathrm{c}} \right) \right). \tag{9}$$

In these equations, intersection and union of fuzzy sets are implemented as t-norms $\top$ and t-conorms $\bot$ (min and max for instance). The fuzzy complementation used in the following is always $c(a) = 1 - a$, but other forms could be employed as well. Eq. (8) then reads:

$$IZ_{\mathrm{dil}}(\mu_1) = \sup_\lambda \top[\delta_{\lambda v}(\mu_1), 1 - \delta_{\lambda v}(\mu_2)]. \tag{10}$$

In the continuous case, if v denotes the elementary structuring element of size 1, then λv denotes the corresponding structuring element of size λ (for instance v is a ball of some distance of radius 1, then λv is the ball of radius λ). In the digital case, the operations performed using λv as structuring elements (λ being an integer in this case) are simply the iteration of λ operations performed with v (iterativity property of fuzzy erosion and dilation [7]):

$$\delta_{2v}(\mu) = \delta_v(\delta_v(\mu)), \quad \delta_{\lambda v}(\mu) = \delta_v(\delta_{(\lambda-1)v}(\mu)),$$

$$\varepsilon_{2v}(\mu) = \varepsilon_v(\varepsilon_v(\mu)), \quad \varepsilon_{\lambda v}(\mu) = \varepsilon_v(\varepsilon_{(\lambda-1)v}(\mu)).$$

Note that the number of dilations to be performed to compute influence zones in a digital bounded space $\mathcal{S}$ is always finite (and bounded by the length of the largest diagonal of $\mathcal{S}$).

Definition 4. The fuzzy SKIZ is then defined as:

$$SKIZ\left(\bigcup_i \mu_i\right) = \left(\bigcup_i IZ(\mu_i) \right)^{\mathrm{c}}. \tag{11}$$

This expression also defines a fuzzy (generalized) Voronoï diagram. Although the notion of Voronoï diagram has already been used in fuzzy systems (an example can be found e.g. in [17]), to our knowledge, no fuzzy version of it was defined until now.

3.3. Definitions based on distances

Another approach consists in extending the definition in terms of distances (Eq. (1)) and defining a degree to which the distance to one of the sets is lower than the distance to the other sets. Several definitions of the distance of a point to a fuzzy set have been proposed in the literature. Some of them provide real numbers and Eq. (1) can then be applied directly. But then the imprecision in the object definition is lost. Definitions providing fuzzy numbers are therefore more interesting, since if the sets are imprecise, it may be expected that distances are imprecise too, as also underlined e.g. in [2,14,21,24]. In particular, as will be seen next, it may be interesting to use the distance proposed in [2], based on fuzzy dilation:

$$d(x, \mu)(n) = \top[\delta_{nv}(\mu)(x), 1 - \delta_{(n-1)v}(\mu)(x)]. \tag{12}$$

It expresses, in the digital case, the degree to which x is at a distance n of μ ($\top$ is a t-norm, and $n \in \mathbb{N}^*$). For $n = 0$, the degree becomes

$$d(x, \mu)(0) = \mu(x).$$

This expression can be generalized to the continuous case as

$$d(x, \mu)(\lambda) = \inf_{\lambda' < \lambda} \top[\delta_{\lambda\nu}(\mu)(x), 1 - \delta_{\lambda'\nu}(\mu)(x)], \tag{13}$$

where $\lambda \in \mathbb{R}^{+*}$, and $d(x, \mu)(0) = \mu(x)$.

3.3.1. First method: comparing fuzzy numbers

When distances are fuzzy numbers, the fact that $d(x, \mu_1)$ is lower than $d(x, \mu_2)$ becomes a matter of degree. The degree to which this relation is satisfied can be computed using methods for comparing fuzzy numbers (see e.g. [31]). Let us consider the definition in [13], which expresses the degree $\mu(d_1 < d_2)$ to which $d_1 < d_2$, d_1 and d_2 being two fuzzy numbers, using the extension principle:

$$\mu(d_1 < d_2) = \sup_{a<b} \min(d_1(a), d_2(b)). \tag{14}$$

Definition 5. The influence zone of μ_1 based on the comparison of fuzzy numbers (using Eq. (14)) is defined as

$$\begin{aligned} IZ_{\text{dist1}}(\mu_1)(x) &= \mu(d(x, \mu_1) < d(x, \mu_2)) \\ &= \sup_{n<n'} \min[d(x, \mu_1)(n), d(x, \mu_2)(n')]. \end{aligned} \tag{15}$$

Note that this approach can be applied whatever the chosen definition of fuzzy distances.

Other methods for comparing fuzzy numbers could be used as well, and it could then be interesting to compare their impact on the resulting influence zones.

3.3.2. Second method: direct approach

When distances are more specifically derived from a dilation, as the ones in Eqs. (12) and (13), a more direct approach can be proposed, taking into account explicitly this link between distances and dilations. Indeed, in the binary case, the following equivalences hold:

$$\begin{aligned} (d(x, X_1) \leqslant d(x, X_2)) &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_2) \Rightarrow x \in \delta_\lambda(X_1)) \\ &\Leftrightarrow (\forall\lambda, x \in \delta_\lambda(X_1) \vee x \notin \delta_\lambda(X_2)). \end{aligned} \tag{16}$$

This means that if x is closer to X_1 than to X_2, x is reached faster by dilating X_1 than by dilating X_2, as illustrated in Fig. 3.

This expression extends to the fuzzy case as follows.

Definition 6. The degree $\mu(d(x, \mu_1) \leqslant d(x, \mu_2))$ to which $d(x, \mu_1)$ is less than $d(x, \mu_2)$ is defined as

$$\mu(d(x, \mu_1) \leqslant d(x, \mu_2)) = \inf_{\lambda} \bot(\delta_{\lambda\nu}(\mu_1)(x), 1 - \delta_{\lambda\nu}(\mu_2)(x)), \tag{17}$$

where $\bot$ is a t-conorm.

This equation defines a new way to compare fuzzy numbers representing distances.

The comparison of fuzzy numbers representing distances, as given by Eq. (17) is reflexive ($\mu(d(x, \mu_1) \leqslant d(x, \mu_1)) = 1$) if and only if $\bot$ is a t-conorm verifying the excluded middle law (Lukasiewicz t-conorm for instance). Moreover, in case the fuzzy numbers are usual numbers, the comparison reduces to the classical comparison between numbers.

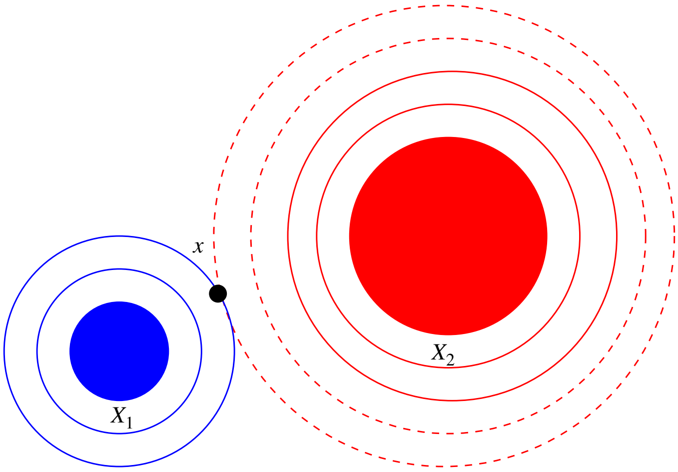

Fig. 3. Two sets X_1 and X_2 and their successive dilations. Point x is closer to X_1 than to X_2 and therefore it is reached faster from X_1 than from X_2 using dilations.

Defining influence zones requires a strict inequality between distances, which is deduced by complementation:

$$\mu(d(x,\mu_1) < d(x,\mu_2)) = 1 - \mu(d(x,\mu_2) \leqslant d(x,\mu_1)). \tag{18}$$

Definition 7. The influence zone of μ_1 using the comparison introduced in Definition 6 is defined as

$$IZ_{\text{dist2}}(\mu_1)(x) = 1 - \inf_\lambda \bot(\delta_{\lambda\nu}(\mu_2)(x), 1 - \delta_{\lambda\nu}(\mu_1)(x)). \tag{19}$$

Whatever the chosen definition of *IZ*, the SKIZ is always defined by Eq. (11).

3.4. Comparison and properties

It is easy to show that the definitions derived from the dilation approach and from the direct distance approach are equivalent.

Proposition 3. *Definitions* 3 *and* 7 *are equivalent*:

$$IZ_{\text{dil}}(\mu_1) = IZ_{\text{dist2}}(\mu_1). \tag{20}$$

Proof. Using the expression of IZ_{dil} given by Eq. (10) and Definition 6, the following equalities are derived:

$$\begin{aligned} IZ_{\text{dil}}(\mu_1) &= \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1), 1 - \delta_{\lambda\nu}(\mu_2)] \\ &= 1 - \inf_\lambda \bot[1 - \delta_{\lambda\nu}(\mu_1), \delta_{\lambda\nu}(\mu_2)] \\ &= 1 - \mu(d(x,\mu_2) \leqslant d(x,\mu_1)) \\ &= \mu(d(x,\mu_1) < d(x,\mu_2)) = IZ_{\text{dist2}}(\mu_1). \quad \square \end{aligned}$$

Although this result is not surprising, both interpretations in terms of dilation and distance remain interesting.

However, the two distance-based approaches are not equivalent, since they rely on different orderings between fuzzy sets. Actually the direct approach always provides a larger result.

Proposition 4. $\forall x \in \mathcal{S}, \quad IZ_{\text{dist1}}(\mu_1)(x) \leqslant IZ_{\text{dist2}}(\mu_1)(x). \tag{21}$

Proof. Let us consider the discrete case (the proof in the continuous case is similar):

$$\begin{aligned} IZ_{\text{dist1}}(\mu_1)(x) &= \sup_{n<n'} \min[d(x,\mu_1)(n), d(x,\mu_2)(n')] \\ &= \sup_{n<n'} \min[\top(\delta_{n\nu}(\mu_1)(x), 1-\delta_{(n-1)\nu}(\mu_1)(x)), \top(\delta_{n'\nu}(\mu_2)(x), 1-\delta_{(n'-1)\nu}(\mu_2)(x))] \\ &\leqslant \sup_{n<n'} \min[\delta_{n\nu}(\mu_1)(x), 1-\delta_{(n'-1)\nu}(\mu_2)(x)]. \end{aligned}$$

Since $n'-1 \geqslant n$, we have $\delta_{(n'-1)\nu}(\mu_2)(x) \geqslant \delta_{n\nu}(\mu_2)(x)$ and

$$IZ_{\text{dist1}}(\mu_1)(x) \leqslant \sup_{n} \min[\delta_{n\nu}(\mu_1)(x), 1-\delta_{n\nu}(\mu_2)(x)]$$

and $IZ_{\text{dist1}}(\mu_1)(x) \leqslant IZ_{\text{dist2}}(\mu_1)(x)$. □

In terms of complexity, the direct approach is computationally less expensive.

Proposition 5. *For N being the size of $\mathcal{S}$ in each dimension, the complexity of computation of IZ_{dist1} is in $O(N^5)$ in 3D and $O(N^4)$ in 2D. The complexity of computation of IZ_{dist2} or IZ_{dil} is at least one order of magnitude less.*

Proof. Let us assume that $\mathcal{S}$ is a bounded space defined by an hypercube of edge length equal to N. In two dimensions, it has a size of N^2, in three dimensions N^3, etc.

In the approach based on comparison of fuzzy numbers, the fuzzy sets $d(x,\mu)(n)$ are computed for a number of values of n equal to at most the length of the hypercube diagonal, i.e. of the order of N. Let us denote by N_ν the size of the support of the structuring element ν ($N_\nu \ll N$ in general). The cost of a dilation in 3D is in $O(N^3 \times N_\nu)$ and the computation of all distances in $O(N^4 \times n)$. The comparison of fuzzy numbers is in $O(N^5)$ for the whole hypercube volume. The computation of IZ_{dist1} is then in $O(N^5)$ in 3D and $O(N^4)$ in 2D.

For the direct approach, the computation is in $O(N^3 \times N_\nu \times N_d)$ in 3D and in $O(N^2 \times N_\nu \times N_d)$ where N_d is the number of dilations to be performed (much smaller than N in general, and bounded by N). Since $N_\nu \ll N$ in general, the computation cost of IZ_{dist2} (and equivalently IZ_{dil}) is at least one order of magnitude less than for IZ_{dist1}. □

An important property concerns the consistency with the crisp case, as generally required when extending an operation on crisp sets to fuzzy sets.

Proposition 6. *Definitions* 3–5 *and* 7 *are equivalent to the classical definitions in case of crisp sets and crisp structuring elements.*

Proof. It is sufficient to prove that Definitions 3 and 5 are consistent with the crisp case.

Since fuzzy dilations and erosions are consistent with the crisp definitions in case the set to be transformed and the structuring element are crisp, Definition 3 reduces to Eq. (3) and hence to Eq. (1) thanks to Proposition 1.

Since the comparison between two fuzzy numbers given by Eq. (14) is consistent with the crisp case (i.e. if d_1 and d_2 are crisp numbers, $\mu(d_1 < d_2) = 1$ iff $d_1 < d_2$ and 0 otherwise), Definition 5 reduces to Eq. (1) if μ_1 and μ_2 are crisp. □

Finally, the SKIZ is symmetrical with respect to the μ_i, hence independent of their order.

3.5. Illustrative example

The notion of fuzzy SKIZ is illustrated on the three objects of Fig. 4. The structuring element ν is a crisp 3×3 square in Fig. 5 and a fuzzy set of paraboloid shape in Fig. 6. The influence zones of each object are displayed, as well as the SKIZ. These results are obtained with the dilation-based definition. Each influence zone is characterized by high membership values close to the corresponding object, and decreasing when the distance to this object increases. The use of a fuzzy structuring element results in more fuzziness in the influence zones and SKIZ. The crest lines of

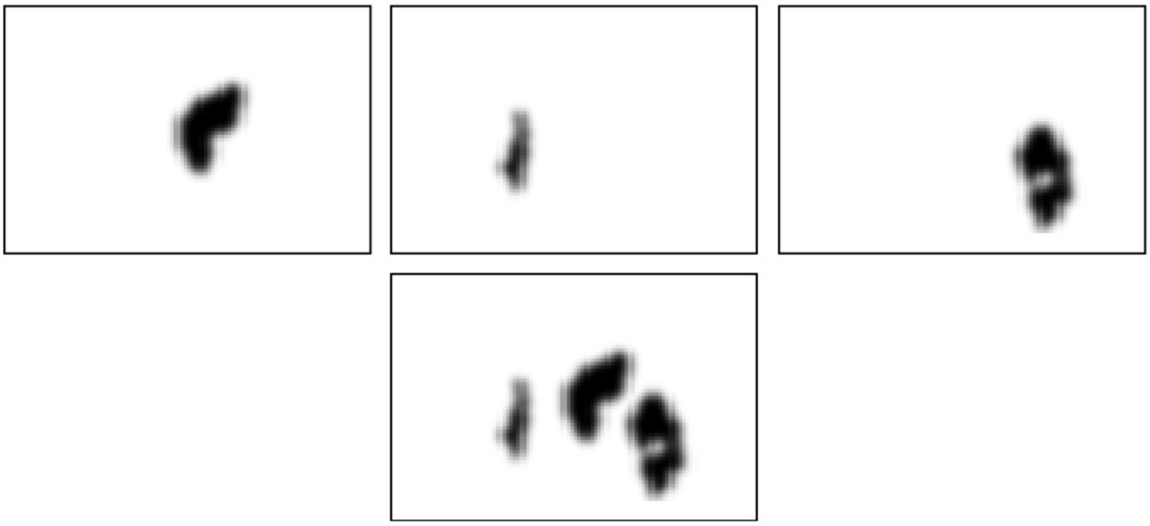

Fig. 4. Three fuzzy objects and their union. Membership degrees range from 0 (white) to 1 (black).

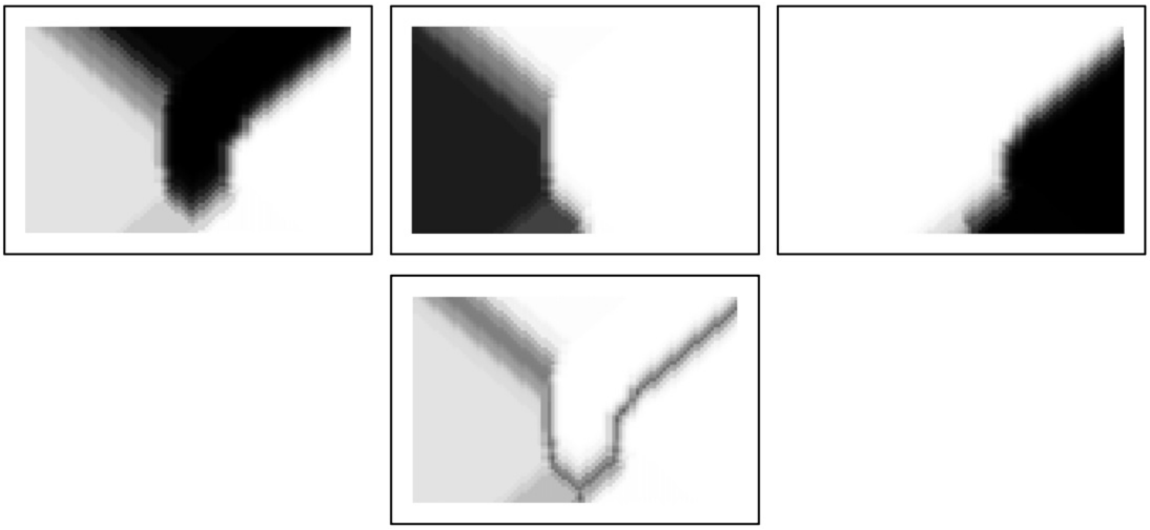

Fig. 5. Influence zones of the three fuzzy objects of Fig. 4 and resulting fuzzy SKIZ, obtained using a binary structuring element (3 × 3 square) and the dilation-based approach.

the fuzzy SKIZ are similar to the crisp SKIZ obtained for crisp version of the objects (Fig. 1), which was an expected result.

A binary decision can be made in order to obtain a crisp SKIZ of fuzzy objects. An appropriate approach is suggested by the previous comment about the crest lines of the result, and consists in computing the watershed lines of the fuzzy SKIZ. It is appropriate in the sense that it provides spatially consistent lines, without holes, and going through the crest lines of the membership function of the SKIZ. A result is provided in Fig. 7. For a fuzzy structuring element ν, the lines can go through the objects (Fig. 7(b)). While this is impossible in the binary case, in the fuzzy case this is explained by the fact that an object can, to some degree, be built of several connected components, linked together by points with low membership degrees. The values of the SKIZ at those points are low too. This is the case for the third object in Fig. 4. The low values of the SKIZ along the line traversing this object are in accordance with the fact that the object has only one connected component with some degree, and two components with some degree. The line separating the third object can be suppressed by eliminating the parts of the watersheds having a very low degree in the fuzzy SKIZ (Fig. 7(c)). This requires to set a threshold value.

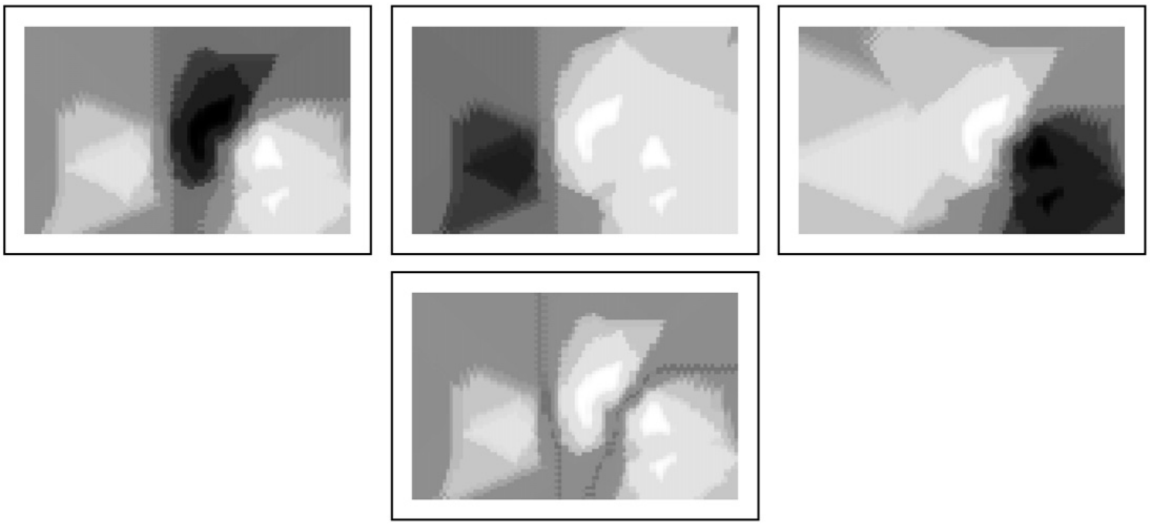

Fig. 6. Influence zones of the three fuzzy objects of Fig. 4 and resulting fuzzy SKIZ, obtained using a fuzzy structuring element (paraboloid shaped) and the dilation-based approach.

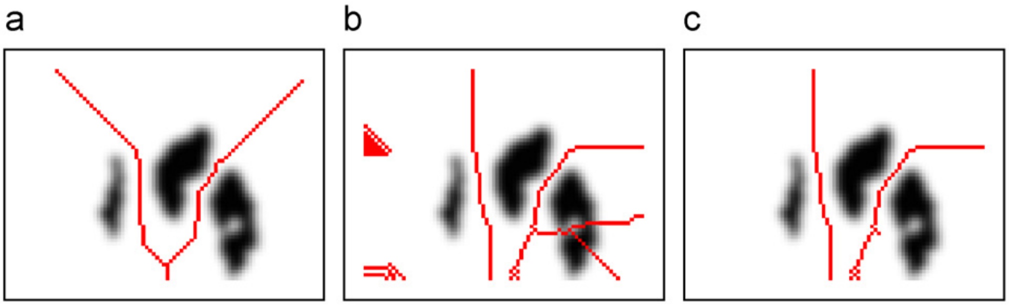

Fig. 7. Binary decision using watershed for v crisp (a) and fuzzy (b). Lines with a very low membership degree in the SKIZ of (b) have been suppressed in (c).

4. Fuzzy median set and interpolation between fuzzy sets

In the mathematical morphology community, two types of approaches have been considered to define the median set of two crisp sets, or to interpolate between two sets. The first one relies on the SKIZ [1,30], while the second one relies on the notion of geodesics of some distance [9,15,16,26,28]. Here, we propose to extend the first approach to the case of fuzzy sets, based on the definitions of the fuzzy SKIZ proposed in Section 3.

The morphological definition of the median set from the SKIZ applies originally to sets X and Y having a nonempty intersection. The SKIZ is computed for $X_1 = X \cap Y$ and $X_2 = (X \cup Y)^c$. The median set of X and Y is defined as the influence zone of $X_1 = X \cap Y$, i.e. as the set of points which are closer to $X \cap Y$ than to the complement of $X \cup Y$. Fig. 8 illustrates this notion.

4.1. Definitions

Let us consider two fuzzy objects with membership functions μ_1 and μ_2 and with intersecting supports. Two definitions can be given for the fuzzy median set, depending on the chosen definition for the influence zones.

Definition 8. Based on the definition of influence zones from dilations, or equivalently the direct approach from distances, the median fuzzy set of μ_1 and μ_2 is defined as the influence zone of $\mu_1 \cap \mu_2$ with respect to $(\mu_1 \cup \mu_2)^c$

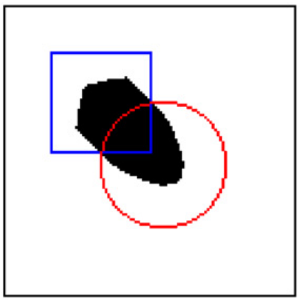

Fig. 8. Median set (in black) between a square and a disk, defined from the SKIZ between the intersection and the complement of the union of the two objects.

(intersection is still defined by a t-norm and union by a t-conorm):

$$\forall x \in \mathcal{S}, \quad M(\mu_1, \mu_2)(x) = \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), 1 - \delta_{\lambda\nu}((\mu_1 \cup \mu_2)^c)(x)]$$
$$= \sup_\lambda \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x)]. \quad (22)$$

Definition 9. By using the definition of influence zones based on comparison of fuzzy distances, the median set is defined as

$$\forall x \in \mathcal{S}, \quad M'(\mu_1, \mu_2)(x) = \sup_{n<n'} \min[d(x, \mu_1 \cap \mu_2)(n), d(x, (\mu_1 \cup \mu_2)^c)(n')]. \quad (23)$$

Proposition 7. *For any two fuzzy sets μ_1 and μ_2, we always have*

$$\forall x \in \mathcal{S}, \quad M'(\mu_1, \mu_2)(x) \leqslant M(\mu_1, \mu_2)(x). \quad (24)$$

Proof. This proposition directly derives from the relation between IZ_{dist1} and IZ_{dist2} (or equivalently IZ_{dil}), expressed in Proposition 4. □

This notion of median set can be exploited to derive a series of interpolating sets between μ_1 and μ_2, by applying recursively the median computation in a dichotomic process.

Definition 10. Let μ_1 and μ_2 be two fuzzy sets. A series of interpolating sets is defined by recursive application of the median computation:

$$Interp(\mu_1, \mu_2)_{1/2} = M(\mu_1, \mu_2),$$
$$Interp(\mu_1, \mu_2)_{1/4} = M(Interp(\mu_1, \mu_2)_{1/2}, \mu_1),$$
$$Interp(\mu_1, \mu_2)_{1/8} = M(Interp(\mu_1, \mu_2)_{1/4}, \mu_1),$$
$$Interp(\mu_1, \mu_2)_{3/8} = M(Interp(\mu_1, \mu_2)_{1/4}, Interp(\mu_1, \mu_2)_{1/2}),$$
$$\vdots$$
$$Interp(\mu_1, \mu_2)_{3/4} = M(Interp(\mu_1, \mu_2)_{1/2}, \mu_2),$$
$$\vdots$$

This process can be written in a general recursive form as

$$Interp_0 = \mu_1,$$
$$Interp_1 = \mu_2,$$
$$Interp_{(i+j)/2} = M(Interp_i, Interp_j) \quad \text{for } 0 \leqslant i \leqslant 1, \quad 0 \leqslant j \leqslant 1.$$

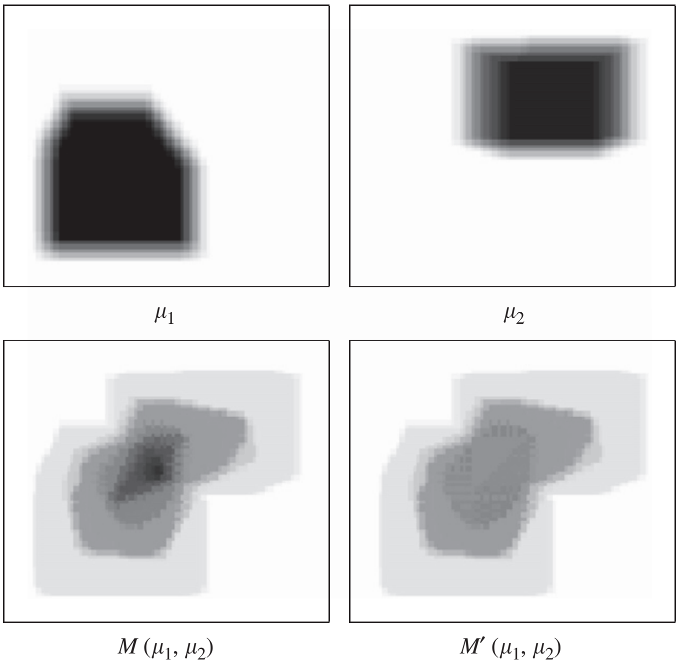

Fig. 9. Two fuzzy objects μ_1 and μ_2 and their median set, computed using Eqs. (22) and (23), with a paraboloid shaped structuring element. The second result provides lower membership values.

This sequence allows transforming progressively μ_1 into μ_2. These two fuzzy sets can represent spatial objects, different situations, sets of constraints or preferences, etc. For instance the sequence allows building intermediate estimates between distant observations or pieces of information.

4.2. *Examples*

As an illustration, let us consider two fuzzy objects in a 2D space. The median sets M and M' computed, respectively, by Eqs. (22) and (23) are illustrated in Fig. 9. It can be actually observed that M' leads to lower membership values than M (Proposition 7). The result provided by M is visually more satisfactory and is moreover faster to compute. Therefore, in the following the chosen definition is the one given by Eq. (22).

Fig. 10 illustrates an example of interpolation between two fuzzy sets. The series of interpolating fuzzy sets is computed recursively from the median set (the fourth set in the sequence displayed in the figure). It is clear on this example that the shape of interpolating sets evolves progressively from the one of the first object towards the one of the second object. This evolution is in accordance with the expected interpolation notion.

Let us now consider real objects, from medical images. We consider the putamen (a brain structure) in different subjects, obtained from the IBSR database. [2] The images are registered, which guarantees a good correspondence between the different instances. Fuzziness at the boundary of the objects is introduced to represent spatial imprecision due to partial volume effect or imprecise segmentation, using a fuzzy dilation. Four examples of the resulting fuzzy objects are illustrated in Fig. 11.

The fuzzy median set has been computed between the two first instances, then between this result and the third instance, etc. Results are displayed in Fig. 12. Using this iterative approach, the fuzzy median set between the 18 instances of this structure has been computed (corresponding to the 18 normal subjects of the IBSR database). Fig. 13 shows the fuzzy median sets for four structures (thalamus and putamen in both hemispheres) on the 18 normal subjects of the database. Such results could be used for instance for representing the inter-individual variability, or to build anatomical atlases.

[2] http://www.cma.mgh.harvard.edu/ibsr/.

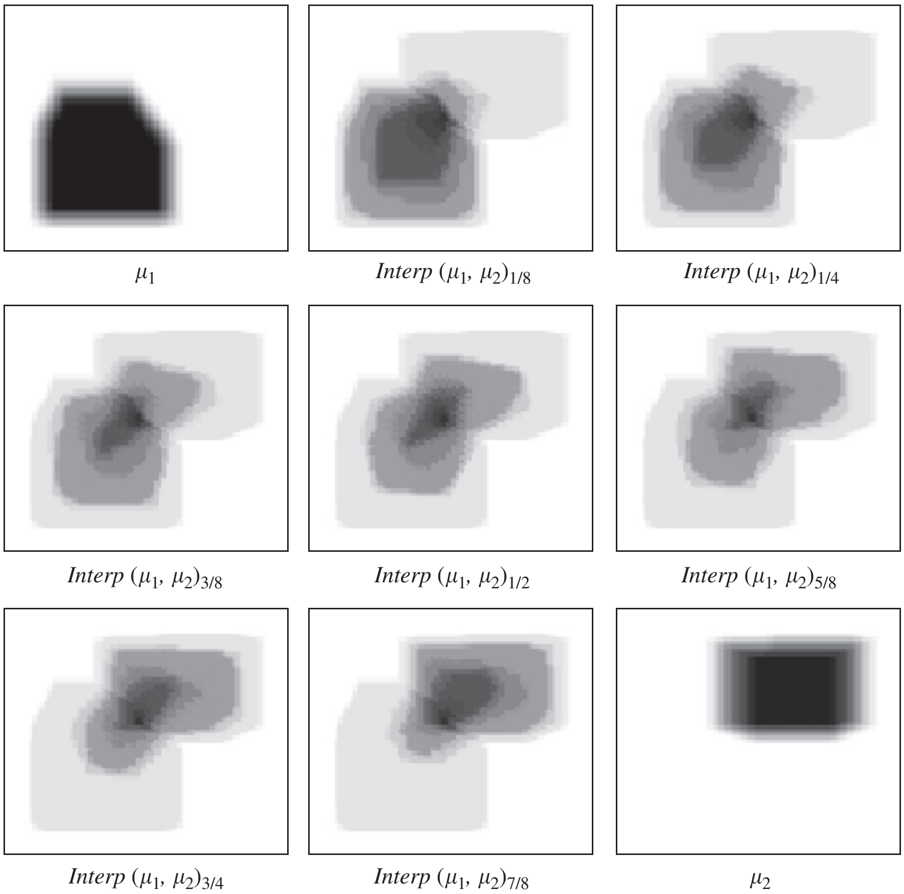

Fig. 10. Interpolation between two fuzzy sets μ_1 and μ_2.

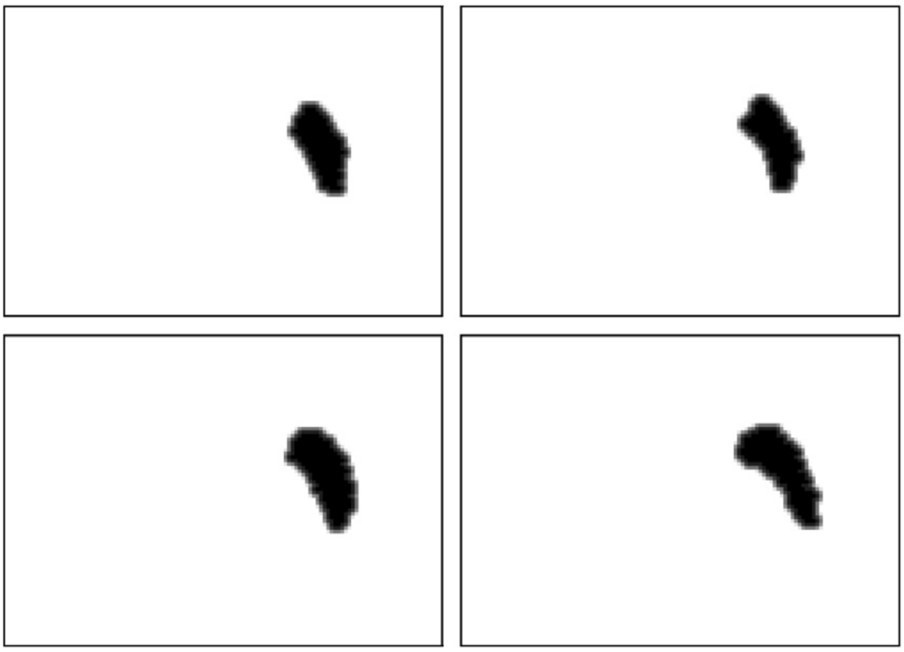

Fig. 11. Four instances of a brain structure from four different subjects.

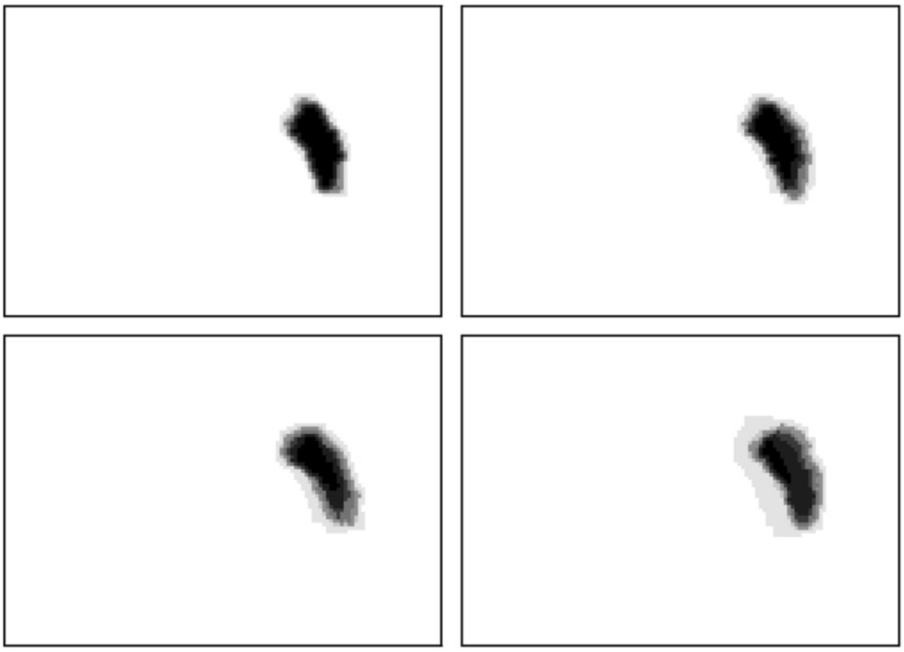

Fig. 12. Median set between two, three, four instances (shown in Fig. 11) and between the 18 instances of the IBSR database.

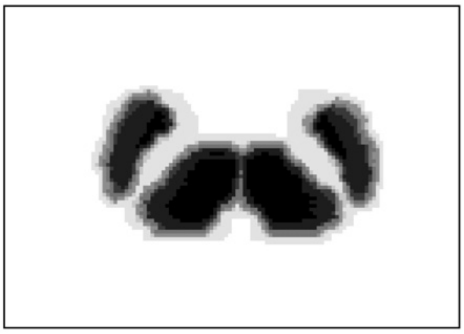

Fig. 13. Median sets between the 18 instances of the IBSR database for four structures (thalamus and putamen in both hemispheres).

Let us now consider another example, in the domain of preference modeling, as in [20]. We consider a propositional language based on a finite set of propositional symbols, on which formulas are defined. Morphological operators can be defined on such formulas [6]. We denote by Ω the set of all interpretations. The models of a formula are considered as a fuzzy subset of Ω. To illustrate the application of the median operator, we consider a simple example, with three propositional symbols a, b, c, and two formulas φ_1 and φ_2, expressing, respectively, preferences for $\neg ab\neg c$ with a degree 0.2, and preferences for anything except abc with degrees as given in Table 1. For defining the morphological operators, we use the Hamming distance (i.e. two models are at a distance equal to the number of symbols instantiated differently), and the structuring elements are the balls of this distance. The conjunction of φ_1 and φ_2 is equal to φ_1 and their disjunction is equal to φ_2.

The successive steps of the computation of the median set are illustrated in Table 1. The models of the median set also constitute a fuzzy set of Ω. Note that dilation and erosion by a ball of radius 0 is the identity mapping. On this example, the classical fusion, according to [19] would lead to the intersection of the sets of models, i.e. φ_1. But this might not be fair, as illustrated on a simple crisp example in Fig. 14.

The result of the median is somewhat larger, since it includes also a model of φ_2 that was not a model of φ_1 ($\neg a\neg b\neg c$), and gives a more fair point of view expressing an intermediate solution between both sets of preferences. This can be interpreted as follows: if an individual has a set of preferences described by φ_1, which is very strict and constraining,

Table 1
Fuzzy sets of Ω representing the preferences expressed by φ_1 and φ_2, and derivation of $M(\varphi_1, \varphi_2)$

Models	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
φ_1	0	0	0	0	0	0.2	0	0
φ_2	0	0.5	0.5	0.5	0.5	0.8	0.5	0.7
$\delta_1(\varphi_1)$	0	0.2	0	0.2	0	0.2	0	0.2
$\varepsilon_1(\varphi_2)$	0	0	0	0	0.5	0.5	0.5	0.5
$\delta_2(\varphi_1)$	0.2	0.2	0	0.2	0.2	0.2	0.2	0.2
$\varepsilon_2(\varphi_2)$	0	0	0	0	0	0	0	0.5
$M(\varphi_1, \varphi_2)$	0	0	0	0	0	0.2	0	0.2

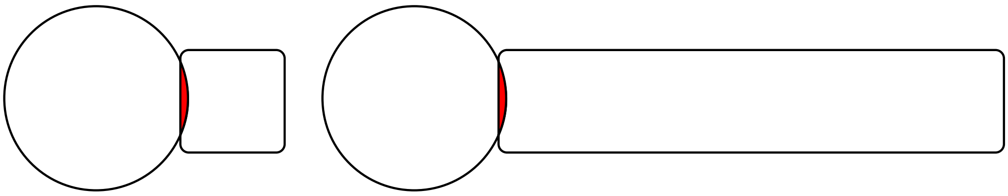

Fig. 14. An example illustrating why the intersection is not necessarily the best median set: in the second situation the intersection leads to exactly the same result as in the first one, while the second set expresses preferences that are very far away of the first one and are not taken into account.

he will be tempted to extend his preferences to obtain a better agreement with the preferences of the second individual. On the other hand, the second individual is ready to restrict his choices to achieve a consensus with the first one, and will be more satisfied if a fair account of all his preferences is obtained. Note that the fact that the median is included in the disjunction (see Proposition 8) guarantees that it does not contain a solution that nobody wants to accept. The resulting membership degrees reflect the low consistency that exists between both sets of preferences on this example.

4.3. Some properties

Proposition 8. *If the origin belongs to the structuring element ν with a membership value equal to* 1 *or if ν represents a reflexive relation* (*which is the condition to have extensive dilations*), *then the median set is included in the union of the two objects*:

$$\forall x \in \mathcal{S}, \quad M(\mu_1, \mu_2)(x) \leqslant (\mu_1 \cup \mu_2)(x). \tag{25}$$

Proof. If $\nu(0) = 1$, then erosion is anti-extensive, and

$$\forall x, \forall \lambda, \quad \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x) \leqslant (\mu_1 \cup \mu_2)(x).$$

From increasingness of the supremum and any t-norm $\top$, we derive

$$\forall x, \quad \sup_{\lambda} \top[\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x), \varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)(x)] \leqslant (\mu_1 \cup \mu_2)(x). \qquad \square$$

Proposition 9. *Under the same condition* ($\nu(0) = 1$), *the cores verify the following inclusion relations*:

$$Core(\mu_1 \cap \mu_2) \subseteq Core(M(\mu_1, \mu_2)) \subseteq Core(\mu_1 \cup \mu_2). \tag{26}$$

Note that the core of the median set can be empty, as in the example of Fig. 9.

Proof. If $x \in Core(\mu_1 \cap \mu_2)$ and $\nu(0) = 1$, then $x \in Core(\delta_{\lambda\nu}(\mu_1 \cap \mu_2))$, and $M(\mu_1, \mu_2)(x) = 1$. This shows that $Core(\mu_1 \cap \mu_2) \subseteq Core(M(\mu_1, \mu_2))$.

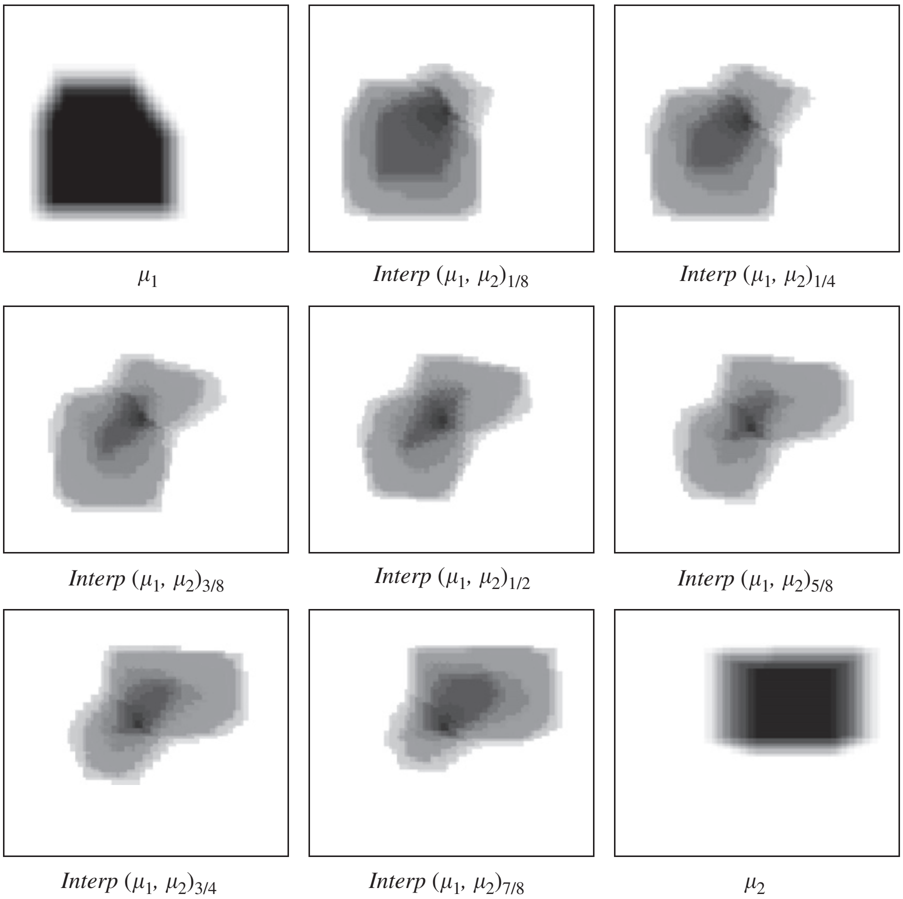

Fig. 15. Interpolation between two fuzzy sets μ_1 and μ_2, by eliminating low membership values in the result, thus reducing the support of the interpolated sets.

Since the erosion is anti-extensive if $v(0) = 1$, we have

$$\forall x, \forall \lambda, \quad \varepsilon_{\lambda v}(\mu_1 \cup \mu_2)(x) \leqslant (\mu_1 \cup \mu_2)(x).$$

From the definition of M, it directly follows that $M(\mu_1, \mu_2)(x) = 1 \Rightarrow \varepsilon_{\lambda v}(\mu_1 \cup \mu_2)(x) = 1$ and thus $(\mu_1 \cup \mu_2)(x) = 1$. □

Proposition 10. *If additionally the origin is the only modal value of* v ($v(0) = 1$ *and* $\forall x \in \mathcal{S} \setminus \{0\}, v(x) < 1$), *then the median set and the union of the two sets have the same support*:

$$Supp(M(\mu_1, \mu_2)) = Supp(\mu_1 \cup \mu_2), \tag{27}$$

and the cores of the median set and of the intersection are equal:

$$Core(\mu_1 \cap \mu_2) = Core(M(\mu_1, \mu_2)). \tag{28}$$

In particular, in the case where the structuring element is crisp (for instance a square of size 3×3), this property does not hold, while it holds for the paraboloid shaped structuring element used in the presented results.

Proof. If the origin is the only modal value, i.e. $v(y) = 1 \Leftrightarrow y = 0$, then the following equivalences hold:

$$\begin{aligned} \varepsilon_v(\mu_1 \cup \mu_2)(x) = 0 &\Leftrightarrow \inf_y \bot[(\mu_1 \cup \mu_2)(y), 1 - v(y - x)] = 0 \\ &\Leftrightarrow \exists\, y \in \mathcal{S}, (\mu_1 \cup \mu_2)(y) = 0 \text{ and } v(y - x) = 1 \\ &\Leftrightarrow (\mu_1 \cup \mu_2)(x) = 0 \end{aligned}$$

(since necessarily $x = y$). This shows that the erosion has the same support as the original fuzzy set: $Supp(\varepsilon_\nu(\mu_1 \cup \mu_2)) = Supp(\mu_1 \cup \mu_2)$. Similarly, $Supp(\varepsilon_{\lambda\nu}(\mu_1 \cup \mu_2)) = Supp(\mu_1 \cup \mu_2)$ for any λ. For λ large enough (assuming that ν is not reduced to the origin, which would not provide anything), we have $\delta_{\lambda\nu}(\mu_1 \cap \mu_2)(x) \neq 0$ for $x \in Supp(\mu_1 \cup \mu_2)$. This shows that

$$M(\mu_1, \mu_2)(x) = 0 \quad \Leftrightarrow \quad \varepsilon_\nu(\mu_1 \cup \mu_2)(x) = 0$$

from which we derive $Supp(M(\mu_1, \mu_2)) = Supp(\mu_1 \cup \mu_2)$.

The proof for the core is similar. □

Fig. 10 illustrates that the median set and the union have the same support. It should be noticed that in a large part of the support, the membership values are very low (0.1 in this example), and that it would be very easy to eliminate these low values if a more reduced support is desired, as could be intuitively preferable. This is illustrated in Fig. 15.

4.4. Nonintersecting fuzzy sets

Let us briefly comment possible extensions of the proposed approach to nonintersecting objects. In the spatial domain, the approach proposed in [30] in the case of crisp objects can be easily extended. It consists in performing a translation of each object by a translation vector equal to the half of the vector joining the center of gravity of this object to the center of gravity of the other one. This ensures that the translated objects intersect each other. The median set is then computed on these translated sets, leading to a result that is a median in terms of shape and located midway between the two original objects.

This approach is intuitively very satisfactory and can be extended directly to the case of fuzzy objects. It can be applied as soon as the underlying space $\mathcal{S}$ is endowed with a affine structure (as the usual spatial domain), in order to define translations. If this is not the case, the question remains open and is left for future work.

5. Conclusion

In this paper, novel notions of fuzzy SKIZ, median and interpolation were introduced, based on mathematical morphology concepts. The proposed definitions are applicable whatever the dimension of the underlying space $\mathcal{S}$ and whatever the semantics attached to the fuzzy sets. The only hypothesis is that it should be possible to define a structuring element, either from a distance on $\mathcal{S}$ or from a binary symmetrical relation.

The definitions of median set and interpolation can be extended to nonintersecting fuzzy sets if a translation on $\mathcal{S}$ can be defined. The cases where $\mathcal{S}$ does not have an affine structure are planned for future work.

Another approach for defining median sets in the crisp case is based on geodesic distances [26]. Extension of this approach to the fuzzy case could be another interesting research direction. Extensions to a logical framework for mediation applications were proposed in [8], but not to the fuzzy sets framework until now.

Extensions of the median set to more than two fuzzy sets could be interesting too, for instance for deriving generic models from different instances. A typical application could be the generation of anatomical atlases from several individual observations. In the brain structure example, the median has been computed iteratively, a process which depends on the order. A direct method involving all objects simultaneously deserves to be developed.

Finally, applications of the propositions of this paper could be further explored, for instance for fusion, with a comparison to other operators also based on distance [19,20], for the definition of compromises or negotiations, for smoothing fuzzy sets representing preferences, observations, etc., or for finding the fuzzy sets in between two sets [5].

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